An Application of Zero Forcing using Power Propagation Time

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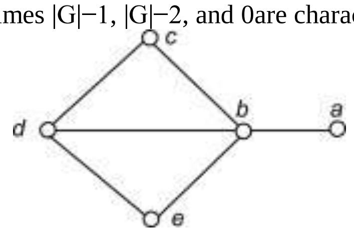
Abstract: ***Zero forcing and power domination are iterative processes on graphs where an initial set of vertices are observed The K-zero forcing number of a graph G is the minimum cardinality of a K- zero forcing set of G. In this paper we determine the K-zero forcing number of CSK- pyramid networks denoted by $CSKP_{(C,L)}$,for all positive values of k except for k = C − 1, C ≥ 2, for which we give an upper bound. The k−propagation radius of a graph G is the minimum number of propagation steps needed to monitor the graph G over all minimum k-PDS. We give a relationship between the k-forcing and the k-power domination numbers of a graph that bounds one in terms of the other***

Keywords: zero forcing, *propagation* time, CSK-pyramid, power domination

1. Introduction

Zero forcing (also called graph infection) on a simple, undirected graph G is based on the color-change rule: if each vertex of G is colored either white or black, and vertex v is a black vertex with only one white neighbor w, then change the color of w to black. A minimum zero forcing set is a set of black vertices of minimum cardinality that can color the entire graph black using the color change rule. The propagation time of a zero forcing set B of graph G is the minimum number of steps that it takes to force all the vertices of G black, starting with the vertices in B black and performing independent forces simultaneously. The minimum and maximum propagation times of a graph are taken over all minimum zero forcing sets of the graph. It is shown that a connected graph of order at least two has more than one minimum zero forcing set realizing minimum propagation time. Graphs G having extreme minimum propagation times |G|−1, |G|−2, and 0are characterized,



The minimum propagation time of a graph G. For example, it is easy to see that the 4-cycle has Z(C_4)=2 and pt(C_4)=1. By deleting one edge of C_4, it becomes a path P_3, which has Z(P_3)=1 and pt(P_3)=2.

Zero forcing was introduced as a process to obtain an upper bound for the maximum nullity of real symmetric matrices whose nonzero pattern of off-diagonal entries is described by a given graph [1]. The minimum rank problem was motivated by the inverse eigenvalue problem of a graph. Independently, zero forcing was introduced by mathematical physicists studying quantum systems [2]. Since its introduction, zero forcing has attracted the attention of a Mathematics, large number of researchers who find the concept useful to model processes in a broad range of disciplines. The need for a uniform framework for the analysis of the diverse processes where the notion of zero forcing appears led to the introduction of a generalization of zero forcing called k-forcing [3]. Amos et al. proposed k-forcing in [3] as the following graph coloring game. Assume the vertices of a graph are colored in two colors, say white and blue. Iteratively apply the following color change rule: if u is a blue vertex with at most k white neighbors, then change the color of all the neighbors of u to blue. Once this rule does not change the color of any vertex, if all vertices are blue, the original set of blue vertices is a k-forcing set of G. The original zero forcing is 1-forcing under this definition. Because the problem of deciding whether a graph admits a 1-forcing set of a given maximum size is NP-complete even if restricted to planar graphs the general problem of finding forcing sets cannot be solved algorithmically for large graphs without the development of further theoretical tools.

Power domination was introduced by Haynes et al. in [4] when using graph models to study the monitoring process of electrical power networks. When a power network is modeled by a graph, a power dominating set provides the locations where monitoring devices (Phase Measurement Units, or PMUs for short) can be placed in order to monitor the power network. Finding optimal PMU placements is an important practical problem in electrical engineering due to the cost of PMUs and network size. Although power domination is substantially different from standard graph domination, the notion of k-power domination was proposed as a generalization of both power domination (k = 1) and standard graph domination (k = 0) [5]. Chang et al. defined k-power domination in [5] using sets of observed vertices. Given a graph G and a set of vertices S, initially all vertices in S and their neighbors are observed; all other vertices are unobserved. Iteratively apply the following propagation rule: if there exists an observed vertex u that has k or fewer unobserved neighbors, then all the neighbors of u are observed. Once this rule does not produce any additional observed vertices, if all vertices of G are observed, S is a k-power dominating set of G. Many problems outside graph theory can be formulated in terms of minimum k-power dominating sets [5] so methods to obtain them are highly desired. An algorithmic approach has been attempted, but the problem of deciding if a graphs admits a k-power dominating set of a given maximum size is NP-complete [5].

Although k-forcing and k-power domination have been studied independently, an in-depth analysis of k-power domination leads to the study of k-forcing. Indeed, after the initial step in which a set observes itself and its neighbors, the observation process in k power domination proceeds exactly as the color changing process in k-forcing. The aim of this paper is to establish a precise connection between k-forcing and k-power domination to facilitate the transference of results, proofs, and methods between them, and ultimately to advance research on both problems.

Throughout this paper we work on k-forcing and k-power domination concurrently, using results in one process as stepping stones for results in the other one. In Section 2 we present the definitions and notation that we use in the rest of the paper. In Section 3 we give some core results and remarks that we use in the sections that follow. In Section 4 we examine the effect of a a relationship between the k-zero forcing that CSKP-networks.

Preliminaries

A graph is an ordered pair G = (V, E) where V = V (G) is a finite nonempty set of vertices and E = E(G) is a set of unordered pairs of distinct vertices called edges (i.e., in this work graphs are simple and undirected). The order of G is |G|:= |V (G)|. Two vertices u and v are adjacent or neighbors in G if {u, v} ∈ E(G). The (open) neighborhood of a vertex v is the set N_G(v) = {u ∈ V: {u, v} ∈ E}, and the closed neighborhood of v is the set N_G[v] = N_G(v) ∪ {v}. Similarly ,for any set of vertices S, N_G(S) = $\cup_{v \in S} N_G$(v) and N_G[S] = $\cup_{v \in S} N_G$[v]. The degree of a vertex v is deg_G(v) := |N(v)|.The maximum and minimum degree of G are Δ(G) := max{deg_G((v) : v ∈ V }and δ(G) := min{deg_G((v) :v ∈ V }, respectively; a graph G is regular if δ(G) = Δ(G). We will omit the subscript G when the graph G is clear from the context.

A path joining u, v ∈ V is a sequence of vertices u = x_0 ,x_1, $x_{2,\ldots\ldots,} x_r$ = v such that {x_i , x_{i+1}} ∈ E for each i = 0, . . ., r − 1. A graph G is connected if there is a path joining every pair of different vertices. If a graph is not connected, each maximal connected sub graph is a component of G. In this paper, c(G) denotes the number of components of G and G_1, . . . , $G_{c(G)}$denote the components of G. Most of the results in this work are given for connected graphs, since if a graph is not connected, we can apply the results to each component. If X is a set of vertices of G, the subgraph induced by X (*in* G) is denoted as G[X]; it has vertex set X and edge set {{u, v} ∈ E : u, v ∈ X}. The graph G − X is defined as G[V \ X]. The contraction of X in G is the graph G/X obtained by adding a vertex $v_X$to G− X with NG/X(v_X) = NG[X] \ X. Note that G/X does not require G[X] to be connected whereas the standard use of graph contraction does.

2. Basic Rules

The following Basic Rules follow directly from the definitions of k-power domination and k-forcing, and provide the initial connection between both concepts.

Rule 1: In any graph G, if T is a k-forcing set of G then T is also a k-power dominating set. The converse is not necessarily true, but S is a k-power dominating set if and only if N[S] is a k-forcing set. As a consequence, ${\gamma_{P,k}}^G \leq Z_K$(G) $\leq {\gamma_{P,k}}^G$ (Δ(G)+1).

Rule 2: (Monitored vertices). Let G be a graph, S ⊆ V (G) and k ≥ 0. The sets ($(P^i_{G,K}(S))_i$ ≥0 of vertices monitored by S at step i are defined as follows:

$P^0_{G,K}$ (S) = N_G [S], and

$P^{i+1}_{G,K}$ (S) = ∪{N_G [v]: v∈ $P^i_{G,K}$(S) such that$|N_G [v] \setminus P^i_{G,K} (S), k(S)|$ ≤ k}.

The second part represents the propagation rule. Since $P^i_{G,K}$ (S) is always a union of neighbourhoods, $P^{i_0}_{G,K}$(S) ⊆ $P^{i_0+1}_{G,K}$ (S). If $P^{i_0}_{G,K}$(S) = $P^{i_0+1}_{G,K}$ (S).for some i_0, then $P^j_{G,K}$(S) = $P^{i_0}_{G,K}$(S) for any j ≥ i_0. We thus define$P^{\infty}_{G,K}$(S) = $P^{i_0}_{G,K}$(S) When the graph G is clear from the context, we will simplify the notations to P^i_K(S) and P^{∞}_K(S).

Rule 3: A k−power dominating set of G (k−PDS) is a set S ⊆ V (G) such that $P^{\infty}_{G,K}$(S) = V (G). The k−power domination number, $\gamma P,k^{(G)}$, of G is the least cardinality of a k−power dominating set of G. A ${\gamma_{P,K}}^{(G)}$set is a k-PDS in G of cardinality $\gamma P,k^{(G)}$. Generalized power domination reduces to the usual power domination when k = 1 and to the domination when k = 0.

Rule 4: In a graph G, S (V (G)) is a k-power dominating set of G if and only if N[S] \ S is a k-forcing set of G – S

Observation 1. Let T be a k-forcing set of a graph G. Let A ⊆ T.

1) If A is k-forcing set of T in G, then A is a k-forcing set of G;
2) If A is k-power dominating set of T in G, then A is a k-power dominating set of G.

Observation 2. Let S be a k-power dominating set of a graph G .Let A ⊆ S.

1) If A is k-forcing set of N[S] in G, then A is a k-forcing set of G;
2) If A is k-power dominating set of N[S] in G, then A is a k-power dominating set of G.

If A is k-power dominating set of N[S] in G, then A is a k-power dominating set of G.

Observation 3. Let G be a graph that has an edge. Then$\left\lceil \frac{Z(G)}{\Delta(G)} \right\rceil$ ≤ γP (G), and this bound is tight.

3. k- zero forcing number of CSK- pyramid network

Theorem 3.1 ***Chandrasekaran, Sulthana* (CS)**
For C≥3 and k∈[c-1] , $\gamma_{p,k}{}^{(CSP_{(C,2)})} \leq$**c-k then** $Z_{p,k}{}^{(CSP_{(C,2)})} \leq$c-k.

Proof **CS** –pyramid network denoted by $csp_{(c,2)}$consist of a set of vertices, we know that v($CSP_{(C,L)}$)={(r,($a_r, a_{r-1}, a_{r-2}\ldots\ldots a_{r1}$));r∈**[L],** $a_i \in$**[c]$_0$,for i∈[r]}U{(0,(1))}.**

Let S = {(1,(i)): k ≤ i ≤ C − 1}. (The vertices in S are coloured in black for K=1,C=5).

Here S is K-PDS OF $CSP_{(C,2)}$ such that s∈ **S** has atleast one neighbor not in N[S/{s}] .We first show that $Z_{p,k}{}^{(CSP_{(C,2)})} \leq |N[S]|$ $-\gamma_{p,k}{}^{(CSP_{(C,2)})}$ .Recall that $N[S]$ is a zeroforcing set of CS-Pyramid For each s∈S,choose a $v_s \in N[S]$ such that v_s N[S/{s}] .Then N[S]/{ $v_1, v_2 \ldots\ldots v_{|S|}$} is also a zero forcing set since s will forcev_s in step one so, $Z_{p,k}{}^{(CSP_{(C,2)})} \leq |N[S]|$-$\gamma_{p,k}{}^{(CSP_{(C,2)})}$ ≤C-K. we get the result.

Theorem 3.2
For C≥3 and k∈[c-2] , $\gamma_{p,k}{}^{(CSP_{(C,2)})} \leq$c-k then $Z_{p,k}{}^{(CSP_{(C,2)})} \geq$c-k.

Proof
We know that S be a minimum k-PDS of $CSP_{(C,2)}$ of $\gamma_{p,k}{}^{(CSP_{(C,2)})} \leq$c-k [theorem 11] therefore N[S] is a zero forcing set $CSP_{(C,2)}$ Here $Z_{p,k}{}^{(CSP_{(C,2)})}$ atleast one force must be performed at each time step , v∈N[S] and v∉N[S] /S , suppose v∉N[S]/S ,atmost v∉N[S]/S forces can be performed ,at any one time step ,It's clearly graphs $CSP_{(C,2)}$ therefore $Z_{p,k}{}^{(CSP_{(C,2)})} \geq$c-k.

Theorem 3.3
For C≥3 and k∈[c-2] , $\gamma_{p,k}{}^{(CSP_{(C,2)})} \leq$c-k then $Z_{p,k}{}^{(CSP_{(C,2)})}$ =c-k.

Proof
Clearly, $Z_{p,k}{}^{(CSP_{(C,2)})}$ =c-k Let C≥3 For k=c-1,any vertex in level 1 forms a K-PDS of $CSP_{(C,2)}$,For k∈[C-2] the result from theorem 3.1 and 3.2 we get the result.

4. Conclusion

In this paper, we have determined the k−power domination number and zero forcing of CSK Pyramid networks, CSKP(C,L) , for all positive values of k except for k = C−1, C ≥ 2, for which we give an upper bound. k−propagation radius of CSKP(C,L) in some cases. The k−power domination number of other pyramid networks such as grid pyramids, torus pyramids , k−propagation radius of CSKP(C,L) in some cases can be studied in future.

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