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Zweier Infinite Matrices of Interval Numbers

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Abstract: ***In this paper, Zweier interval null, Zweier interval convergent and Zweier interval bounded sequence spaces of interval numbers are introduced and proved some inclusion relations on them. Additionally, an isomorphism is constructed on these interval sequence spaces. Besides, definition of infinite dimensional Zweier interval matrix and its left and right parts are introduced.***

Keywords: Interval number, interval sequence space, interval Zweier matrix, isomorphism

1. Introduction

Interval arithmetic was introduced by Dwyer [2]. In, [1], Chiao established sequence of interval numbers and gave the definition of usual convergence of sequence of interval numbers. Bounded and convergent sequence spaces of interval numbers are studied by Şengönül and Eryılmaz [9]. In recent years, Esi [3] introduced lacunary sequence spaces of interval numbers. Hansen and Smith [4] make matrix calculations by means of interval arithmetic, firstly. After, many others such as Neumaier [6], Jaulin et al [5] and Rohn [8], etc. have worked on interval matrices. Furthermore, Ng and Lee [7] and Wang [11] build new spaces in terms of the matrix domain of a limitation methods. In addition this, in [10] Şengönül and Zararsız give the definition of complete fuzzy module space of fuzzy numbers and the sequence spaces of fuzzy numbers with fuzzy metric are introduced in this work. Besides, the set of all null, convergent and bounded sequences of interval valued fuzzy numbers are defined with respect to modulus function M, by Zararsız and Taş [13]. Furthermore, in [15] generalized intervals are studied some new and original sets are defined. After, α-,β- and γ- duals are determined and matrix transformations are given on these original sets. In addition these, [16] and [17] will be investigated for further information.

In the past decades, modal analysis has become a major technology in the quest for determining, improving and optimizing dynamic characteristics of engineering structures. Since modals are used in different branches of engineering, Zararsız and Şengönül [14] have constructed some sequence spaces of modals and introduced the null, convergent and bounded sequence spaces of interval numbers.

The rest of our paper is organized, as follows: In Section 2, some basic definitions and theorems related with the interval numbers are given. Also, definitions of interval metric space, sequence of interval numbers, interval Cauchy sequence are given. In Section 3, we have introduced Zweier interval null, Zweier interval convergent and Zweier interval bounded sequence spaces of interval numbers as the set of all sequences such that $\mathcal{Z}$ - transforms of them are in the spaces c_0^i, c^i and l_∞^i, respectively, and proved some relations on these interval sequence spaces.

2. Preliminaries, Background and Notation

In this section, we recall some of the basic definitions and notions in the theory of interval numbers and sequence spaces such as notions of interval metric space, algebraic operations on the set of interval numbers as in the following:

Let suppose that $\mathbb{N},\mathbb{R}$ and E be the set of all positive integers, all real numbers and the set of interval numbers, respectively. We denote the set of all sequences with complex terms by w which is a linear space with addition and scalar multiplication of sequences. Each linear subspace of w is called a sequence space and write ℓ_∞, c and c_0 for the classical sequence spaces of all bounded, convergent and null sequences, respectively. For brevity in notation, through all the text, we shall write $\sum_n$, $\sup_n$, and $\lim_n$ instead of $\sum_{n=0}^{\infty}$, $\sup_{n\in\mathbb{N}}$ and $\lim_{n\to\infty}$. Further, we define addition, scalar multiplication and multiplication calculations as below:

$$+: E\times E\to E, +(u,v)=u+v=[u^-+v^-,u^++v^+],$$

$$\cdot:\mathbb{R}\times E\to E, +(u,v)=\alpha u=\begin{cases}[\alpha u^-,\alpha u^+], & \alpha\geq 0\\ [\alpha u^+,\alpha u^-], & \alpha<0,\end{cases}$$

$$\cdot: E\times E\to E, \cdot(u,v)=[minR, maxR],$$
$$R=\{u^-v^-,u^-v^+,u^+v^-,u^+v^+\}.$$

Let λ and μ be two sequence spaces and $\mathbb{A}=(a_{nk})$ be an infinite matrix of real or complex numbers a_{nk}, where n, $k\in\mathbb{N}$. Then, we can say that $\mathbb{A}$ defines a matrix mapping from λ to μ and we denote it by writing $\mathbb{A}\in(\lambda:\mu)$, if for every sequence $x=(x_k)$ is in λ and the sequence $\mathbb{A}x=\{(\mathbb{A}x)_n\}$, the $\mathbb{A}$- transform of x is in μ where k runs from 0 to ∞. The domain $\lambda_{\mathbb{A}}$ of an infinite matrix $\mathbb{A}$ in a sequence space λ is defined by

$$\lambda_{\mathbb{A}}=\{x=(x_k)\in w:\mathbb{A}x\in\lambda\} \qquad (1.1)$$

which is a sequence space. If assume λ as c, then $c_{\mathbb{A}}$ is called convergence field of $\mathbb{A}$. We write the limit of $\mathbb{A}x$ as $\mathbb{A}-\lim_n x_n=\lim_n\sum_{k=0}^{\infty}a_{nk}x_k$, and the $\mathbb{A}$ is called regular if $\lim_{\mathbb{A}} x=\lim x$ for every $x\in c$. A matrix $\mathbb{A}=(a_{nk})$ is called triangle if $a_{nk}=0$ for $k>n$ and $a_{nn}\neq 0$ for all $n\in\mathbb{N}$.

The set

$$w(E)=\{([u_k^-,u_k^+]):f,g:\mathbb{N}\to\mathbb{R},$$
$$f(k)=u_k^-,g(k)=u_k^+,u_k^-\leq u_k^+\}.$$

is called sequence of interval number sets. If we take $u_k^- = u_k^+$ then w(E) is reduced to the ordinary sequence space of real numbers.

Definition 2.1: [12] The matrix

$$A = \begin{cases} [a_{nk}^-, a_{nk}^+], & n \leq k \ ise \\ [0,0], & otherwise \end{cases}$$

is defined as lower triangular interval matrix. Besides, if there is no element on the principal diagonal, then A is called normal interval matrix.

Definition 2.2: Let us give the definitions of some triangle, regular matrices, which are necessary for the text. The Cesàro ($C = (c_{nk})$) and Zweier ($Z = (Z_{nk})^p$) matrices of order one and p, respectively, which are lower triangular matrix defined by the following for all $n, k \in \mathbb{N}$:

$$c_{nk} = \begin{cases} \frac{1}{n+1}, & 0 \leq k \leq n, \\ 0, & k > n, \end{cases}$$

$$Z = (z_{nk})^p = \begin{cases} p, n = k \ ise \\ 1 - p, n - 1 = k \ ise \\ 0, otherwise \end{cases}$$

Here, Z is called as Zweier matrix with the degree of $p \neq 1$. Through the text, we make all calculations by taking p as $\frac{1}{2}$.

Definition 2.3: [1]. An interval sequence $([u_k^-, u_k^+]) \in w(E)$ is convergent to $[u_0^-, u_0^+] \Leftrightarrow$ for every $\epsilon > 0$ there exists a $k_0 \in \mathbb{N}$ such that for every$k \geq k_0$, $d([u_k^-, u_k^+], [u_0^-, u_0^+]) < \epsilon$. Additionally, $([u_k^-, u_k^+]) \in w(E)$ is Cauchy sequence if for every $\epsilon > 0$ there exists a $k_0 \in \mathbb{N}$ such that for every $k, n \geq k_0$, $d([u_k^-, u_k^+], [u_n^-, u_n^+]) < \epsilon$.

3. Material and Method

In the following, we give the sequence spaces by means of [9] named interval convergent, null interval convergent and interval bounded sequence spaces, respectively:

$$c^i = \left\{[u_k^-, u_k^+] : \lim_k [u_k^-, u_k^+] = [u_0^-, u_0^+]\right\}$$

$$c_0^i = \left\{[u_k^-, u_k^+] : \lim_k [u_k^-, u_k^+] = [0,0]\right\}$$

$$l_\infty^i = \left\{[u_k^-, u_k^+] : \sup_k M[u_k^-, u_k^+] < \infty\right\}$$

In [12], Zararsız give the definition of Cesàro interval matrix showed by $\mathbb{C} = (\mathbb{C}_{nk})$. After, Zararsız [12] divide Cesàro interval matrix into two parts named left Cesàro interval matrix and right Cesàro interval matrix represented by $\mathbb{C}^- = (\mathbb{C}_{nk}{}^-)$ and $\mathbb{C}^+ = (\mathbb{C}_{nk}{}^+)$, respectively. Here, we write $\mathbb{C} = (\mathbb{C}_{nk})$, $\mathbb{C}^- = (\mathbb{C}_{nk}{}^-)$ and $\mathbb{C}^+ = (\mathbb{C}_{nk}{}^+)$, respectively as follows:

$$\mathbb{C} = (\mathbb{C}_{nk}) = \begin{cases} \left[\frac{-1}{n+1}, \frac{1}{n+1}\right], n \geq k \ ise \\ [0,0], \quad otherwise \end{cases}$$

$$\mathbb{C}^- = (\mathbb{C}_{nk}{}^-) = \begin{cases} \left[\frac{-1}{n+1}, 0\right], n \geq k \ ise \\ [0,0], \quad otherwise \end{cases}$$

$$\mathbb{C}^+ = (\mathbb{C}_{nk}{}^+) = \begin{cases} \left[0, \frac{1}{n+1}\right], n \geq k \ ise \\ [0,0], \quad otherwise. \end{cases}$$

Additionally, Zararsız [12] calculate norm of these matrices. It means that $\|\mathbb{C}^-\| = \|\mathbb{C}^+\| = \|\mathbb{C}\| = 1$.

Theorem 3.1: [12] If there exists both left and right Cesàro limits of real sequence (r_k), i.e. that $\lim_n (\mathbb{C}^- r)_n = [L_1^-, L_1^+]$ and $\lim_n (\mathbb{C}^+ r)_n = [L_2^-, L_2^+]$ are present then $\lim_n (\mathbb{C}^- r)_n + \lim_n (\mathbb{C}^+ r)_n = [L_1^- + L_2^-, L_1^+ + L_2^+]$.

Now, we give definition of Zweier interval matrix, left and right Zweier interval matrices showed by $\mathcal{Z}$, $\mathcal{Z}^-$, $\mathcal{Z}^+$, respectively, as in the following:

$$\mathcal{Z} = (\mathcal{Z}_{nk})^p = \begin{cases} [-p, p], & n = k \\ [p - 1, 1 - p], & n - 1 = k \\ [0,0], otherwise \end{cases}$$

$$\mathcal{Z}^- = \begin{cases} [-p, 0], & n = k \\ [p - 1, 0], & n - 1 = k \\ [0,0], & otherwise \end{cases}$$

$$\mathcal{Z}^+ = \begin{cases} [0, p], & n = k \\ [0, 1 - p], & n - 1 = k \\ [0,0], & otherwise \end{cases}$$

where $p \neq 1$.

3.1 Zweier Interval Matrices and Zweier Interval Sequence Spaces

Definition 3.1:

$$\mathbb{N}_n = \{0,1,2,\ldots,n\}, \mathbb{N}_m = \{0,1,2,\ldots,m\} \text{ and } f : \mathbb{N}_n \times \mathbb{N}_m \to E$$
$$(i,j) \to f(i,j) = a_{ij} \in E, (1 \leq i \leq n, 1 \leq j \leq m)$$

We show the set of all interval matrices that have $n \times m$ dimension by $E^{n \times m}$.

Let $A, B \in E^{n \times m}$, $\alpha \in \mathbb{R}$ and $D \in E^{m \times r}$, $(1 \leq i \leq n, 1 \leq j \leq r)$. Addition, scalar multiplication and multiplication calculations are defined, respectively, as below:

$$A + B = \left[[a_{ij}^-, a_{ij}^+]\right]_{n \times m} + \left[[b_{ij}^-, b_{ij}^+]\right]_{n \times m}$$

$$\alpha A = \begin{cases} \left[[\alpha a_{ij}^-, \alpha a_{ij}^+]\right]_{n \times m}, & \alpha \geq 0 \\ \left[[\alpha a_{ij}^+, \alpha a_{ij}^-]\right]_{n \times m}, & \alpha < 0. \end{cases}$$

$$A \cdot D = \left[[a_{ij}^-, a_{ij}^+]\right]_{n \times m} + \left[[d_{ij}^-, d_{ij}^+]\right]_{m \times r} = \left[[c_{ij}^-, c_{ij}^+]\right]_{n \times r}$$

where

$$c_{ij}^- = \sum_{k=1}^{m} min\{a_{ik}^- d_{kj}^-, a_{ik}^- d_{kj}^+, a_{ik}^+ d_{kj}^-, a_{ik}^+ d_{kj}^+\}$$

$$c_{ij}^+ = \sum_{k=1}^{m} max\{a_{ik}^- d_{kj}^-, a_{ik}^- d_{kj}^+, a_{ik}^+ d_{kj}^-, a_{ik}^+ d_{kj}^+\}$$

Definition 3.2:
Let $([u_k^-,u_k^+]) = ([u_1^-,u_1^+],[u_2^-,u_2^+],\ldots,[u_k^-,u_k^+],\ldots)$ be a sequence of interval numbers. If series in the form $\sum_k [a_{nk}^-,a_{nk}^+][u_k^-,u_k^+]$ are convergent for all $n \in \mathbb{N}$ then $Au^k = v^k$ is called as A-transformation of u^k.

Definition 3.3: A be an infinite dimensional interval matrix then norm of A is defined as follow:

$$\|A\| = \sup_i \sum_j M([a_{ij}^-,a_{ij}^+])$$

where

$$M([a_{ij}^-,a_{ij}^+]) = max\{|a_{ij}^-|,|a_{ij}^+|\}$$

3.2 The Zweier Interval Sequence Spaces $\mathcal{Z}_0^E$, $\mathcal{Z}_c^E$ and $\mathcal{Z}_b^E$

Now, we give the Zweier interval null, Zweier interval convergent and Zweier interval bounded sequence spaces of interval numbers, as the set of all sequences such that Z -transforms of these interval sequence spaces are in the sequence spaces c_0^i, c^i and l_∞^i, respectively, i.e.

$$\mathcal{Z}_0^E = \{u^k = [u_k^-,u_k^+] \in w(E) : \mathcal{Z}u^k \in c_0^i\}$$

$$\mathcal{Z}_c^E = \{u^k = [u_k^-,u_k^+] \in w(E) : \mathcal{Z}u^k \in c^i\}$$

$$\mathcal{Z}_b^E = \{u^k = [u_k^-,u_k^+] \in w(E) : \mathcal{Z}u^k \in l_\infty^i\}$$

It is clear that Zweier interval matrix is lower triangular and regular interval matrix of type $\infty \times \infty$.

Define the interval sequence v^k which will be necessary as the Z transform of a interval sequence u^k, i.e.

$$[v_k^-,v_k^+] = p[u_k^-,u_k^+] + (1-p)[u_{k-1}^-,u_{k-1}^+]$$

Where

$$[u_{-1}^-,u_{-1}^+] = [0,0], p \neq 1$$

Theorem 3.2.1: There is an linear isomorphism between the spaces $\mathcal{Z}_b^E$ and l_∞^i. It means that $\mathcal{Z}_b^E \cong l_\infty^i$.

Proof: For making proof of theorem, we should show the existence of a linear bijection between the spaces $\mathcal{Z}_b^E$ and l_∞^i. Let T be a transformation as $T : \mathcal{Z}_b^E \to l_\infty^i$, $u^k \to v^k = \mathrm{T}u^k = \frac{1}{2}([u_k^-,u_k^+] + [u_{k-1}^-,u_{k-1}^+])$. The linearity of T is clear. It means that $T(u^k + v^k) = \mathrm{T} + \mathrm{T}v^k$ and $\mathrm{T}(\alpha u^k) = \alpha\mathrm{T}(u^k)$. Let us take $v^k \in l_\infty^i$ and define the sequence $u^k = ([u_k^-,u_k^+])$ as below:

$$[u_k^-,u_k^+] = 2\sum_{j=0}^{k}(-1)^{k-j}u^k[v_j^-,v_j^+], k \in \mathbb{N}$$

Then we have

$$\sup_{k\in\mathbb{N}} d(\mathcal{Z}u^k,\theta) = \sup_{k\in\mathbb{N}} d\left(\frac{[u_k^-,u_k^+]}{2} + \frac{[u_{k-1}^-,u_{k-1}^+]}{2},\theta\right)$$

$$= \sup_{k\in\mathbb{N}} d\left(\sum_{j=0}^{k}(-1)^{k-j}u^k[v_j^-,v_j^+] + \sum_{j=0}^{k-1}(-1)^{k-j}u^k[v_j^-,v_j^+],\theta\right)$$

$$= \sup_{k\in\mathbb{N}} d([v_k^-,v_k^+],\theta) < \infty$$

Therefore, $u^k \in \mathcal{Z}_b^E$. Additionally,

$$\|u^k\|_{\mathcal{Z}_b^E} = \sup_{k\in\mathbb{N}} d\left(\sum_{j=0}^{k}(-1)^{k-j}u^k[v_j^-,v_j^+] + \sum_{j=0}^{k-1}(-1)^{k-j}u^k[v_j^-,v_j^+],\theta\right)$$

$$= \sup_{k\in\mathbb{N}} d([v_k^-,v_k^+],\theta)$$

$$= \|v^k\|_{l_\infty^i} < \infty$$

which means that T is norm preserving. Consequently, T is linear bijection. It means that $\mathcal{Z}_b^E \cong l_\infty^i$.

Theorem 3.2.3: $\mathcal{Z}_b^E$ is complete metric space with the metric given as in the following:

$$\sup_{k\in\mathbb{N}} \max\left(\left|\sum_{j=1}^{k} z_{kj}^- u_j^- - \sum_{j=1}^{k} z_{kj}^- v_j^-\right|, \left|\sum_{j=1}^{k} z_{kj}^+ u_j^+ - \sum_{j=1}^{k} z_{kj}^+ v_j^+\right|\right)$$

Proof: The interval sequence space l_∞^i is isomorphic to the space $\mathcal{Z}_b^E$. Besides, Zweier interval matrix is normal [sen] and l_∞^i is complete sequence space, then it is easy to see that the interval sequence spaces l_∞^i and $\mathcal{Z}_b^E$ are complete metric spaces with the metric defined above.

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