Numerical Method for Weakly Coupled Nonlinear Parabolic System

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Abstract: The aim of the paper is to study a system of finite difference equations corresponding to the weakly coupled nonlinear parabolic system with nonlinear boundary conditions in a bounded domain. Method of upper – lower solutions and monotone iteration process are studied. We develop new iteration scheme. Discrete initial boundary value problems are studied by applying the method of upper lower solutions, existence comparison and uniqueness of solutions.

Keywords: Diffusion convection system, Initial boundary value Problem, Upper lower solution, Iteration scheme, Existence and uniqueness results

Subject Classification: MSC: 65N06,65N12,65N22,65H10, 65 F10

1. Introduction

The finite difference numerical method is one of the powerful and well developed numerical method employed in the study of differential equations. The stability and convergence of this method for parabolic differential equation is discussed in [3]. the coupled system of partial differential equation are well studied by many researchers for both continuous problems [1,2,5] as well as discrete problems[1,4,5]. Pao [5,6] studied the weakly coupled system of reaction diffusion convection equation. We generalize the result recently obtained by Pao [6] for discrete problems for weakly coupled system with linear boundary conditions in two ways on one hand new iteration scheme is developed which is similar to the iteration scheme for algebraic system and monotone convergent sequence are constructed by choosing suitable initial iteration which leads to existence-comparison as well as uniqueness results and on the other hand the results are obtained with nonlinear boundary conditions, under new iteration scheme.

The out line of the paper is as: The formulation of the system of finite difference equations corresponding to the weakly coupled system of parabolic equations with nonlinear boundary conditions by implicit method and the development of new iteration scheme for the construction of two monotone convergent sequences when both reaction and boundary functions are quasi-monotone non-decreasing . In the last existence-comparison as well as uniqueness of the solution of the system under consideration are studied.

2. Upper-Lower Solutions

Consider the following initial boundary value problem

\[\begin{align*}
  &u_t - L^{(1)}[u] = f^{(1)}(x, t, u, v) \text{ in } \Omega_T, \\
  &v_t - L^{(2)}[v] = f^{(2)}(x, t, u, v) \text{ in } \Omega_T
\end{align*}\]

With nonlinear boundary conditions

\[\begin{align*}
  &\frac{\partial u}{\partial n} + g^{(1)}(x, t, u, v) \text{ on } S_T, \\
  &\frac{\partial v}{\partial n} + g^{(2)}(x, t, u, v) \text{ on } S_T
\end{align*}\]

And initial conditions

Now let \(k_n = t_n - t_{n-1}\) be the nth time increment for \(n = 1, 2, \ldots, N\) and \(h_n\) be the spatial increment in the \(x_i\) coordinate direction. Let \(e_0 = (0, \ldots, 1, \ldots, 0)\) be the unit vector in \(R^p\), where constant 1 appears in the \(V^0\) component and zero elsewhere.

The standard first and second order difference approximations are

\[u(x_i, t_n) = u(x_i, h_n, e_p, t_n) - u(x_i - h_n, e_p, t_n)\]

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\[ \Delta^{(p)} u (x_i, t_n) = h_p^{-1} [u (x_i + h_p \sigma_p, t_{n+1}) - 2u (x_i, t_n) + u (x_i - h_p \sigma_p, t_{n-1})] \]

Respectively (for details see [1,6]), also note that \( k_n^{-1}(u_{i,n} - u_{i,n-1}) \) is the backward approximation for \( u_n \). Using these notations, the continuous IBVP (2.1)-(2.3) is transformed to following discrete initial boundary value problem

\[ k_n^{-1}(u_{i,n} - u_{i,n-1}) - L^{(2)}[u_{i,n}](i, n) \in \Lambda_p \]

\[ B^{(2)}[u_{i,n}](i, n) \in S_p \]

3. Monotone Iteration Scheme

We consider (f\(^{(1)}\), f\(^{(2)}\)) and (g\(^{(1)}\), g\(^{(2)}\)) as quasimonotone nondecreasing function in the sector \( S_p \); therefore, there exists nonnegative function \( y_{i,n}^{(1)}, y_{i,n}^{(2)}, \sigma_{i,n}^{(1)} \) and \( \sigma_{i,n}^{(2)} \) such that for any pair \( (u_{i,n}, v_{i,n}) \) and \( (u'_{i,n}, v'_{i,n}) \) in the sector \( S_p \) formed from the ordered upper-lower solutions, the functions \( f^{(l)} \), \( g^{(l)} \), \( l = 1, 2 \) satisfy the following one-sided Lipschitz condition, which ensure the existence of solutions of the discrete IBVP (2.4)-(2.6).

\[ f^{(1)}(u_{i,n}, v_{i,n}) - f^{(1)}(u'_{i,n}, v'_{i,n}) \geq -y_{i,n}^{(1)}(u_{i,n} - u'_{i,n}) \]

\[ f^{(2)}(u_{i,n}, v_{i,n}) - f^{(2)}(u'_{i,n}, v'_{i,n}) \geq -y_{i,n}^{(2)}(v_{i,n} - v'_{i,n}) \]

\[ g^{(1)}(u_{i,n}, v_{i,n}) - g^{(1)}(u'_{i,n}, v'_{i,n}) \geq -\sigma_{i,n}^{(1)}(u_{i,n} - u'_{i,n}) \]

\[ g^{(2)}(u_{i,n}, v_{i,n}) - g^{(2)}(u'_{i,n}, v'_{i,n}) \geq -\sigma_{i,n}^{(2)}(v_{i,n} - v'_{i,n}) \]

Now we start from the suitable initial iteration \( (u_{i,n}^{(0)}, v_{i,n}^{(0)}) \) as either \( (\bar{u}_{i,n}^{(0)}, \bar{v}_{i,n}^{(0)}) \) or \( (\overline{u}_{i,n}, \overline{v}_{i,n}) \) and construct a sequence \( \{u_{i,n}^{(m)}, v_{i,n}^{(m)}\} \) from the following iteration process.

Iteration process

\[ k_n^{-1}(u_{i,n}^{(m)} - u_{i,n-1}^{(m)}) - L^{(1)}[u_{i,n}^{(m)}] + y_{i,n}^{(1)}u_{i,n}^{(m)} = F^{(1)}(u_{i,n}^{(m)}, v_{i,n}^{(m)}) \]

\[ B^{(1)}[u_{i,n}^{(m)}] + \sigma_{i,n}^{(1)}u_{i,n}^{(m)} = G^{(1)}(u_{i,n}^{(m)}), v_{i,n}^{(m)} \]

\[ u_{i,0}^{(m)} = \psi_i^{(m)} \]

\[ k_n^{-1}(v_{i,n}^{(m)} - v_{i,n-1}^{(m)}) - L^{(2)}[v_{i,n}^{(m)}] + y_{i,n}^{(2)}v_{i,n}^{(m)} = G^{(2)}(u_{i,n}^{(m)}, v_{i,n}^{(m)}) \]

\[ B^{(2)}[v_{i,n}^{(m)}] + \sigma_{i,n}^{(2)}v_{i,n}^{(m)} = G^{(2)}(u_{i,n}^{(m)}), v_{i,n}^{(m)} \]

Where \( m = 1, 2 \).

For each \( m = 1, 2 \) the above system consist of two linear finite difference parabolic problems. So the sequence constructed from the above iteration process are well defined. Define

\[ L^{(1)}[u_{i,n}] \equiv k_n^{-1}(u_{i,n} - u_{i,n-1}) - L^{(1)}[u_{i,n}] \]

\[ L^{(2)}[v_{i,n}] \equiv k_n^{-1}(v_{i,n} - v_{i,n-1}) - L^{(2)}[v_{i,n}] \]

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We choose \((\tilde{u}_{1,n}, \tilde{v}_{1,n})\) and \((\tilde{u}_{1,n}, \tilde{v}_{1,n})\) as initial iteration and obtain the sequences of iteration by splitting the above iteration process in the following sub-iteration process 1. - 1.  

Sub - iteration process A. 

\[
L^{(1)}[\tilde{u}_{1,m}] = \gamma^{(1)} \tilde{u}_{1,m-1} + f^{(1)}(\tilde{u}_{1,m-1}, \tilde{v}_{1,m-1}) \\
\tilde{u}_{1,0} = \psi^{(1)} \quad \text{where} \quad m = 1, 2, \ldots
\]

Sub - iteration process A. 

\[L^{(2)}[\tilde{v}_{1,m}] = \gamma^{(2)} \tilde{v}_{1,m-1} + f^{(2)}(\tilde{u}_{1,m}, \tilde{v}_{1,m}) \\
\tilde{u}_{1,0} = \psi^{(2)} \quad \text{where} \quad m = 1, 2, \ldots
\]

Sub - iteration process A. 

\[L^{(3)}[\tilde{u}_{1,m}] = \gamma^{(3)} \tilde{u}_{1,m-1} + f^{(3)}(\tilde{u}_{1,m-1}, \tilde{v}_{1,m}) \\
\tilde{u}_{1,0} = \psi^{(3)} \quad \text{where} \quad m = 1, 2, \ldots
\]

We start \((\tilde{u}_{1,0}, \tilde{v}_{1,0}) = (\tilde{u}_{1,0}, \tilde{v}_{1,0})\) with as initial iteration in the sub-iteration process A. 1. for \(m = 1\) and obtain \(\tilde{u}_{1,1}\) then we use this value in the sub-iteration process A. 2. for \(m = 1\) and obtain \(\tilde{v}_{1,1}\) in this way we obtain the first iteration \((\tilde{u}_{1,1}, \tilde{v}_{1,1})\). Then using \((\tilde{u}_{1,1}, \tilde{v}_{1,1})\) as initial iteration we find the next iteration \((\tilde{u}_{1,1}, \tilde{v}_{1,1})\) as above from the sub-iteration process A. 1, A. 2. \(m = 2\) and so on. We denote the sequence of these iteration by \((\tilde{u}_{1,m}, \tilde{v}_{1,m})\). 

Now we start with \((\tilde{u}_{1,0}, \tilde{v}_{1,0})\) as initial iteration in the sub - iteration process A. 3. for \(m = 1\) and obtain \(\tilde{u}_{1,1}\) thus we obtain the first iteration \((\tilde{u}_{1,1}, \tilde{v}_{1,1})\). using this iterat as initial iteration we obtain the next iteration \((\tilde{u}_{1,0}, \tilde{v}_{1,0})\) as above from sub - iteration process A. 3 and A. 4 for \(m = 2\) and so on. We denote the sequence of these iteration by \((\tilde{u}_{1,m}, \tilde{v}_{1,m})\). 

To prove monotone property of these sequence we required the monotone property of the functions \(F^{(1)}, G^{(1)}\), \(l = 1.2, \ldots, 8\) as well as positivity result. The monotone property and positivity results are proved in [3.8] and [6], we state without proof these results as lemma 3.1 and lemma 3.2 respectively. 

**Lemma 3.1** Suppose \((u_{i,n}, v_{1,i})\) and \((u'_{i,n}, v'_{1,i})\) are two functions in \(S_{1,n}\) such that \((u_{i,n}, v_{1,i}) \geq (u'_{i,n}, v'_{1,i})\) and suppose that Lipschitz condition (3.1) 

\[\text{Holds,}\]

If \((F^{(1)}, F^{(2)})\) and \((G^{(1)}, G^{(2)})\) are quasimonotone non-decreasing in \(S_{1,n}\) then 

\[F^{(1)}(u_{i,n}, v_{1,i}) \geq F^{(1)}(u'_{i,n}, v'_{1,i})\]

\[F^{(2)}(u_{i,n}, v_{1,i}) \geq F^{(2)}(u'_{i,n}, v'_{1,i})\]

\[G^{(1)}(u_{i,n}, v_{1,i}) \geq G^{(1)}(u'_{i,n}, v'_{1,i})\]

\[G^{(2)}(u_{i,n}, v_{1,i}) \geq G^{(2)}(u'_{i,n}, v'_{1,i})\]

**Lemma 3.2** Let \(w_{i,n}\) be function defined in \(\Lambda_{p}\) such that 

\[L[w_{i,n}] + C_{i,n}w_{i,n} \geq 0 (i, n) \in \Lambda_{p}\]

\[B[w_{i,n}] \geq 0 (i, n) \in \Lambda_{p}\]

Where \(C_{i,n} \geq 0\). then \(w_{i,n} \geq 0, i \in \Lambda_{p}\). 

**Lemma 3.3** Let \((F^{(1)}, F^{(2)})\) and \((G^{(1)}, G^{(2)})\) be quasimonotone nondecreasing \(C^{1} + \ldots + \text{functions} \in S_{1,n}\). 

Then the sequence \((\tilde{u}_{1,m}, \tilde{v}_{1,m})\) \((\tilde{u}_{1,m}, \tilde{v}_{1,m})\) obtained from the iteration process (3.3) with initial iterations \((\tilde{u}_{1,0}, \tilde{v}_{1,0}) = (\tilde{u}_{1,0}, \tilde{v}_{1,0})\) \((\tilde{u}_{1,0}, \tilde{v}_{1,0})\) respectively possess the monotone property 

\[\tilde{u}_{1,m} \geq \tilde{u}_{1,m-1} \geq \tilde{u}_{1,m-1} \quad \text{and} \quad \tilde{v}_{1,m} \geq \tilde{v}_{1,m-1} \geq \tilde{v}_{1,m-1}\]

\[\text{where} \quad m = 1, 2, \ldots\]

**Proof.**

Let \(w_{i,n} = \tilde{u}_{1,0} \in \tilde{v}_{1,0} \in \tilde{v}_{1,0} \in \tilde{v}_{1,0} \in \tilde{v}_{1,0} \in \tilde{v}_{1,0}\)

Then by (2.9), (2.10), (2.12), iteration process (3.3) and (3.4)

\[L^{(1)}[w_{i,n}] = L^{(1)}[\tilde{u}_{1,0}] - L^{(1)}[\tilde{u}_{1,0}, \tilde{v}_{1,0}] \geq 0\]

\[B^{(1)}[\tilde{u}_{1,0}] - B^{(1)}[\tilde{u}_{1,0}, \tilde{v}_{1,0}] \geq 0\]

\[B^{(1)}[\tilde{u}_{1,0}] - B^{(1)}[\tilde{u}_{1,0}, \tilde{v}_{1,0}] \geq 0\]

By (2.11) and iteration process (3.3)

\[w_{i,0} \geq \tilde{u}_{1,0} \geq \tilde{u}_{1,0} \geq \tilde{u}_{1,0} \geq \tilde{u}_{1,0} \geq \tilde{u}_{1,0}\]

By lemma 3.2, \(w_{i,0} \geq 0\). This gives \(\tilde{u}_{1,0} \geq 0\).

Now by (2.9), (2.10), (2.12), iteration process (3.3) and (3.4)

\[L^{(2)}[\tilde{v}_{1,0}] = L^{(2)}[\tilde{v}_{1,0}] - L^{(2)}[\tilde{v}_{1,0}, \tilde{v}_{1,0}] \geq 0\]

\[B^{(2)}[\tilde{v}_{1,0}] - B^{(2)}[\tilde{v}_{1,0}, \tilde{v}_{1,0}] \geq 0\]

\[B^{(2)}[\tilde{v}_{1,0}] - B^{(2)}[\tilde{v}_{1,0}, \tilde{v}_{1,0}] \geq 0\]

\[\geq 0 \quad \text{if} \quad f^{(2)} \text{is quasimonotone decreasing and} \quad \tilde{u}_{1,0} \geq \tilde{u}_{1,0}\]

\[B^{(2)}[\tilde{v}_{1,0}] - B^{(2)}[\tilde{v}_{1,0}, \tilde{v}_{1,0}] \geq 0\]

\[B^{(2)}[\tilde{v}_{1,0}] - B^{(2)}[\tilde{v}_{1,0}, \tilde{v}_{1,0}] \geq 0\]

\[\geq 0 \quad \text{if} \quad f^{(2)} \text{is quasimonotone decreasing and} \quad \tilde{u}_{1,0} \geq \tilde{u}_{1,0}\]
Then we obtain iteration $z_i,0 = v_i,0 - \psi_i,1 = \theta_i,0 - \psi_i,2 \geq 0$

It follows from lemma 3.2 that $z_i,0 \geq 0$ in $\Lambda p$, which gives $v_i,0 \geq v_i,1$; thus we get $(u_i,0, v_i,0) \leq (u_i,1, v_i,1)$. Similarly letting $w_i,0 = u_i,0 - u_i,1 = \tilde{u}_i,0 - \tilde{u}_i,1$,

$z_i,0 = \tilde{v}_i,0 - \tilde{v}_i,1 = \tilde{\theta}_i,0 - \tilde{\psi}_i,1 \geq 0$

We obtain $(\tilde{u}_i,0, \tilde{v}_i,0) \geq (u_i,0, v_i,0)$.

Now let

$w_i,1 = \tilde{u}_i,1 - u_i,1$

$z_i,1 = \tilde{v}_i,1 - v_i,1$

Then by iteration process (3.3) and lemma 3.1,

$L(1)[w_i,1] = L(1)[\tilde{u}_i,1] - L(1)[u_i,1]$

$= F(1)(\tilde{u}_i,0, v_i,0) - F(1)(u_i,0, \tilde{u}_i,1) \geq 0$

$B(1)[w_i,1] = B(1)[\tilde{u}_i,1] - B(1)[v_i,1]$

$= G(1)(\tilde{u}_i,1, \tilde{v}_i,0) - G(1)(u_i,1, v_i,0) \geq 0$

By iteration process (3.3), $w_i,1 = \tilde{v}_i,0 - \tilde{v}_i,1 = \psi_i,1 - \psi_i,2 = 0$

From lemma 3.2, it shows that $w_i,0 \geq 0$ on $\Lambda p$. It leads to $\tilde{u}_i,0 \geq u_i,0$.

Again by iteration process (3.3) and lemma 3.1,

$L(2)[\tilde{u}_i,1] = L(2)[\tilde{v}_i,1] - L(2)[v_i,1]$

$= F(2)(\tilde{u}_i,1, \tilde{v}_i,0) - F(2)(u_i,1, v_i,0) \geq 0$

$B(2)[\tilde{u}_i,1] = B(2)[\tilde{v}_i,1] - B(2)[v_i,1]$

$= G(2)(\tilde{u}_i,1, \tilde{v}_i,0) - G(2)(u_i,1, v_i,0) \geq 0$

By iteration process (3.3), $z_i,1 = \tilde{v}_i,0 - \tilde{v}_i,1 = \psi_i,1 - \psi_i,2 = 0$

Using Lemma 3.2, we get $z_i,1 \geq 0$ on $\Lambda p$, which gives $\tilde{v}_i,1 \geq v_i,1$.

Thus we get $(\tilde{u}_i,0, \tilde{v}_i,0) \geq (u_i,1, v_i,1)$.

The above conclusion show that $(u_i,0, v_i,0) \leq (u_i,1, v_i,1) \leq (u_i,2, v_i,2)$.

Assume (3.5) holds for some integer $m$, suppose that $w_i,m = \tilde{u}_i,m - u_i,m+1$ and $z_i,m = \tilde{v}_i,m - v_i,m+1$.

Then by iteration process (3.3) and lemma 3.1,

$L(1)[w_i,m] = F(1)(\tilde{u}_i,m, \tilde{v}_i,m) - F(1)(u_i,m, v_i,m) \geq 0$

$B(1)[w_i,m] = G(1)(\tilde{u}_i,m, \tilde{v}_i,m) - G(1)(u_i,m, v_i,m) \geq 0$

$w_i,0 = \tilde{u}_i,0 - u_i,0 = \psi_i,1 - \psi_i,2 = 0$

By lemma 3.2, it leads to the conclusion $w_i,0 \geq 0$, $\tilde{u}_i,1 \geq u_i,1$.

$L(2)[z_i,m] = F(2)(\tilde{u}_i,m, \tilde{v}_i,m) - F(2)(u_i,m, v_i,m) \geq 0$

$B(2)[z_i,m] = G(2)(\tilde{u}_i,m, \tilde{v}_i,m) - G(2)(u_i,m, v_i,m) \geq 0$

Again by Lemma 3.2, $z_i,m \geq 0$ i.e. $\tilde{v}_i,m \geq v_i,m$. So $(\tilde{u}_i,m, \tilde{v}_i,m) \leq (u_i,m, v_i,m)$.

A similar argument gives $w_i,(m+1) \geq 0$. By induction principle the result (3.5) follows for all $m$. This complete the proof. The results in Lemma 3.3 shows that

$\lim(\tilde{u}_i,m, \tilde{v}_i,m) = (u_i,m, v_i,m)$ as $m \rightarrow \infty$.

Exits in $\Lambda p$. Taking $m \rightarrow \infty$ in the iteration process (3.3), it implies that both $(\tilde{u}_i,m, \tilde{v}_i,m)$.

And $(w_i,m, v_i,m)$ are solutions of the discrete IBVP (2.4) – (2.6). These two solutions may not be equal. They are equal if for $(u_i,m, v_i,m)(u_i,m, v_i,m)$ in the sector $S_i$ with $(u_i,m, v_i,m) \geq (\tilde{u}_i,m, \tilde{v}_i,m)$ ; functions $(f(1), f(2))$ and $(g(1), g(2))$ satisfy the following one sides Lipchitz conditions:

$f(1)(\tilde{u}_i,m, \tilde{v}_i,m) - f(1)(u_i,m, v_i,m)$

$\leq \gamma(1)(u_i,m - u_i,m)$

$\gamma(1)(u_i,m - u_i,m)$

$f(2)(\tilde{u}_i,m, \tilde{v}_i,m) - f(2)(u_i,m, v_i,m)$

$\leq \gamma(2)(u_i,m - u_i,m)$

$\gamma(2)(u_i,m - u_i,m)$

Where the function $\gamma(1), \gamma(2), \sigma(1), \sigma(2)$ for $l = 1, 2$ are given by

$\gamma(1) = \max \left\{ \frac{\partial f(1)}{\partial u} : (i, n) \in \Lambda p, (u_i,m, v_i,m) \in S_i \right\}$

$\gamma(2) = \max \left\{ \frac{\partial f(2)}{\partial u} : (i, n) \in \Lambda p, (u_i,m, v_i,m) \in S_i \right\}$

$\sigma(1) = \max \left\{ \frac{\partial g(1)}{\partial u} : (i, n) \in S_p, (u_i,m, v_i,m) \in S_i \right\}$

$\sigma(2) = \max \left\{ \frac{\partial g(2)}{\partial u} : (i, n) \in S_p, (u_i,m, v_i,m) \in S_i \right\}$

From (3.8) it is clear that $\gamma(1) \geq 0, \gamma(2) \geq 0, \sigma(1) \geq 0, \sigma(2) \geq 0$.

The conditions in (3.1) and (3.7) leads to the following results.
4. Existence-comparison and uniqueness results

In this section, we prove existence-comparison and uniqueness of the solution for the discrete IBVP (2.4) – (2.6).

Theorem 4.1. Suppose that both \( f^{(1)}, f^{(2)} \) and \( g^{(1)}, g^{(2)} \) are quasimonotone non-decreasing \( C^1 \) functions in the sector \( S_{i,n} \) and satisfies Lipschitz conditions (3.1), (3.7).

Suppose pairs of functions \( (\bar{u}_{i,n}, \bar{v}_{i,n}) \) and \( (\tilde{u}_{i,n}, \tilde{v}_{i,n}) \) are ordered upper-lower solutions of the IBVP (2.4) – (2.6). Then the sequences \( \left\{ \bar{u}_{i,n}^{(m)}, \bar{v}_{i,n}^{(m)} \right\} \), \( \left\{ \bar{u}_{i,n}^{(m)}, \bar{v}_{i,n}^{(m)} \right\} \) obtained from the iteration process (3.3) with initial iterations \( (\bar{u}_{i,n}, \bar{v}_{i,n}) = (\bar{u}_{i,n}, \bar{v}_{i,n}) \) and \( (\bar{u}_{i,n}, \bar{v}_{i,n}) = (\bar{u}_{i,n}, \bar{v}_{i,n}) \) converge monotonically from above and below respectively to the solutions \( (\bar{u}_{i,n}, \bar{v}_{i,n}) \) and \( (\tilde{u}_{i,n}, \tilde{v}_{i,n}) \) of the discrete IBVP (2.4) – (2.6).

Moreover

\[
(\bar{u}_{i,n}, \bar{v}_{i,n}) \leq (\bar{u}_{i,n}, \bar{v}_{i,n}) \leq (\tilde{u}_{i,n}, \tilde{v}_{i,n}), (i, n) \in \mathbb{N}_p \quad (4.1)
\]

If, in addition

\[
k_n > \max\{\gamma^{(1)} + \gamma^{(21)} + \sigma^{(11)} + \sigma^{(21)} + \gamma^{(12)} + \gamma^{(22)} + \sigma^{(12)} + \sigma^{(22)}\} \quad (4.2)
\]

then \( (\bar{u}_{i,n}, \bar{v}_{i,n}) \) is unique solution of the discrete initial boundary value problem (2.4) – (2.6) in \( S_{i,n} \).

Proof. It is seen from the Lemma 3.3, that the limits \( \lim_{n \to \infty} (\bar{u}_{i,n}, \bar{v}_{i,n}) \) and \( \lim_{n \to \infty} (\tilde{u}_{i,n}, \tilde{v}_{i,n}) \) in (3.6) are solutions of the discrete IBVP (2.4) – (2.6). Clearly they satisfy the relation (4.1).

Now we show that \( (\bar{u}_{i,n}, \bar{v}_{i,n}) = (\tilde{u}_{i,n}, \tilde{v}_{i,n}) \) in \( \mathbb{N}_p \). Let \( k_n = k \) be constant time step for \( n = 1, 2, ..., n_1 \) where \( 1 \leq n_1 \leq N \) and \( k \) be a fixed number such that

\[
\gamma^{(11)} + \gamma^{(21)} + \sigma^{(11)} + \sigma^{(21)} + \gamma^{(12)} + \gamma^{(22)} + \sigma^{(12)} + \sigma^{(22)} < k^{-1} \quad (4.3)
\]

and

\[
\gamma^{(12)} + \gamma^{(22)} + \sigma^{(12)} + \sigma^{(22)} < k^{-1} \quad (4.4)
\]

Clearly, existence of such a \( k \) follows from [4,2]. Let

\[
U_{i,n} = (1 - ak)^{i,n,k} (\bar{u}_{i,n} - \tilde{u}_{i,n}) \quad (4.4)
\]

\[
V_{i,n} = (1 - ak)^{i,n,k} (\bar{v}_{i,n} - \tilde{v}_{i,n})
\]

for \( n = 1, 2, ..., n_1 \).

Then observe that \( U_{i,n} \geq 0, V_{i,n} \geq 0 \) and

\[
(\bar{u}_{i,n} - \tilde{u}_{i,n}) - (\bar{u}_{i,n-1} - \tilde{u}_{i,n-1}) = (1 - ak)^{i,n,k} [U_{i,n} - (1 - ak)U_{i,n-1}]
\]

\[
(\bar{v}_{i,n} - \tilde{v}_{i,n}) - (\bar{v}_{i,n-1} - \tilde{v}_{i,n-1}) = (1 - ak)^{i,n,k} [V_{i,n} - (1 - ak)V_{i,n-1}]
\]

By (2.4), (3.7) and \( k_n = k \), we have

\[
k_n^{-1} [ (\bar{u}_{i,n} - \tilde{u}_{i,n}) - (\bar{u}_{i,n-1} - \tilde{u}_{i,n-1}) ] - L^{i,n} [ (\bar{u}_{i,n} - \tilde{u}_{i,n}) ]
\]

\[
\leq (\bar{u}_{i,n} - \tilde{u}_{i,n}) - (\bar{u}_{i,n-1} - \tilde{u}_{i,n-1})\quad (4.6)
\]

and

\[
k_n^{-1} [ (\bar{v}_{i,n} - \tilde{v}_{i,n}) - (\bar{v}_{i,n-1} - \tilde{v}_{i,n-1}) ] - L^{i,n} [ (\bar{v}_{i,n} - \tilde{v}_{i,n}) ]
\]

\[
\leq (\bar{v}_{i,n} - \tilde{v}_{i,n}) - (\bar{v}_{i,n-1} - \tilde{v}_{i,n-1}) \quad (4.7)
\]

Multiplying (4.6) and (4.7) by \( (1 - ak)t_n/k \) and using (4.4) and (4.5), it gives

\[
k_n^{-1} (U_{i,n} - (1 - ak)U_{i,n-1}) - L^{i,n} [U_{i,n}] \leq (\bar{u}_{i,n} - \tilde{u}_{i,n}) + (\bar{u}_{i,n-1} - \tilde{u}_{i,n-1}) \quad (4.8)
\]

Thus

\[
B^{i,n} [U_{i,n}] \leq \sigma^{(11)} (\bar{u}_{i,n} - \tilde{u}_{i,n}) + \sigma^{(12)} (\bar{v}_{i,n} - \tilde{v}_{i,n}) \quad (4.10)
\]

On similar lines, we get

\[
B^{i,n} [V_{i,n}] \leq \sigma^{(11)} (\bar{u}_{i,n} + \bar{u}_{i,n}) + \sigma^{(12)} (\bar{v}_{i,n} + \bar{v}_{i,n}) \quad (4.11)
\]

Clearly

\[
U_{i,0} = (1 - ak)^{i,0,k} (\bar{u}_{i,0} - \tilde{u}_{i,0})
\]

\[
V_{i,0} = (1 - ak)^{i,0,k} (\bar{v}_{i,0} - \tilde{v}_{i,0})
\]

\[
= (1 - ak)^{i,0,k} (\bar{v}_{i,0} - \bar{v}_{i,0}) = (1 - ak)^{i,0,k} \psi_i = 0 \quad (4.12)
\]

Let \( (i', n') \) and \( (i'', n'') \) be in \( \mathbb{N}_p \), \( i \in \mathbb{N}_p \), \( n \in I, I, P, \) \( n \neq 0, n^* \neq 0 \) unless

\[
||U||_1 = ||V||_1 = 0
\]

Using \( n_1 \) as the initial time step and considering \( k_n = k \) for \( n = n_1 + 1, n_2 \) where \( n_1 + 1 \leq n_2 \leq N \)

The same reasoning gives

\[
(\bar{u}_{i,n}, \bar{v}_{i,n}) = (\bar{u}_{i,n}, \bar{v}_{i,n}) \quad (4.9)
\]

Continuing the same procedure leads to

\[
(\bar{u}_{i,n}, \bar{v}_{i,n}) = (\bar{u}_{i,n}, \bar{v}_{i,n}) \quad (4.10)
\]

Next to prove uniqueness, let \( (u^{i,n}_1, u^{i,n}_2) \) be any other solution of the discrete problem(2.4) – (2.6)

in \( S_{i,n} \), Lemma 3.3 implies that \( (\bar{u}_{i,n}, \bar{v}_{i,n}) \) in (3.6) are solutions of the...
discrete problem (2.4) − (2.6), then considering 
\((\bar{u}_{i,n}, \bar{v}_{i,n})\) and \((u_{i,n}^*, v_{i,n}^*)\) as pair of ordered upper- lower solutions, we have
\[(\bar{u}_{i,n}, \bar{v}_{i,n}) \geq (u_{i,n}^*, v_{i,n}^*)\]
Next considering \((u_{i,n}^*, v_{i,n}^*)\), \((\bar{u}_{i,n}, \bar{v}_{i,n})\) as pair of ordered upper- lower solutions, we have
\[(u_{i,n}^*, v_{i,n}^*) \geq (\bar{u}_{i,n}, \bar{v}_{i,n})\]
This shows that the solution \((u_{i,n}^*, v_{i,n}^*)\) in \(S_{i,n}\) satisfy
\[(u_{i,n}, v_{i,n}) \leq (u_{i,n}^*, v_{i,n}^*) \leq (\bar{u}_{i,n}, \bar{v}_{i,n})\]
Since \((u_{i,n}, v_{i,n}) = (\bar{u}_{i,n}, \bar{v}_{i,n})\) this implies that
\[(u_{i,n}^*, v_{i,n}^*) = (\bar{u}_{i,n}, \bar{v}_{i,n})\] This completes the proof.

5. Conclusions

Discrete initial boundary value problems are studied by applying the method of upper lower solutions. We developed the results of existence comparison and uniqueness of solutions. We conclude that these results also develop by the integro parabolic initial boundary value problems.

References