International Journal of Science and Research (IJSR)
ISSN: 2319-7064
Index Copernicus Value (2016): 79.57 | Impact Factor (2017): 7.296

Properties of Combination Using Double Factorial

G. Babu

Assistant Professor, Department of Mathematics, Sri Akilandeshwari Women's College Wandiwash

Abstract: ***In this paper contains to calculation some combination formulae using the techniques of double factorial. Applying this concept I have find out and proved the results of combination which had been already calculated.***

Keywords: Combination, Factorial, Double factorial

1. Introduction

The properties of combination is already proved using some technique of factorial and sub factorial. In same concept I decided to verify and proved the properties of permutation using this double factorial.

2. Basic Definitions

2.1 Definition of Combination

A permutation is an arrangement of all or part of a set of objects, where the order doesn't matter.
Computing the number of combination of r objects chosen from n objects is $nC_r = \frac{n!}{r!(n-r)!}$

2.2 Definition of factorial

Factorial says to multiply all whole numbers from the chosen number down to 1. The symbol is "!" The formula is $n! = n(n-1)(n-2) \ldots \ldots .. \times 3 \times 2 \times 1$

2.3 Definition of double factorial

The double factorial or semi factorial of a number n(denoted by n!!) is the product of all the integers from 1 up to n that have the same parity (odd or even) as n. Formula is

For n is even, then
$n!! = n(n-2)(n-4) \ldots \ldots \times 6 \times 4 \times 2$

For n is odd , then
$n!! = n(n-2)(n-4) \ldots \ldots \times 5 \times 3 \times 1$

Example:
$10!! = 10 \times 8 \times 6 \times 4 \times 2$
$= 3840$
$9!! = 9 \times 7 \times 5 \times 3 \times 1$
$= 945$
Note : As $(n!)!$ and not $n!!$

3. Relation between factorial and double factorial

For even n=2x, x≥ 0 then
$n!! = 2^x\, x!$
For odd n=2x-1 , x≥ 1 then
$$n!! = \frac{(2x)!}{2^x\, x!}$$

Example:
For n=10=2(5)
$10!! = 2^5\, 5!$
$= 32 \times 120$
$= 3840$
For n=9=2(5)-1
$$9!! = \frac{10!}{2^5\, 5!}$$
$$= \frac{3628800}{3840}$$
$= 945$

3.1 Useful theorem

For any non negative integer n, then $\frac{n!}{n!!} = (n-1)!!$ (or) $n! = (n-1)!! \times n!!$

Proof

1) If n is odd
$$\frac{n!}{n!!} = \frac{n(n-1)(n-2) \ldots \ldots \times 3 \times 2 \times 1}{n(n-2)(n-4) \ldots \ldots \times 5 \times 3 \times 1}$$
$= (n-1)(n-3) \ldots \ldots \times 4 \times 2$
$= (n-1)!!$

2) If n is even
$$\frac{n!}{n!!} = \frac{n(n-1)(n-2) \ldots \ldots \times 3 \times 2 \times 1}{n(n-2)(n-4) \ldots \ldots \times 6 \times 4 \times 2}$$
$= (n-1)(n-3) \ldots \ldots \times 3 \times 1$
$= (n-1)!!$

Combining this cases for any non negative integer n
$$\frac{n!}{n!!} = (n-1)!!$$
$\Leftrightarrow n! = (n-1)!! \times n!!$

Example
For n= 6
$n! = 6!$
$= 6 \times 5 \times 4 \times 3 \times 2 \times 1$
$= 720$
$(n-1)!! \times n!! = 5!! \times 6!!$
$= 5 \times 3 \times 1 \times 6 \times 4 \times 2$
$= 720$

Result 1
$0!! = 1$ and $(-1)!! = 1$

Proof
w.k.t $n!! = 2^x\, x!$

$0!! = 2^0\, 0!$
$= 1$

Result 2
$n!! = n(n-2)!!$
$(n+1)!! = (n+1)(n-1)!!$

Example:
$6!! = 6 \times 4!!$
$= 6 \times 4 \times 2!!$
$= 6 \times 4 \times 2$
$= 48$

Result 3

$$nC_r = \frac{n!! \times (n-1)!!}{r!! \times (r-1)!!\,(n-r-1)!! \times (n-r)!!}$$

Proof:

$$nC_r = \frac{n!}{r!\,(n-r)!}$$

Since, $n! = (n-1)!! \times n!!$
Therefore,

$$nC_r = \frac{n!! \times (n-1)!!}{r!! \times (r-1)!!\,(n-r-1)!! \times (n-r)!!}$$

Example:

$$5C_2 = \frac{5!! \times 4!!}{2!! \times (1)!!\,(2)!! \times (3)!!}$$
$$= \frac{5 \times 3 \times 1 \times 4 \times 2}{2 \times 1 \times 2 \times 3}$$
$$= 10$$

Result 4
$nC_0 = nC_n = 1$

Proof:

$$nC_r = \frac{n!! \times (n-1)!!}{r!! \times (r-1)!!\,(n-r-1)!! \times (n-r)!!}$$
$$nC_0 = \frac{n!! \times (n-1)!!}{0!! \times (-1)!!\,(n-1)!! \times (n)!!}$$
$$= 1$$
$$nC_n = \frac{n!! \times (n-1)!!}{n!! \times (n-1)!!\,(-1)!! \times (0)!!}$$
$$= 1$$

Result 5
$nC_1 = n$
Proof:

$$nC_1 = \frac{n!! \times (n-1)!!}{1!! \times (0)!!\,(n-2)!! \times (n-1)!!}$$
$$= \frac{n(n-2)!!}{1!! \times (0)!!\,(n-2)!!}$$
$$= n$$

Example

$$6C_1 = \frac{6!! \times (5)!!}{1!! \times (5)!!\,(0)!! \times (4)!!}$$
$$= \frac{6 \times 4 \times 2}{4 \times 2}$$
$$= 6$$

Result 6
$nC_r = nC_p$ then $r = p$ or $r + p = n$

Proof:

$$nC_r = \frac{n!! \times (n-1)!!}{r!! \times (r-1)!!\,(n-r-1)!! \times (n-r)!!} \quad \text{——(1)}$$
$$nC_p = \frac{n!! \times (n-1)!!}{p!! \times (p-1)!!\,(n-p-1)!! \times (n-p)!!} \quad \text{——(2)}$$

Equating (1) and (2)
$r!! = p!!$ then r=p or r+p=n

Example:

$$6C_1 = \frac{6!! \times (5)!!}{1!! \times (5)!!\,(0)!! \times (4)!!}$$
$$= \frac{6 \times 4 \times 2}{4 \times 2}$$
$$= 6$$
$$6C_5 = \frac{6!! \times (5)!!}{5!! \times (1)!!\,(4)!! \times (0)!!}$$
$$= \frac{6 \times 4 \times 2}{4 \times 2}$$
$$= 6$$

Result 7

$$nC_r = \frac{np_r}{r!}$$

Proof:

$$nC_r = \frac{n!! \times (n-1)!!}{r!! \times (r-1)!!\,(n-r-1)!! \times (n-r)!!}$$
$$= \frac{np_r}{r!! \times (r-1)!!}$$
$$= \frac{np_r}{r!}$$

Example:

$$6C_3 = \frac{6!! \times 5!!}{3!! \times 2!! \times 2!! \times 3!!}$$
$$= \frac{6 \times 4 \times 2 \times 5 \times 3}{3 \times 2 \times 2 \times 3}$$
$$= 20$$

$$6p_3/3! = \frac{6!! \times 5!!}{2!! \times 3!!} / 3!$$
$$= \frac{6 \times 4 \times 2 \times 5 \times 3 \times 1}{2 \times 3 \times 1} / 6$$
$$= 120/6$$
$$=20$$

Result 8
$nC_r + nC_{r-1} = (n+1)C_r$

Proof:
w.k.t
$(n+1)C_r$

$$= \frac{(n+1)!! \times (n)!!}{r!! \times (r-1)!!\,(n-r+1)!! \times (n-r)!!} \quad \text{————(1)}$$

LHS=$nC_r + nC_{r-1}$

$$= \frac{n!! \times (n-1)!!}{r!! \times (r-1)!!\,(n-r-1)!! \times (n-r)!!} + \frac{n!! \times (n-1)!!}{(r-2)!! \times (r-1)!!\,(n-r+1)!! \times (n-r)!!}$$

$$= \frac{(n+1)n!! \times (n-1)!!}{(n+1)r!! \times (r-1)!!\,(n-r-1)!! \times (n-r)!!} + \frac{(n+1)n!! \times (n-1)!!}{(n+1)(r-2)!! \times (r-1)!!\,(n-r+1)!! \times (n-r)!!}$$

$$= \frac{n!! \times (n+1)!!}{(n+1) \times (r-1)!! \times (n-r)!!}\left[\frac{1}{r!! \times (n-r-1)!!} + \frac{1}{(r-2)!! \times (n-r+1)!!}\right]$$

$$= \frac{n!! \times (n+1)!!}{(n+1) \times (r-1)!! \times (n-r)!!}\left[\frac{(n-r+1)(n-r-1)!!\,(r-2)!!}{r!! \times (n-r-1)!!}\frac{+\,r(r-2)!!\,(n-r-1)!!}{(r-2)!! \times (n-r+1)!!}\right]$$

$$= \frac{n!! \times (n+1)!!\,(n-r-1)!!\,(r-2)!!}{(n+1) \times (r-1)!! \times (n-r)!!}\left[\frac{(n-r+1)}{r!! \times (n-r-1)!!}\frac{+r}{(r-2)!! \times (n-r+1)!!}\right]$$

$$= \frac{n!! \times (n+1)!!\,(n+1)}{(n+1) \times (r-1)!! \times (n-r)!!\,r!!\,(n-r+1)!!}$$

$$= \frac{(n+1)!! \times (n)!!}{r!! \times (r-1)!!\,(n-r+1)!! \times (n-r)!!}$$

$$= (n+1)C_r \;(\text{by } (1))$$

Example:

$nC_r + nC_{r-1} = (n+1)C_r$

Where n=5, r=2

LHS=$5C_2 + 5C_1$

$$= \frac{5!! \times (4)!!}{2!! \times (3)!!\,(1)!! \times (2)!!} + \frac{5!! \times (4)!!}{1!! \times (4)!!\,(0)!! \times (3)!!}$$

$$= \frac{4 \times 2 \times 5 \times 3}{3 \times 2 \times 2} + \frac{4 \times 2 \times 5 \times 3}{3 \times 4 \times 2}$$

$$= 10 + 5 = 15$$

$$\text{RHS} = 6C_2 = \frac{6!! \times (5)!!}{2!! \times (4)!!\,(1)!! \times (3)!!}$$

$$= \frac{6 \times 4 \times 2 \times 5 \times 3}{4 \times 2 \times 2 \times 3}$$

$$= 15$$

Result 9

$nC_0 + nC_1 + nC_2 + \cdots + nC_n = 2^n$

Proof

w.k.t

$$nC_0 = \frac{n!! \times (n-1)!!}{0!! \times (-1)!!\,(n-1)!! \times (n)!!} = 1$$

$$nC_1 = \frac{n!! \times (n-1)!!}{1!! \times (0)!!\,(n-2)!! \times (n-1)!!} = n$$

$$nC_2 = \frac{n!! \times (n-1)!!}{2!! \times (1)!!\,(n-3)!! \times (n-2)!!} = \frac{n(n-1)}{2!}$$

$$nC_3 = \frac{n!! \times (n-1)!!}{3!! \times (2)!!\,(n-3)!! \times (n-4)!!} = \frac{n(n-1)(n-2)}{3!}$$

Etc…

$$nC_{n-2} = \frac{n!! \times (n-1)!!}{2!! \times (1)!!\,(n-3)!! \times (n-2)!!} = \frac{n(n-1)}{2!}$$

$$nC_{n-1} = \frac{n!! \times (n-1)!!}{1!! \times (0)!!\,(n-2)!! \times (n-1)!!} = n$$

$$nC_n = \frac{n!! \times (n-1)!!}{0!! \times (-1)!!\,(n-1)!! \times (n)!!} = 1$$

$$nC_0 + nC_1 + nC_2 + \cdots + nC_n = \left(1 + n + \frac{n(n-1)}{2!} + \cdots + \frac{n(n-1)}{2!} + n + 1\right)$$

$$= (1+1)^n$$

$$= 2^n$$

Since using binomial formula

$$(a+b)^n = a^n + na^{n-1}b^1 + \frac{n(n-1)}{2!}a^{n-2}b^2 + \cdots + b^n$$

We observe this a=1, b=1

Result 10

$$nC_0 + nC_2 + nC_4 + \cdots + \cdots = nC_1 + nC_3 + nC_5 + \cdots + \cdots = 2^{n-1}$$

Proof:

$$nC_0 + nC_2 + nC_4 + \cdots + \cdots = 1 + \frac{n(n-1)}{2!} + \frac{n(n-1)(n-2)(n-3)}{4!} + \cdots = \frac{1}{2}((1+1)^n + 0^n)$$

$$= 2^{n-1}$$

$$nC_1 + nC_3 + nC_5 + \cdots + \cdots = n + \frac{n(n-1)(n-2)}{3!} + \frac{n(n-1)(n-2)(n-3)(n-4)}{5!} + \cdots = \frac{1}{2}((1+1)^n + 0^n)$$

$$= 2^{n-1}$$

Since binomial formula

$$\frac{(a+b)^n + (a-b)^n}{2} = a^n + \frac{n(n-1)}{2!}a^{n-2}b^2 + \frac{n(n-1)(n-2)(n-3)}{4!}a^{n-4}b^4 + \cdots$$

$$\frac{(a+b)^n - (a-b)^n}{2} = na^{n-1}b^1 + \frac{n(n-1)(n-2)}{3!}a^{n-3}b^3 + \frac{n(n-1)(n-2)(n-3)(n-4)}{5!}a^{n-5}b^5 + \cdots$$

4. Conclusion

In this paper I have find out and proved some results with example, properties of combinations applying the technique of double factorial. It is used in many areas. Already I have calculated and proved some results with example, properties of permutation applying the technique of double factorial and I extend this paper to solve the same concept to find out the properties of combinations using double factorial.

References

[1] Ramesh Chandra, "Permutation and combinations" book.
[2] Anglani R, Barlie M. Factorials as Sums. arXiv:Math/0702010v1 [Math.HO] 2007.
[3] Bhargava M. The Factorial Function and Generalizations. Math. Assoc. America. Am. Math. Monthly. 107:783–799. 2000.
[4] Culham JR. Factorial, Gamma and Beta Functions. Engineering Course ME755- Special Functions. University of Waterloo, Waterloo. 2014.
[5] Dartmouth College . Correspondance Between Leonhard Euler and Chr. Goldbach, 1729–1763. 2014. pp. 1–59.
[6] Davis PJ. Leonhard Euler's Integral: A Historical Profile of the Gamma Function. In Memorium: Milton Abramowitz. Math. Assoc. America. Amer Math Monthly. 66:849–869. 1959.