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Solution of Linear and Non-Linear Ordinary Differential Equations Using the Variational Iteration Method

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Abstract: ***In this paper, we apply He's variational iteration method to compute approximate solutions of linear and nonlinear systems of Ordinary Differential Equations (ODEs). The results show that the exact solution and the solution of the variational iteration method are in complete agreement.***

1. Introduction

The Variational Iteration Method (VIM), first envisioned by He [2] modifying the approach by Inokuti *et al* [3], has been applied to many linear and nonlinear ODEs and PDEs. For example, He [4] employed VIM to give approximate solutions for some well – known nonlinear problems and in [5], he successfully applied VIM to autonomous systems of ODEs. Also, He [6] gave a solution for a seepage flow problem with fractional derivatives in porous media using VIM. Other researchers demonstrated further applications of VIM. For instance, Soliman [7] applied VIM to solve KdV – Burger's and Lax's seventh – order KdV equations. Momani and Abuasad [8] also used VIM to solve the Helmholtz equation and Odibat and Momani [9] solved nonlinear fractional differential equations via VIM; while Batiha *et al* [10] solved the general Riccati equation using VIM. Bildik [11] used VIM for solving different types of nonlinear partial differential equations, while Abbasbandy [12] solved the quadratic Riccati differential equations by He's VIM with considering Adomian polynomials. Jufeng [13] applied VIM to solve two – point boundary value problems.

The paper is organized as follows: In section 2, we give a brief description of VIM and in section 3, some illustrative examples are given. Some concluding remarks are given in section 4.

2. Description of the Variational Iteration Method (VIM)

To illustrate the basic concept of the Variational Iteration Method, we consider the following general differential equation:

$$Lu + Nu = g(x,t) \qquad (1)$$

where L and N are linear and non-linear operators respectively, and $g(x,t)$ is the source inhomogeneous term. He proposed the variational iteration method where a correction functional for (1) can be written as

$$u_{i,n+1}(x,t) = u_{i,n}(x,t) + \int_{t_0}^{t} \lambda\{Lu_{i,n}(s) + N\tilde{u}_{i,n}(s) - g(s)\}ds, \qquad (2)$$

where $i = 1,2,\ldots,m$, λ is a general Lagrange multiplier, which can be identified optimally via the variational theory, n denotes the nth approximation and $\tilde{u}_{i,n}$ is considered as a restricted variation which means $\delta\tilde{u}_{i,n} = 0$. It is required first to determine the Lagrange multiplier λ [3] that will be identified optimally via integration by parts. The successive approximations $u_{i,n+1}(x,t), n \geq 0$, of the solution $u(x,t)$ will be readily obtained upon using the Lagrange multiplier obtained and by using any selective function u_0. Having $\boldsymbol{\lambda}$ determined, then several approximations $u_j(x,t), j \geq 0$, can be determined. Consequently, the solution is given by

$$u = \lim_{n\to\infty} u_n$$

3. Illustrative Examples

In this section, three examples as considered by [1] using the differential transform method will be solved by the VIM and in each case, comparison will be made with exact solution to show the effectiveness of the method.

Example 1

$$y_1'(x) = y_3(x) - \cos(x)$$
$$y_2'(x) = y_3(x) - e^x$$
$$y_3'(x) = y_1(x) - y_2(x)$$
$$y_1(0) = 0, y_2(0) = 0, y_3(0) = 2$$

The first step of the VIM in solving this example and the subsequent ones is to construct the correction functional as follows

$$y_1^{n+1}(x) = y_1^n(x) + \int_0^x \lambda_1(s)\left[\frac{d}{ds}y_1^n(s) - y_3^n(s) + \cos(s)\right]ds$$

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$$y_2^{n+1}(x) = y_2^n(x) + \int_0^x \lambda_2(s)\left[\frac{d}{ds}y_2^n(s) - y_3^n(s) + e^s\right]ds$$

$$y_3^{n+1}(x) = y_3^n(x) + \int_0^x \lambda_3(s)\left[\frac{d}{ds}y_3^n(s) - y_1^n(s) + y_2^n(s)\right]ds$$

Making the functional stationary, we obtain

$$\delta y_1^{n+1}(x) = \delta y_1^n(x) + \delta\int_0^x \lambda_1(s)\left[\frac{d}{ds}y_1^n(s) - \widetilde{y_3^n}(s) + \cos(s)\right]ds$$

$$\delta y_2^{n+1}(x) = \delta y_2^n(x) + \delta\int_0^x \lambda_2(s)\left[\frac{d}{ds}y_2^n(s) - \widetilde{y_3^n}(s) + e^s\right]ds$$

$$\delta y_3^{n+1}(x) = \delta y_3^n(x) + \delta\int_0^x \lambda_3(s)\left[\frac{d}{ds}y_3^n(s) - \widetilde{y_1^n}(s) + \widetilde{y_2^n}(s)\right]ds$$

where λ_1, λ_2 and λ_3 are general Lagrange multipliers, n denotes the nth approximation and $\delta\widetilde{y_3^n}$, $\delta\widetilde{y_1^n}$ and $\delta\widetilde{y_2^n}$ are considered as restricted variations, i.e. $\delta\widetilde{y_3^n} = 0$, $\delta\widetilde{y_1^n} = 0$, $\delta\widetilde{y_2^n} = 0$.

From here, we obtain the Lagrange multipliers as:

$1 + \lambda_1(s)\big|_{s=x} = 0$ and $\lambda_1'(s) = 0$

$1 + \lambda_2(s)\big|_{s=x} = 0$ and $\lambda_2'(s) = 0$

$1 + \lambda_3(s)\big|_{s=x} = 0$ and $\lambda_3'(s) = 0$

$\Rightarrow \lambda_1(s) = -1$, $\lambda_2(s) = -1$ and $\lambda_3(s) = -1$

Substituting these Lagrange multipliers into the correction functional to obtain

$$y_1^{n+1}(x) = y_1^n(x) - \int_0^x \left[\frac{d}{ds}y_1^n(s) - y_3^n(s) + \cos(s)\right]ds$$

$$y_2^{n+1}(x) = y_2^n(x) - \int_0^x \left[\frac{d}{ds}y_2^n(s) - y_3^n(s) + e^s\right]ds$$

$$y_3^{n+1}(x) = y_3^n(x) - \int_0^x \left[\frac{d}{ds}y_3^n(s) - y_1^n(s) + y_2^n(s)\right]ds$$

By using the proposed method, the computed approximate solutions for the second iteration are given by:

$y1_3 = -1 - \frac{1}{2}x^2 + e^x$

$y2_3 = sinx - \frac{1}{2}x^2$

$y3_3 = -x + cosx + e^x$

The exact and the approximate solutions (VIM) are depicted in Table 1 and Figures 1, 2 & 3. As we see, the VIM converged to the exact solution at $n = 2$ showing the effectiveness of the method.

Example 2

$y_1'(x) = y_1(x) + y_2(x)$

$y_2'(x) = -y_1(x) + y_2(x)$

$y_1(0) = 0, y_2(0) = 1$

Constructing the correction functional as follows

$$y_1^{n+1}(x) = y_1^n(x) + \int_0^x \lambda_1(s)\left[\frac{d}{ds}y_1^n(s) - y_1^n(s) - y_2^n(s)\right]ds$$

$$y_2^{n+1}(x) = y_2^n(x) + \int_0^x \lambda_2(s)\left[\frac{d}{ds}y_2^n(s) + y_1^n(s) - y_2^n(s)\right]ds$$

Making the functional stationary, we obtain

$$\delta y_1^{n+1}(x) = \delta y_1^n(x) + \delta\int_0^x \lambda_1(s)\left[\frac{d}{ds}y_1^n(s) - y_1^n(s) - \widetilde{y_2^n}(s)\right]ds$$

$$\delta y_2^{n+1}(x) = \delta y_2^n(x) + \delta\int_0^x \lambda_2(s)\left[\frac{d}{ds}y_2^n(s) + \widetilde{y_1^n}(s) - y_2^n(s)\right]ds$$

where λ_1 and λ_2 are general Lagrange multipliers, n denotes the nth approximation and $\delta\widetilde{y_1^n}$ and $\delta\widetilde{y_2^n}$ are considered as restricted variations, i.e. $\delta\widetilde{y_1^n} = 0$, $\delta\widetilde{y_2^n} = 0$.

Remark 1. Although, the terms y_1 and y_2 are linear, we identify the multipliers approximately considering them as restricted variations. If we consider them as linear terms, then the Lagrange multipliers would be $\lambda_1(s) = \lambda_2(s) = -e^{(s-x)}$.

From here, we obtain the stationary conditions as:

$1 + \lambda_1(s)\big|_{s=x} = 0$ and $\lambda_1'(s) = 0$

$1 + \lambda_2(s)\big|_{s=x} = 0$ and $\lambda_2'(s) = 0$

$\Rightarrow \lambda_1(s) = -1$ and $\lambda_2(s) = -1$

Substituting these Lagrange multipliers into the correction functional to obtain

$y_1^{n+1}(x) = y_1^n(x) - \int_0^x \left[\frac{d}{ds}y_1^n(s) - y_1^n(s) - y_2^n(s)\right]ds$

$y_2^{n+1}(x) = y_2^n(x) - \int_0^x \left[\frac{d}{ds}y_2^n(s) + y_1^n(s) - y_2^n(s)\right]ds$

The exact and approximate solutions are plotted in Fig. 4 & 5 and their values in Table 2. As the figures show, the method gives a very good approximation of the exact solution.

Example 3

$y_1'(x) = 2e^{4x}y_4^2(x)$

$y_2'(x) = y_1(x) - y_3(x) + cosx - e^{2x}$

$y_3'(x) = y_2(x) - y_4(x) + e^{-x} - sinx$ $y_4'(x) = -e^{-5x}y_1^2(x)$

$y_1(0) = 1, y_2(0) = 1, y_3(0) = 0, y_4(0) = 1$

Constructing the correction functional as follows

$$y_1^{n+1}(x) = y_1^n(x) + \int_0^x \lambda_1(s)\left[\frac{d}{ds}y_1^n(s) - 2e^{4s}(y_4^n)^2(s)\right]ds$$

$$y_2^{n+1}(x) = y_2^n(x) + \int_0^x \lambda_2(s)\left[\frac{d}{ds}y_2^n(s) - y_1^n(s) + y_3^n(s) - \cos(s) + e^{2s}\right]ds$$

$$y_3^{n+1}(x) = y_3^n(x) + \int_0^x \lambda_3(s)\left[\frac{d}{ds}y_3^n(s) - y_2^n(s) + y_4^n(s) + \sin(s) - e^{s}\right]ds$$

$$y_4^{n+1}(x) = y_4^n(x) + \int_0^x \lambda_4(s)\left[\frac{d}{ds}y_4^n(s) + e^{-5s}(y_1^n)^2(s)\right]ds$$

Making the functional stationary, we obtain

$$\delta y_1^{n+1}(x) = \delta y_1^n(x) + \delta\int_0^x \lambda_1(s)\left[\frac{d}{ds}y_1^n(s) - 2e^{4s}\left(\widetilde{y_4^n}\right)^2(s)\right]ds$$

$$\delta y_2^{n+1}(x) = \delta y_2^n(x) + \delta\int_0^x \lambda_2(s)\left[\frac{d}{ds}y_2^n(s) - \widetilde{y_1^n}(s) + \widetilde{y_3^n}(s) - \cos(s) + e^{2s}\right]ds$$

$$\delta y_3^{n+1}(x) = \delta y_3^n(x) + \delta\int_0^x \lambda_3(s)\left[\frac{d}{ds}y_3^n(s) - \widetilde{y_2^n}(s) + \widetilde{y_4^n}(s) + \sin(s) - e^{s}\right]ds$$

$$\delta y_4^{n+1}(x) = \delta y_4^n(x) + \delta\int_0^x \lambda_4(s)\left[\frac{d}{ds}y_4^n(s) + e^{-5s}(\widetilde{y_1^n})^2(s)\right]ds$$

where $\lambda_1, \lambda_2, \lambda_3$ and λ_4 are general Lagrange multipliers, n denotes the nth approximation and $\delta\widetilde{y_1^n}$, $\delta\widetilde{y_2^n}, \delta\widetilde{y_3^n}$ and $\delta\widetilde{y_4^n}$ are considered as restricted variations, i.e. $\delta\widetilde{y_1^n} = \delta\widetilde{y_2^n} = \delta\widetilde{y_3^n} = \delta\widetilde{y_4^n} = 0$.

From here, we obtain the stationary conditions as:

$1 + \lambda_1(s)\big|_{s=x} = 0$ and $\lambda_1'(s) = 0$

$1 + \lambda_2(s)\big|_{s=x} = 0$ and $\lambda_2'(s) = 0$

$1 + \lambda_3(s)\big|_{s=x} = 0$ and $\lambda_3'(s) = 0$

$1 + \lambda_4(s)\big|_{s=x} = 0$ and $\lambda_4'(s) = 0$

$\Rightarrow \lambda_1(s) = \lambda_2(s) = \lambda_3(s) = \lambda_4(s) = -1$

Substituting these Lagrange multipliers into the correction functional to obtain

$$y_1^{n+1}(x) = y_1^n(x) - \int_0^x \left[\frac{d}{ds}y_1^n(s) - 2e^{4s}(y_4^n)^2(s)\right]ds$$

$$y_2^{n+1}(x) = y_2^n(x) - \int_0^x \left[\frac{d}{ds}y_2^n(s) - y_1^n(s) + y_3^n(s) - \cos(s) + e^{2s}\right]ds$$

$$y_3^{n+1}(x) = y_3^n(x) - \int_0^x \left[\frac{d}{ds}y_3^n(s) - y_2^n(s) + y_4^n(s) + \sin(s) - e^{s}\right]ds$$

$$y_4^{n+1}(x) = y_4^n(x) - \int_0^x \left[\frac{d}{ds}y_4^n(s) + e^{-5s}(y_1^n)^2(s)\right]ds$$

The graphs of the approximate solution and the exact solutions are displayed in Figures 6, 7, 8 & 9. These figures show the efficiency of the method. The values of both solutions for some points are given in Table 3.

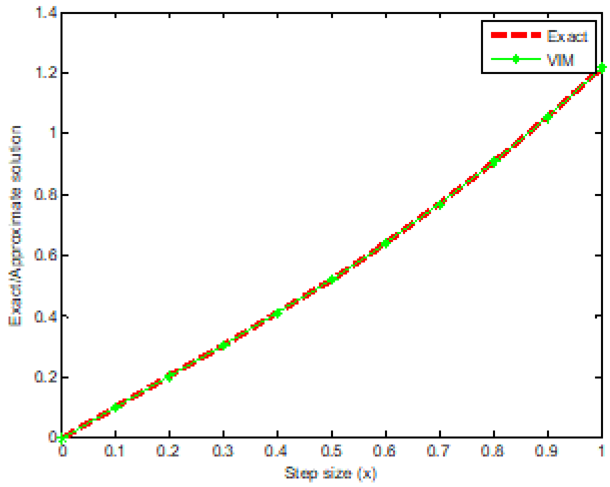

Figure 1: Exact and approximate solutions for y_1 (Example 1)

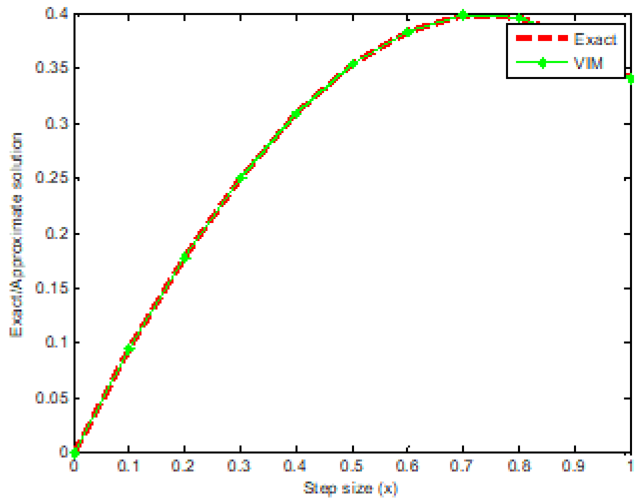

Figure 2: Exact and approximate solutions for y_2 (Example 1)

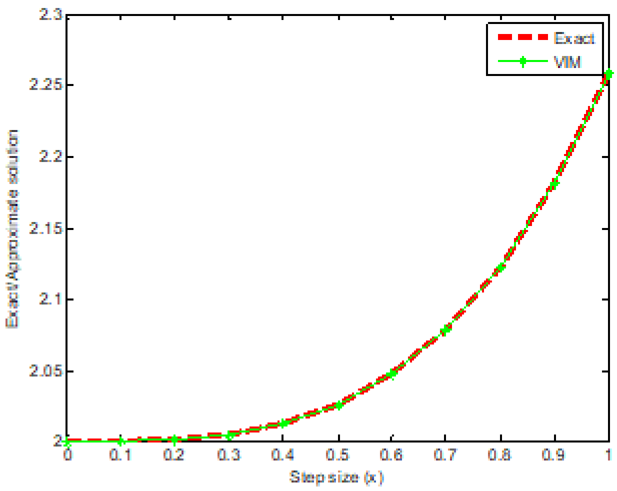

Figure 3: Exact and approximate solutions for y_3 (Example 1)

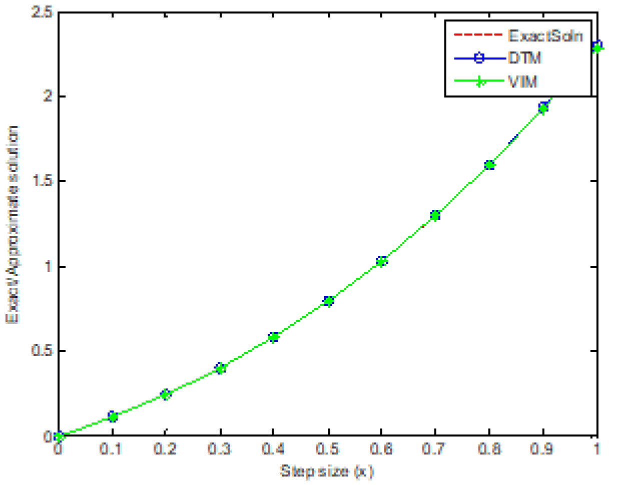

Figure 4: Exact and approximate solutions for y_1 (Example 2)

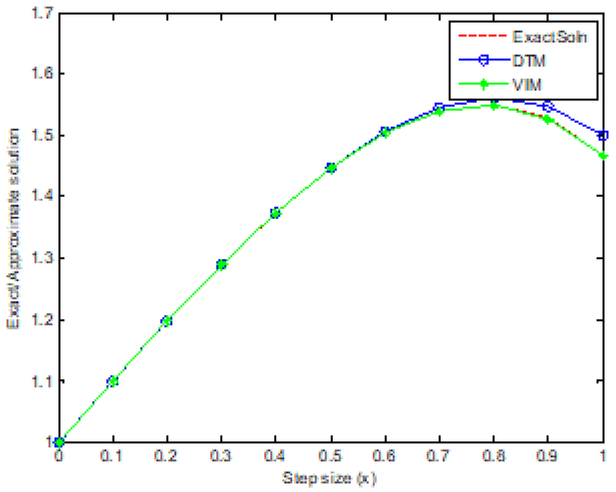

Figure 5: Exact and approximate solutions for y_2 (Example 2)

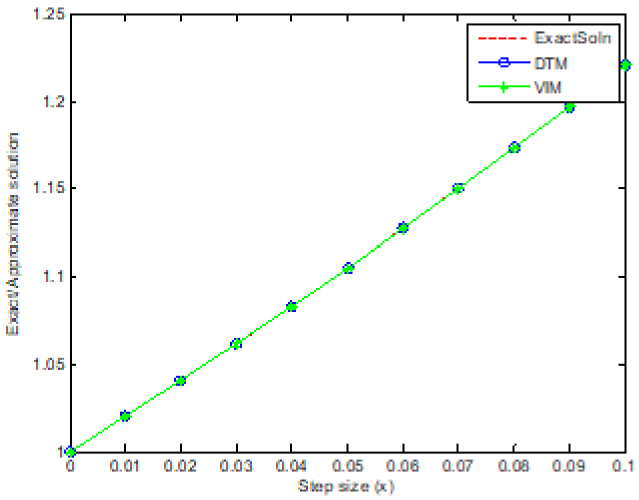

Figure 6: Exact and approximate solutions for y_1 (Example 3)

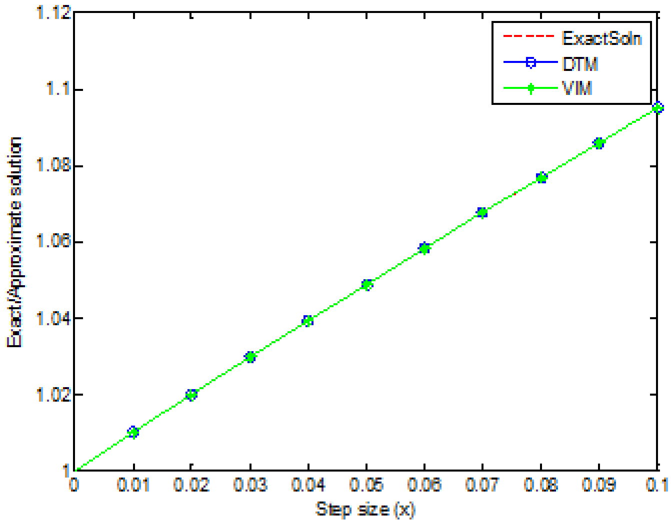

Figure 7: Exact and approximate solutions for y_2 (Example 3)

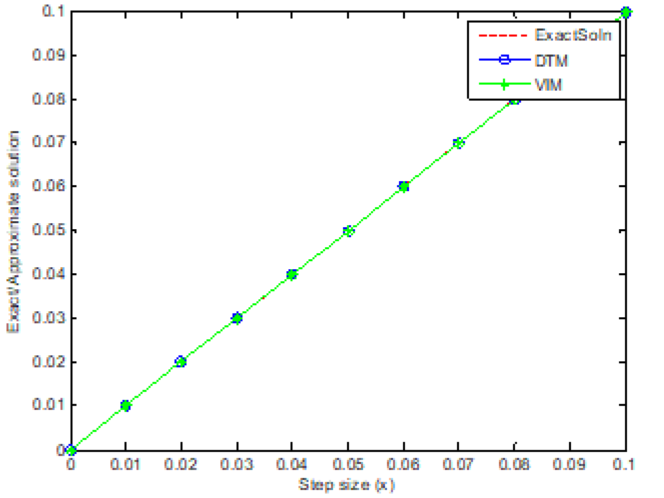

Figure 8: Exact and approximate solutions for y_3 (Example 3)

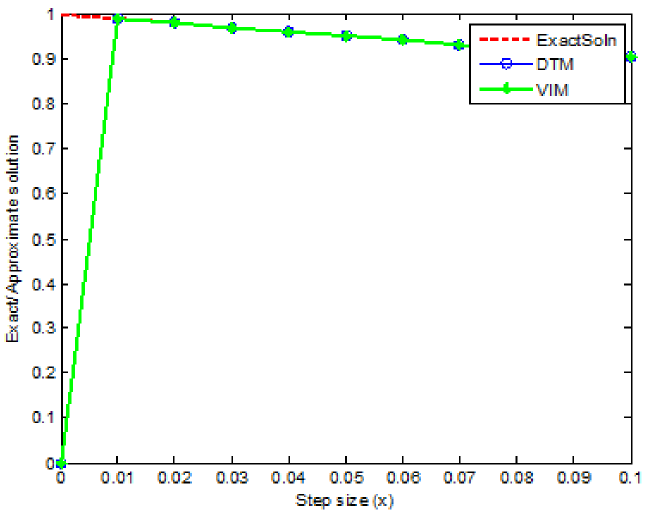

Figure 9: Exact and approximate solutions for y_4 (Example 3)

Table 1 for Example 1

	Exact solution			VIM		
x	y_1	y_2	y_3	y_1	y_2	y_3
0	0	0	2	0	0	2
0.1	0.100170918	0.09483341665	2.000175083	0.100170918	0.09483341665	2.000175083

0.2	0.201402758	0.1786693308	2.001469336	0.201402758	0.1786693308	2.001469336
0.3	0.304858808	0.2505202067	2.005195297	0.304858808	0.2505202067	2.005195297
0.4	0.411824698	0.3094183423	2.012885692	0.411824698	0.3094183423	2.012885692
0.5	0.523721271	0.3544255386	2.026303833	0.523721271	0.3544255386	2.026303833
0.6	0.642118800	0.3846424734	2.047454415	0.642118800	0.3846424734	2.047454415
0.7	0.768752707	0.3992176872	2.078594894	0.768752707	0.3992176872	2.078594894
0.8	0.905540928	0.3973560909	2.122247637	0.905540928	0.3973560909	2.122247637
0.9	1.054603111	0.3783269096	2.181213079	1.054603111	0.3783269096	2.181213079
1.0	1.218281828	0.3414709848	2.258584134	1.218281828	0.3414709848	2.258584134

Table 2 for Example 2

	Exact solution		DTM[1]		VIM	
x	y_1	y_2	y_1	y_2	y_1	y_2
0	0	1	0	1	0	1
0.1	0.1103329887	1.099649667	0.1103330000	1.099650000	0.1103329887	1.099649667
0.2	0.2426552686	1.197056021	0.2426560000	1.197066666	0.2426552686	1.197055999
0.3	0.3989105540	1.289569374	0.3989190000	1.289650000	0.3989105529	1.289569000
0.4	0.5809439009	1.374061539	0.5809920000	1.374400000	0.5809438883	1.374058667
0.5	0.7904390834	1.446889037	0.7906250000	1.447916666	0.7904389881	1.446874999
0.6	1.028845666	1.503859540	1.029408000	1.506400000	1.028845166	1.503808000
0.7	1.297295111	1.540203025	1.298731000	1.545650000	1.297293068	1.540047667
0.8	1.596505340	1.550549296	1.599744000	1.561066666	1.596498408	1.550143999
0.9	1.926673304	1.528913812	1.933317000	1.547650000	1.926652899	1.527967000
1.0	0.287355287	1.468693940	2.300000000	1.500000000	2.287301587	1.466666667

Table 3 for Example 3

	Exact solution				VIM			
x	y_1	y_2	y_3	y_4	y_1	y_2	y_3	y_4
0	1	1	0	1	1	1	0	0
0.01	1.020201340	1.009949834	0.00999983333	0.9900498337	1.020201339	1.009949833	0.00999983063	0.9900498334
0.02	1.040810774	1.019798673	0.01999866669	0.9801986733	1.040810776	1.019798673	0.01999866732	0.9801986728
0.03	1.061836547	1.029545534	0.02999550020	0.9704455335	1.061836548	1.029545531	0.02999549864	0.9704455336
0.04	1.083287068	1.039189441	0.03998933419	0.9607894392	1.083287069	1.039189440	0.03998933366	0.9607894404
0.05	1.105170918	1.048729430	0.04997916927	0.9512294245	1.105170923	1.048729427	0.04997916734	0.9512294283
0.06	1.127496852	1.058164546	.05996400648	0.9417645336	1.127496871	1.058164535	0.05996399799	0.9417645490
0.07	1.150273799	1.067493848	0.06994284734	0.9323938199	1.150273845	1.067493815	0.06994282844	0.9323938566
0.08	1.173510871	1.076716400	0.07991469397	0.9231163464	1.173510977	1.076716332	0.07991465219	0.9231164276
0.09	1.197217363	1.085831282	0.08987854920	0.9139311853	1.197217585	1.085831141	0.08987846896	0.9139313476
0.1	1.221402758	1.094837582	0.09983341665	0.9048374180	1.221403181	1.094837319	0.09983326455	0.9048377188

Table 4 for Example 3

	Exact solution				DTM[1]			
x	y_1	y_2	y_3	y_4	y_1	y_2	y_3	y_4
0	1	1	0	1	1	1	0	0
0.01	1.020201340	1.009949834	0.00999983333	0.9900498337	1.020201333	1.009949833	0.00999983333	0.9900498333
0.02	1.040810774	1.019798673	0.01999866669	0.9801986733	1.040810667	1.019798674	0.01999866670	0.9801986667
0.03	1.061836547	1.029545534	0.02999550020	0.9704455335	1.061836000	1.029545534	0.02999550020	0.9704455000
0.04	1.083287068	1.039189441	0.03998933419	0.9607894392	1.083285333	1.039189441	0.03998933418	0.9607893333
0.05	1.105170918	1.048729430	0.04997916927	0.9512294245	1.105166667	1.048729429	0.04997916927	0.9512291667
0.06	1.127496852	1.058164546	0.05996400648	0.9417645336	1.127488000	1.058164546	0.05996400648	0.9417640000
0.07	1.150273799	1.067493848	0.06994284734	0.9323938199	1.150257333	1.067493847	0.06994284734	0.9323928333
0.08	1.173510871	1.076716400	0.07991469397	0.9231163464	1.173482667	1.076716401	0.07991469398	0.9231146667
0.09	1.197217363	1.085831282	0.08987854920	0.9139311853	1.197172000	1.085831283	0.08987854921	0.9139285000
0.1	1.221402758	1.094837582	0.09983341665	0.9048374180	1.221333333	1.094837584	0.09983341666	0.9048333333

4. Conclusion

We have successfully applied the variational iteration method to solve some systems of ordinary differential equations. The results obtained by the method confirm the efficiency and effectiveness of the method.

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