# Partial Integral Equations

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**Abstract:** ***In this paper we give a study of the partial integral equations and discuss the eigenvalue problems related to the linear partial integral equations. Also we use the collection and Galerkin's methods to find an approximated solution for the eigen value problems related to the linear partial integral equations.***

**Keywords:** partial integral equations, integral equations, collection methods, Galerkin's methods

## 1. Introduction

The eigenvalue problems (linear or nonlinear ) plays an important role in the mathematical modeling of many filed : physical, nuclear rector dynamics, free electron laser dynamics , biological phenomena and engineering sciences in which necessary to take in to account the effect of real world problems,[Fox L.,1964 ] and [ Boyce D.,2001].The mathematical modeling of real life problems may be expressed as differential equations or integral equations or system of equations .

Some integral equations were considered in the early of thr 19th century. While their systematic theory was developed at the end of the 19th and the beginning of 20th century by V. Volterra (1860 -1940 ) and Italian mathematician . He developed the theory of integro- differential equations and integral equations. The integral equations has many real life applications, say

(a) Stability of nuclear vectors [Cordunenu C., 1991 ]
(b) Hydrodynamic stability [Pozrikidis C., 1998 ]
(c) Vibration of circular membrane : to Beat a Drum with a finite difference method [Pozrikidis C., 1998 ]

Therefore, many authors study the subject of the eigenvalue problems. For example [Gourlay A. and Walsh G., 1973 ] and [ Boyce D., 2001 ]give the most important ideas about the standard algebraic linear eigenvalue problem. [Muna M., 1996 ] study the eigenvalue problems related to the ordinary differential equations, [Zeena H., 2002 ] study the eigenvalue problems related to the linear and nonlinear ordinary integral equations.[ S. Iqbal1, I. Siddique ,2012] talk about galerkin's finite element method for solving special forth-order boundary-value problem,[J. E.,2012], solution of weighted residual problems by using galerkin's method, [Erich M., 2012] introduced Lecture Notes on Partial Differential Equations, [F. Hosseini , 2014] used Sinccollocation method solve numerical problems involving integral equations and ordinary differential equations. A Collocation Method for Numerical Solution of the Generalized

Burgers-Huxley Equation [Mohammad Z. and Nastaran A., 2014], [J. APPELL and et al., 2004], a note on the fredholm property of partial integral eguations of romanovskij type, [Ramziddin K., Yusup E., 2016] study Fredholm partial integral equations of second type with degenerate kernel,

In this work, we study the eigenvalue problems related to the linear partial integral equations and we give two methods for solving this type of problems, namely the collection and Galekin's methods. Also, some examples are solved with their computer programs written in MathCad software package.

## 2. The Definition of Integral Equations, [ David K., 1999 ]

### 2.1 What is an integral equation?

An integral equation (IE ) is an equation in which an unknown function appears within an integral , just as a differential equation is an unknown function appears within a derivative . Just as the solution to a differential equation is a function, so too is the solution to an integral equation a function.

### 2.2 Ordinary versus Partial Integral Equations,[David K., 1999 ]

When dealing with the differential equations, there is ordinary differential equations (ODE) and partial differential equations ( PDE ). Here, if the unknown function which appears under the integral sign depends only on one independent variable then this type of the integral equation is said to be ordinary ( OIE ), otherwise it is partial integral equation (PIE ). For example of an ordinary integral equation:

$$f(x) = g(x) + \lambda \int_a^x K(x,y) f(y)\,dy \qquad (1)$$

For an example of partial integral equation:

$$f(x,y) = g(x,y) + \lambda \int_{a2}^{x} \int_{a1}^{y} K(x,y,z,m) f(z,,m)\,dz\,dm \qquad (2)$$

Hopefully we see the analogy between ODE's and PDE's with OIE's and PIE's.

### 2.3 Linearity versus nonlinearity of integral equations, [ Davis K., 1999]

Do integral equations come in linear and nonlinear flavors?

Yes. An integral equation is called linear if linear operations are performed in it upon the unknown function, otherwise it is nonlinear. For example

$$h(x) f(x) = g(x) + \lambda \int_a^x K(x,y) f(y)\,dy \qquad (3)$$

is a linear ordinary integral equation. On the other hand

$$f(x) = g(x) + \lambda \int_a^x k(x, y, f(y))\, dy \qquad (4)$$

Is a nonlinear integral equation in case if K[x, y , f(y) ] has nonlinear function of f(y),for example, if k( x , y , f(y) )=sin($f(y)$)

### 2.4 Categorization of integral equations

The integral equations can be divided into Freholm and Volterra types of the first, second and third types [ David K., 1999 ]. The third type of the linear ordinary integral equation termed as the eigenvalue problems related to the integral equations It takes the form

$$\lambda \int_a^b k(x,t) f(t)\, dt \qquad (5)$$

The values of λ for which nontrivial solutions exist are called the characteristic values of the integral equation.

In this work, we restrict the study to the eigenvalue problems related to the Freholm linear partial integral equations which take the form

$$f(x,y) = \lambda \int_a^b \int_c^d k(x, y, z, m) f(z,m)\, dzdm \quad (6)$$

The eigenvalue problem related to eq.(6) is a problem of finding an eigenvalue λ with the corresponding eigen function f(x, y) in case the kernel k(x,y,z,m) is a given function of x,y,z and m and a ,b , c and d are given constants

**Remarks:**

1) The above equation is said to be a Fredholm linear partial integral equation of the third kind.
2) It is clear that the trivial solution f(x,y)=0 is a solution for every value of λ. But for a certain values of λ a nontrivial solution f(x,y) ≠ 0 may exist.
3) Eq.(6) can be written as

$$\mathrm{Af} = \lambda\, \mathrm{Bf} \qquad (7)$$

Where A=I and $B = \int_a^b \int_c^d k(x, y, z, m)\, dmdz$

## 3. Approximated Solutions to the Eignvalue Problems of the Linear Partial Integral Equations

In this section, we solve the eigenvalue problem related to the partial integral equation given by eq. (7). To do so, two methods, namely collocation method and Galerkin's Method are introduced to solve such problems.

### 3.1 The Collocation Method

This method is one of the most common methods used to approximate the solution of the integral equations, [ Harriet H. and Robert K., 1974 ]. Here, we use this method to solve the eigenvalue problems related to the linear partial integral equation given by eq. (7)

This method is based on approximating the solution f(x,y) of eq. (7) as a linear combination of the function $\phi_{ij}$ (x,y) , that is write

$$f(x,y) \cong \sum_{i=1}^{n} \sum_{j=1}^{m} c_{ij}\, \phi_{ij}(x, y) \qquad (8)$$

By substituting eq. (8) into eq. (7), one can get

$$A\left(\sum_{i=1}^{n} \sum_{j=0}^{n} c_{ij} \phi_{ij} x, y)\right) = \lambda B\left(\sum_{i=0}^{n} \sum_{j=0}^{m} c_{ij}\, \phi_{ij}(z,m)\right) + \varepsilon(x, y, c_{11}, c_{12}, \dots, , c_{nm}\ \lambda) \quad (9)$$

Where ε(x,y,$c_{11}, c_{12}, \dots c_{nm}, \lambda$) is the error in the approximation of eq.(8).

Therefore, the problem here is reduced to finding the coefficients $(c_{ij})_{1.1}^{m.n}$ and λ To do so, this method requires nm conditions by assuming that the error given in eq.(9) vanishes at them. In other words, choose nm points say $(x_k, y_k)_{k=1}^{nm}$ in the domain D where $D = \{(x,y) | a \le x \le b, c \le y \le d\}$ such that ε($x_k, y_k, c_{11}, c_{12}, \dots, , c_{nm}, \lambda$) = 0 and substituting these points into eq.(9) to give the system of the nonlinear equations

$$A\left(\sum_{i=1}^{n} \sum_{j=0}^{m} c_{ij}\ \phi_{ij}\, x_k, y_k)\right) = \lambda B\left(\sum_{i=0}^{n} \sum_{j=0}^{m} c_{ij}\ \phi_{ij}(z,m)\right), k = 1,2, \dots . nm$$

The above system of the nonlinear equations can be solved by using Mathcad software package to find the values of $(c_{ij})_{1.1}^{m.n}$ and λ.

To illustrate collocation methods consider the following example :

<u>**Example (1):**</u> consider the first order homogeneous fredholm linear partial integral equation of the third kind.

$$f(x,y) = \lambda \int_0^1 \int_0^1 (x+y) f(z+m)\, dmdz \quad (10)$$

Approximate the solution f(x+y) of eq.(10) by:

$$f(x,y) = c_1 + c_2 x + c_3 y + c_4 xy$$

And by substituting this solution into eq.(10), one can get

$$c_1 + c_2 x + c_3 y + c_4 xy = \lambda \int_0^1 \int_0^1 (x+y)(c_1 + c_2 m + c_3 z + c_4 zm)\, dmdz + \varepsilon(x, y, c_1, c_2, c_3, c_{4,}, \lambda)$$

(11)

Therefore

$$\varepsilon(x, y, c_{11}, c_{12}, \dots, , c_{nm}\ \lambda) - \lambda(x+y) \int_0^1 \int_0^1 (c_1 + c_2 m + c_3 z + c_4 zm)\, dmdz = \varepsilon(x, y, c_1, c_2, c_3, c_{4,}, \lambda)$$

After simple computations, one can have

$$c_1 + c_2 x + c_3 y + c_4 xy = \lambda\left(c_1 x + \frac{1}{4} c_4 (x+y) + c_1 y + \frac{1}{2} c_2 (x+y) + \frac{1}{2} c_3 (x+y)\right) + \varepsilon(x, y, c_1, c_2, c_3, c_{4,}, \lambda)$$

Now, to find $c_1, c_2, c_3, c_4$ and $\lambda$, we choose four points in the domain D where $D = \{(x,y)|0 \leq x \leq 1, 0 \leq y \leq 1\}$say: the points $(x_1 + y_1) = (0,1), (x_2, y_2) = (1,0)$ ; $(x_3 + y_3) = (0,0)$ and $(x_4 + y_4) = (1,1)$ at which the error defined in eq.(13) will be vanished to get the system of the equations.

$$c1 + c3 - \lambda.\left(\frac{1}{4}.c4 + c1 + \frac{1}{2}.c2 + \frac{1}{2}.c3\right) = 0$$
$$c1 + c3 - \lambda.\left(c1 + \frac{1}{2}.c2 + \frac{1}{2}.c3 + \frac{1}{4}.c4\right) = 0$$
$$c1 = 0$$
$$c1 + c2 + c3 + c4 - \lambda.\left(2.c1 + \frac{1}{2}.c4 + c2 + c3\right) = 0$$

Which can be solved by using Mathcad software package and the results are tabulated down.

**Table 1:** The approximated solution of example (1) by using the collocation method

| The Initial values of $c1, c2, c3, c4$ & $\lambda$ | The approximated values of $c1, c2, c3, c4$ & $\lambda$ |
|---|---|
| -6<br>-8<br>-9<br>-7<br>-5 | 0<br>-4.557<br>-4.557<br>8.529x10$^{-12}$<br>1 |
| 2<br>2<br>4<br>2<br>3 | 0<br>1.803<br>1.803<br>2.042x10$^{-11}$<br>1 |
| 1<br>0<br>2<br>1<br>1 | 0<br>0.4<br>0.4<br>0<br>1 |
| 3<br>3<br>3<br>3<br>2 | 0<br>2.812<br>2.812<br>-5.479x10$^{-12}$<br>1 |

### 3.2 Galerkin's Method

The Galerkin's method is also one of the most important methods that can be used to approximate the solution of the integral equations [ Chambers L., 1976] .

Here, we use this method to solve the eigenvalue problem related to the partial integral equation given by eq. (7).

This method starts by considering eq.(9) and requiring that the error defined in eq.(9) is orthogonal to nm linearly independent functions. That is, choose linearly independent functions $\psi_{ij}(x,y)$ such that

$$\int_a^b \int_c^d \psi_{ij}(x,y)\,\varepsilon(x,y,c_{11},c_{12},\dots,c_{nm}\ \lambda)dxdy = 0 \quad (14)$$

Note that, in general the linearly independent functions $\psi_{ij}(x,y)$ is different from the linearly independent function $\phi_{ij}(x,y)$ which is used for the approximation of the solution given by eq.(8). But sometimes it is convenient to used the same functions.

Therefore eq.(14) can be written as

$$\int_a^b \int_c^d A\left(\sum_{i=1}^{n}\sum_{j=1}^{m} c_{ij}\,\phi_{ij}(x,y)\right) - \lambda B\left(\sum_{i=1}^{n}\sum_{j=1}^{m} c_{ij}\,\phi_{ij}(z,m)\right)\phi_{ij}(x,y)\,dydx = 0$$
$$i=1,2,\dots n, \qquad j=1,2,\dots,m$$

The above system of the nonlinear equations can be solved to find the values of $(c_{ij})_{1.1}^{m.n}$ and λ. To illustrate the Galerkin's method see the following example:

**Example 2:**

Recall example (1)

Next, choose four linearly independent functions to be orthogonal to the error defined in the eq.(13), say: 1, x, y and xy to get the following equations

$$c1 - \lambda.c1 + \frac{1}{2}.c2 - \frac{1}{2}.\lambda.c2 + \frac{1}{2}.c3 + \frac{1}{4}.c4 - \frac{1}{2}.\lambda.c3 - \frac{1}{4}.\lambda.c4 = 0$$

$$\frac{1}{3}c2 + \frac{1}{4}.c3 + \frac{1}{6}.c4 - \frac{7}{24}.\lambda.c2 - \frac{7}{48}.\lambda.c4 - \frac{7}{12}.\lambda.c1 - \frac{7}{24}.\lambda.c3 + \frac{1}{2}.c1 = 0$$

$$\frac{1}{2}.c1 - \frac{7}{12}.\lambda.c1 + \frac{1}{4}.c2 - \frac{7}{24}.\lambda.c2 + \frac{1}{3}.c3 + \frac{1}{6}.c4 - \frac{7}{24}.\lambda.c3 - \frac{7}{48}.\lambda.c4 = 0$$

$$\frac{1}{6}.c2 + \frac{1}{6}.c3 + \frac{1}{9}.c4 - \frac{1}{6}.\lambda.c2 - \frac{1}{12}.\lambda.c4 - \frac{1}{3}.\lambda.c4 - \frac{7}{24}.\lambda.c1 - \frac{1}{6}.\lambda.c3 + \frac{1}{4}.c1 = 0$$

Which is solved by using Mathcad software package and the results tabulated down

**Table 2:** The approximated solution of example (2) by using Galerkin's method

| The Initial values of $c1, c2, c3, c4$ & $\lambda$ | The approximated values of $c1, c2, c3, c4$ & $\lambda$ |
|---|---|
| -6<br>-3<br>-8<br>-9<br>-2 | 1.264x10$^{-11}$<br>-2.68<br>-2.68<br>2.206x10$^{-12}$<br>1 |
| 4<br>2<br>3<br>2<br>3 | -5.207x10$^{-11}$<br>2.024<br>2.024<br>-2.085x10$^{-10}$<br>1 |
| 5<br>3<br>4<br>3<br>4 | 7.411x10$^{-12}$<br>2.876<br>2.876<br>-6.62x10$^{-12}$<br>1 |
| 6<br>4<br>5<br>4<br>5 | 1.315x10$^{-11}$<br>3.635<br>3.635<br>3.946x10$^{-11}$<br>1 |

## 4. Conclusions and Recommendations

From our work, it is convenient to mention that:

1) The collocation method and the Galerkin's method are very active methods when it is applied to solve the eigenvalue problems associated with the linear partial integral equations. These methods can be also applied to solve the eigenvalue problems related to the nonlinear partial integral equations.
2) The above methods can be applied to solve the non-eigenvalue problems related to the linear and nonlinear partial integral equations.
3) Also, for future work, one can introduce the following problems:
4) Find another method for solving the generalized linear eigenvalue problems in partial Integral equations and make a comparison between these methods and the other methods which appeared in this work.
5) Use another type of functions instead of the polynomial to approximate the solution of the linear and nonlinear partial integral equations of the first, second and third kinds.
6) Use the same methods to solve the eigenvalue problems of the linear and nonlinear partial integral equations in which the unknown function depend on more than two variables.

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