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Numerical Method for Weakly Coupled Nonlinear Parabolic System

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Abstract: ***The aim of the paper is to study a system of finite difference equations corresponding to the weakly coupled nonlinear parabolic system with nonlinear boundary conditions in a bounded domain . Method of upper – lower solutions and monotone iteration process are studied. We develop new iteration scheme. Discrete initial boundary value problems are studied by applying the method of upper lower solutions, existence comparison and uniqueness of solutions***

Keywords: Diffusion convection system, Initial boundary value Problem, Upper lower solution, Iteration scheme, Existence and uniqueness results

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1. Introduction

The finite difference method is one of the powerful and well developed numerical method employed in the study of differential equations. The stability and convergence of this method for parabolic differential equation is discussed in [3].the coupled system of partial differential equation are well studied by many researchers for both continuous problems [1,2,5] as well as discrete problems[1,4,5]. Pao [5,6] studied the weakly coupled system of reaction diffusion convection equation. We generalize the result recently obtained by Pao [6] for discrete problems for weakly coupled system with linear boundary conditions in two ways .on one hand new iteration scheme is developed which is similar to the iteration scheme for algebraic system and monotone convergent sequence are constructed by choosing suitable initial iteration which leads to existence-comparison as well as uniqueness results and on the other hand the results are obtained with nonlinear boundary conditions, under new iteration scheme.

The out line of the paper is as; The formulation of the system of finite difference equations corresponding to the weakly coupled system of parabolic equations with nonlinear boundary conditions by implicit method and the development of new iteration scheme for the construction of two monotone convergent sequences when both reaction and boundary functions are quasi-monotone non-decreasing . In the last existence-comparison as well as uniqueness of the solution of the system under consideration are studied.

2. Upper-Lower Solutions

Consider the following initial boundary value problem [IBVP] for weakly coupled Nonlinear parabolic system

$$u_t - L^{(1)}[u] = f^{(1)}(x,t,u,v) \text{ in } D_T \qquad (2.1)$$
$$v_t - L^{(2)}[v] = f^{(2)}(x,t,u,v) \text{ in } D_T$$

With nonlinear boundary conditions

$$\frac{\partial u}{\partial v} + B^{(1)}u = g^{(1)}(x,t,u,v) \text{ on } S_T$$
$$\frac{\partial u}{\partial v} + B^{[2]}u = g^{[2]}(x,t,u,v) \text{ on } S_T \qquad (2.2)$$

And initial conditions

$$u(x,0) = \psi^{(1)}(x) \text{ in } \Omega$$
$$u(x,0) = \psi^{[2]}(x) \text{ in } \Omega \; (2.3)$$

where Ω is a bounded domain in $R^P(p = 1,2,\dots)$ with boundary $\partial\Omega$ and

$$D_T = \Omega \times (0,T], S_T = \partial\,\Omega \times (0,T], T > 0$$
$$L^{(1)} \equiv D^{(l)}(x,t)\nabla^2 + b^{(l)}(x,t)\,.\nabla$$
$$D^{(l)}(x,t) > 0 \; on \; D_T \,, B^{(l)}(x) \geq on \; \partial\,\Omega \; for \; l = 1,2.$$

The function $f^{(l)}$, $g^{(l)}$, $\psi^{(l)}$, $l = 1,2$ are all Holder continues on their respective domains of their definitions.

Now we convert the continuous IBVP (2.1) – (2.3) into the discrete problems. Therefore, we introduce the following notations. Let $L = (i_1, i_2, \dots i_p)$ be a multiple index with $i_v = 0,1,2,\dots M_v + 1$ and (x_i, t_n) be an arbitrary mesh point in Λ_p where $x_i = (x_{i1}, x_{i2}, \dots x_{ip})$ be an arbitrary mesh point in $\Omega_p \, and$ M_v is the total number of interior mesh points in the x_{iv} co-ordinate direction. Denote by $\Omega_p, \overline{\Omega_p}, \partial\Omega_p, \Lambda_p$ and S_p the sets of mesh points in $\Omega, \overline{\Omega}, \partial\,\Omega, \Omega \times (0,T]$ and $\partial\,\Omega \times (0,T]$ respectively and $\overline{\Lambda_p}$ the set of all mesh points in $\overline{\Omega} \times (0,T]$ where $\overline{\Omega}$ is the Closure of Ω. Let (i,n)be used to represent the mesh point (x_i, t_n). Set

$$u_{i,n} \equiv u(x_i, t_n), v_{i,n} \equiv v(x_i, t_n),$$
$$f^{(l)}(u_{i,n}, v_{i,n}) \equiv f^{(l)}(x_i, t_n, u_{i,n}, v_{i,n})$$
$$D_{i,n}^{(l)} \equiv D^{(l)}(x_i, t_n), b_{i,n}^{(l)} \equiv b^{(l)}(x_i, t_n)$$
$$g^{(l)}(u_{i,n}, v_{i,n}) \equiv g^{(l)}(x_i, t_n, u_{i,n}, v_{i,n})$$
$$\psi_i^{(l)} \equiv \psi^{(l)}(x_i) \; l = 1,2. \, and$$
$$u_{i,0} \equiv u(x_{i,0}), v_{i,0} \equiv v(x_{i,0})$$

Let $k_n = t_n - t_{n-1}$ be the nth time increment for $n = 1,2,3 \dots N$ and h_v be the spatial increment in the $x_{i,v}$ coordinate direction. Let $e_v = (0,\dots,1\dots,0)$ be the unit vector in R^p , where constant 1 appears in the V[th] component and zero elsewhere.

The standard first and second order difference approximation are

$$u(x_i, t_n) = 2h_v^{-1}[u(x_i + h_v e_v, t_n) - u(x_i - h_v e_v, t_n)]$$

$$\Delta^{(v)} u\,(x_i, t_n) = h_v^{-2}[\,u\,(x_i + h_v e_v, t_n) - 2u\,(x_i, t_n) + u\,(x_i - h_v e_v, t_n)]$$

Respectively (for details see[1.6]. also note that $k_n^{-1}(u_{i,n} - u_{i,n-1})$ is the backward approximation for u_t.Using these notations, the continues IBVP(2.1)-(2.3) is transformed to following discrete initial boundary value problem

$$k_n^{-1}(u_{i,n} - u_{i,n-1}) - L^{(1)}[u_{i,n}] = f^{(1)}(u_{i,n}, v_{i,n}) (i,n)\, in\, \Lambda_p \quad (2.4)$$

$$k_n^{-1}(u_{i,n} - u_{i,n-1}) - L^{(2)}[u_{i,n}] = f^{(2)}(u_{i,n}, v_{i,n})$$

$$B^{(1)}[u_{i,n}] = g^{(1)}(u_{i,n}, v_{i,n})\,(i,n)\ in\ S_p \quad (2.5)$$

$$B^{(2)}[u_{i,n}] = g^{(2)}(u_{i,n}, v_{i,n})$$

$$u_{i,0} = \psi_i^{(1)}$$

$$i \in \Omega_p \quad (2.6)$$

$$v_{i,0} = \psi_i^{(2)}$$

Where for $l = 1,2$.

$$L^{(l)}[w_{i,n}] \equiv \sum_{v=1}^{p} (D_{i,n}^{(l)} \Delta^{(v)} w_{i,n} + b_{i,n}^{(l)} \delta_{i,n}^{(v)} w_{i,n}) \quad (2.7)$$

$$B^{(l)}[w_{i,n}] \equiv |x_i - \hat{x}_i|^{-1}[w(x_i, t_n) - w(\hat{x}_i, t_n)] + \beta^{(l)}(x_i,) w(x_i, t_n) \quad (2.8)$$

In the above boundary approximation (2.8) , $\hat{x}_i$ is suitable point in Ω_p and $|\,x_i - \hat{x}_i\,|$is the distance between x_i and $\hat{x}_i$.Here boundary surface is assumed to be parallel to the coordinate planes. Now we define quasimonotone nondecreasing functions.

Definition: A function $(f^{(1)}, f^{(2)})$,is said to be quasi monotonic decreasing in $J \subset R^2$ if

$$\frac{\partial}{\partial v} f^{(1)} \geq 0\,, \frac{\partial}{\partial u} f^{(2)} \geq 0, for\ (u,v) \in J$$

Note the Quasimonotone nondecreasing property of boundary function $(g^{(1)}, g^{(2)})$, can be defined in the same way.

The definition of upper- lower solutions for the coupled finite difference system (2.4)-(2.6) depends on both the quasimonotone nondecreasing property of reaction function $(f^{(1)}, f^{(2)})$ and boundary function $(g^{(1)}, g^{(2)})$ in a sector defined below.

Definition 2.2: Let $(\tilde{u}_{i,n}, \tilde{v}_{i,n})$ and $(\hat{u}_{i,n}, \hat{v}_{i,n})$ be any two function in $\overline{\Lambda_p}$ with $(\tilde{u}_{i,n}, \tilde{v}_{i,n}) \geq (\hat{u}_{i,n}, \hat{v}_{i,n})$ then we define the sector $S_{i,n}$ as $S_{i,n} = \{(u_{i,n}, v_{i,n}) \in \overline{\Lambda_p}$: *:ui,n ,vi,n≤ui,n,vi,n≤ ui,n ,vi,n*

Where inequalities for vector functionare both component wise as well as point wise.

Definition 2.3: Two functions $(\tilde{u}_{i,n}, \tilde{v}_{i,n})$, $(\hat{u}_{i,n}, \hat{v}_{i,n})$ in $\overline{\Lambda_p}$ with $(\tilde{u}_{i,n}, \tilde{v}_{i,n}) \geq (\hat{u}_{i,n}, \hat{v}_{i,n})$ are called ordered upper – lower solutions for the system (2.4) - (2.6) if they satisfying the following inequalities

$$k_n^{-1}(\tilde{u}_{i,n} - \tilde{u}_{i,n-1}) - L^{(1)}[\tilde{u}_{i,n}] \geq f^{(1)}(\tilde{u}_{i,n}, \tilde{v}_{i,n})$$

$$k_n^{-1}(\tilde{v}_{i,n} - \tilde{v}_{i,n-1}) - L^{(2)}[\tilde{v}_{i,n}] \geq f^{(2)}(\tilde{u}_{i,n}, \tilde{v}_{i,n})$$

$$k_n^{-1}(\hat{u}_{i,n} - \hat{u}_{i,n-1}) - L^{(1)}[\hat{u}_{i,n}] \leq f^{(1)}(\hat{u}_{i,n}, \hat{v}_{i,n})\ (i,n) \in \Lambda_p$$

$$k_n^{-1}(\hat{v}_{i,n} - \hat{v}_{i,n-1}) - L^{(2)}[\hat{u}_{i,n}] \leq f^{(2)}(\hat{u}_{i,n}, \hat{v}_{i,n}) \quad (2.9)$$

$$B^{(1)}[\tilde{u}_{i,n}] \geq g^{(1)}(\tilde{u}_{i,n}, \tilde{v}_{i,n})$$

$$B^{(2)}[\tilde{v}_{i,n}] = g^{(2)}(\tilde{u}_{i,n}, \tilde{v}_{i,n})$$

$$B^{(1)}[\hat{u}_{i,n}] \leq g^{(1)}(\hat{u}_{i,n}, \hat{v}_{i,n})\ (i,n) \in S_p$$

$$B^{(2)}[\hat{v}_{i,n}] = g^{(2)}(\hat{u}_{i,n}, \hat{v}_{i,n}) \quad (2.10)$$

$$\tilde{u}_{i,0} \geq \psi_i^{(1)} \geq \hat{u}_{i,0}\ i \in \Omega_p$$

$$\tilde{v}_{i,0} \geq \psi_i^{(2)} \geq \hat{v}_{i,0}$$

3. Monotone Iteration Scheme

We concider $(f^{(1)}, f^{(2)})$ and $(g^{(1)}, g^{(2)})$, as quasimonotone nondecreasing function in the sector $S_{i,n}$.therefore ,there exixts nonnegative function$\gamma_{i,n}^{(1)}, \gamma_{i,n}^{(2)}, \sigma_{i,n}^{(1)}$ *and* $\sigma_{i,n}^{(2)}$ such that for any pair $(u_{i,n}, v_{i,n})$ *and* $(u'_{i,n}, v'_{i,n})$ in the sector $S_{i,n}$ formed from the ordered upper-lower solutions, the functions $f^{(l)}$, $g^{(l)}, l = 1,2$. satisfy the following one side Lipchitz condition, which ensure the exixtance of solutions of the discrete IBVP (2.4) –(2.6).

$$f^{(1)}(u_{i,n}, v_{i,n}) - f^{(1)}(u'_{i,n}, v_{i,n}) \geq -\gamma_{i,n}^{(1)}(u_{i,n} - u'_{i,n})\ when\ u_{i,n} \geq u'_{i,n}$$

$$f^{(2)}(u_{i,n}, v_{i,n}) - f^{(2)}(u_{i,n}, v'_{i,n}) \geq -\gamma_{i,n}^{(2)}(v_{i,n} - v'_{i,n})\ when\ v_{i,n} \geq v'_{i,n}$$

$$g^{(1)}(u_{i,n}, v_{i,n}) - g^{(1)}(u'_{i,n}, v_{i,n}) \geq -\sigma_{i,n}^{(1)}(u_{i,n} - u'_{i,n})\ when\ u_{i,n} \geq u'_{i,n}$$

$g^{(2)}(u_{i,n}, v_{i,n}) - g^{(2)}(u_{i,n}, v'_{i,n}) \geq -\sigma_{i,n}^{(2)}(v_{i,n} -$ *v′i,n when vi,n≥ v′i,n* (3.1)

$$Let\ F^{(1)}(u_{i,n}, v_{i,n}) = \gamma_{i,n}^{(1)} u_{i,n} + f^{(1)}(u_{i,n}, v_{i,n})$$

$$F^{(2)}(u_{i,n}, v_{i,n}) = \gamma_{i,n}^{(2)} v_{i,n} + f^{(2)}(u_{i,n}, v_{i,n}) \quad (3.2)$$

$$G^{(1)}(u_{i,n}, v_{i,n}) = \sigma_{i,n}^{(1)} u_{i,n} + f^{(1)}(u_{i,n}, v_{i,n})$$

$$G^{(2)}(u_{i,n}, v_{i,n}) = \sigma_{i,n}^{(2)} v_{i,n} + g^{(1)}(u_{i,n}, v_{i,n})$$

Now we start from the suitable initial iteration $(u_{i,n}^{(0)}, v_{i,n}^{(0)})$ as either $(\tilde{u}_{i,n}, \tilde{v}_{i,n})$ or $(\hat{u}_{i,n}, \hat{v}_{i,n})$ and construct a sequence $\{u_{i,n}^{(m)}, v_{i,n}^{(m)}\}$ from the following iteration process.

Iteration process

$$k_n^{-1}(u_{i,n}^{(m)} - u_{i,n-1}^{(m)}) - L^{(1)}[u_{i,n}^{(m)}] + \gamma_{i,n}^{(1)} u_{i,n}^{(m)} = F^{(1)}(u_{i,n}^{(m-1)}, v_{i,n}^{(m-1)}) \quad (3.3)$$

$$B^{(1)}[u_{i,n}^{(m)}] + \sigma_{i,n}^{(1)} u_{i,n}^{(m)} = G^{(1)}(u_{i,n}^{(m-1)}, v_{i,n}^{(m-1)})$$

$$u_{i,0}^{(m)} = \psi_i^{(1)}$$

$$k_n^{-1}(v_{i,n}^{(m)} - v_{i,n-1}^{(m)}) - L^{(2)}[v_{i,n}^{(m)}] + \gamma_{i,n}^{(2)} v_{i,n}^{(m)} = G^{(1)}(u_{i,n}^{(m)}, v_{i,n}^{(m-1)})$$

$$B^{(2)}[v_{i,n}^{(m)}] + \sigma_{i,n}^{(2)} v_{i,n}^{(m)} = G^{(2)}(u_{i,n}^{(m-1)}, v_{i,n}^{(m-1)})$$

$$v_{i,0}^{(m)} = \psi_i^{(2)}$$

Where $m = 1,2$ ...

For each $m = 1,2$ ... the above system consist of two linear finite difference parabolic problems. So the sequence constructed from the above iteration process are well defined. Define

$$\mathcal{L}^{(1)}[u_{i,n}] \equiv k_n^{-1}(u_{i,n} - u_{i,n-1}) - L^{(1)}[u_{i,n}] + \gamma_{i,n}^{(1)} u_{i,n}$$

$$\mathcal{L}^{(2)}[\, u_{i,n}\,] \equiv\; k_n^{-1}(u_{i,n} - u_{i,n-1}) - L^{(2)}[u_{i,n}] +\; \gamma_{i,n}^{(2)} u_{i,n}$$
$$\mathfrak{B}^{(1)}[\, u_{i,n}\,] \equiv B^{(1)}[\, u_{i,n}^{(m)}] +\; \sigma_{i,n}^{(1)}\; u_{i,n}^{(m)}$$
$$\mathfrak{B}^{(2)}[\, v_{i,n}\,] \equiv B^{(2)}[\, v_{i,n}^{(m)}] +\; \sigma_{i,n}^{(2)}\; v_{i,n}^{(m)}$$

We choose $(\tilde{u}_{i,n}\,, \tilde{v}_{i,n}\,)$ and $(\hat{u}_{i,n}, \hat{v}_{i,n})$ as initial iteration and obtain the sequences of iteration by splitting the above iteration process in to the following sub- iteration process $A_1 - A_4$.

Sub – iteration process A_1

$$\mathcal{L}^{(1)}[\; \bar{u}_{i,n}^{(m)}] = \gamma_{i,n}^{(1)}\; \bar{u}_{i,n}^{(m-1)} + f^{(1)}(\; \bar{u}_{i,n}^{(m-1)}, \bar{v}_{i,n}^{(m-1)})$$
$$\mathfrak{B}^{(1)}[\; \bar{u}_{i,n}^{(m)}\,] =\; \sigma_{i,n}^{(1)}\; \bar{u}_{i,n}^{(m-1)} + f^{(1)}(\; \bar{u}_{i,n}^{(m-1)}, \bar{v}_{i,n}^{(m-1)})$$
$$\bar{u}_{i,0}^{(m)} \;= \psi_i^{(1)} \text{ Where } m = 1,2\ldots$$

Sub – iteration process A_2.

$$\mathcal{L}^{(2)}[\; \bar{v}_{i,n}^{(m)}] = \gamma_{i,n}^{(2)}\; \bar{u}_{i,n}^{(m-1)} + f^{(2)}(\; \bar{u}_{i,n}^{(m-1)}, \bar{v}_{i,n}^{(m-1)})$$
$$\mathfrak{B}^{(2)}[\, \bar{v}_{i,n}^{(m)}\,] =\; \sigma_{i,n}^{(2)} \bar{v}_{i,n}^{(m-1)} + f^{(2)}(\; \bar{u}_{i,n}^{(m)}, \bar{v}_{i,n}^{(m-1)}) \quad (3.4)$$
$$\bar{v}_{i,0}^{(m)} \;= \psi_i^{(2)} \text{ Where } m = 1,2\ldots$$

Sub – iteration process A_3.

$$\mathcal{L}^{(1)}[\, \underline{u}_{i,n}^{(m)}] = \gamma_{i,n}^{(1)}\; \underline{u}_{i,n}^{(m-1)} + f^{(1)}(\, \underline{u}_{i,n}^{(m-1)}, \underline{v}_{i,n}^{(m-1)})$$
$$\mathfrak{B}^{(1)}[\, \underline{u}_{i,n}^{(m)}\,] =\; \sigma_{i,n}^{(1)} \underline{u}_{i,n}^{(m-1)} + f^{(1)}(\, \underline{u}_{i,n}^{(m-1)}, \underline{v}_{i,n}^{(m-1)})$$
$$\underline{u}_{i,0}^{(m)} \;= \psi_i^{(1)} \text{ Where } m = 1,2\ldots$$

Sub – iteration process A_4.

$$\mathcal{L}^{(2)}[\, \underline{v}_{i,n}^{(m)}] = \gamma_{i,n}^{(2)}\; \underline{v}_{i,n}^{(m-1)} + f^{(2)}(\, \underline{u}_{i,n}^{(m-1)}, \underline{v}_{i,n}^{(m-1)})$$
$$\mathfrak{B}^{(2)}[\, \underline{v}_{i,n}^{(m)}\,] =\; \sigma_{i,n}^{(2)} \underline{v}_{i,n}^{(m-1)} + f^{(2)}(\, \underline{u}_{i,n}^{(m)}, \underline{v}_{i,n}^{(m-1)})$$
$$\underline{u}_{i,0}^{(m)} \;= \psi_i^{(2)} \text{ Where } m = 1,2\ldots$$

We start$(\; \bar{u}_{i,n}^{(0)}, \; \bar{u}_{i,n}^{(0)}) = (\tilde{u}_{i,n}\,, \tilde{v}_{i,n}\,)$ with as initial iteration in the sub-iteration process A_1. for $m = 1$ and obtain $\bar{u}_{i,n}^{(1)}$ then we use this value in the sub- iteration process A_2. for $m = 1$ and obtain $\bar{v}_{i,n}^{(1)}$.in this way we obtain the first iteration $(\bar{u}_{i,n}^{(1)}, \; \bar{v}_{i,n}^{(1)})$. Then using $(\; \bar{u}_{i,n}^{(1)}, \; \bar{v}_{i,n}^{(1)})$ as initial iteration we find the next iteration $(\; \bar{u}_{i,n}^{(2)}, \; \bar{v}_{i,n}^{(2)})$ as above from the sub-iteration process A_1 and A_2 $m = 2$ and so on.We denote the sequence of these iteration by $\{\; \bar{u}_{i,n}^{(m)}, \; \bar{v}_{i,n}^{(m)}\}$.

Now we start with $(\; \bar{u}_{i,n}^{(0)}, \; \bar{v}_{i,n}^{(0)}) = (\hat{u}_{i,n}, \hat{v}_{i,n})$ as initial iteration in the sub –iteration process A_3 for $m = 1$ and obtain$\underline{u}_{i,n}^{(1)}$thus we obtain the first iteration$(\underline{u}_{i,n}^{(1)}, \underline{v}_{i,n}^{(1)})$ .using this iteratin as initial iteatration we obtain the next iteration$(\underline{u}_{i,n}^{(2)}, \underline{v}_{i,n}^{(2)})$ as above from sub – iteration process A_3 and A_4 for $m = 2$ and so on. We denote the sequence of these iteration by$\{\; \bar{u}_{i,n}^{(m)}, \; \bar{v}_{i,n}^{(m)}\}$.

To prove monotone property of these sequence we required the monotone property of the functions $F^{(l)}, G^{(l)}, l = 1{,}2$.as well as positivity result . The monotone property and positivity results are proved in [3,8] and [6], we state without proof these results as lemma 3.1 and lemma 3.2 respectively.

LEMMA 3.1 suppose $(u_{i,n}, v_{i,n})$ and $(u'_{i,n}, v'_{i,n})$ are two functions in $S_{i,n}$such that $(u_{i,n}, v_{i,n}) \geq (u'_{i,n}, v'_{i,n})$ and suppose that Lipchitz condition (3.1)

Holds, if $(\, f^{(1)}, f^{(2)})$ and $(\, g^{(1)}, g^{(2)})$are quasimonotone non decreasing in $S_{i,n}$ then

$$F^{(1)}(u_{i,n}, v_{i,n}) \geq F^{(1)}(u'_{i,n}, v'_{i,n})$$
$$F^{(2)}(u_{i,n}, v_{i,n}) \geq F^{(2)}(u'_{i,n}, v'_{i,n})$$
$$G^{(1)}(u_{i,n}, v_{i,n}) \geq G^{(1)}(u'_{i,n}, v'_{i,n})$$
$$G^{(2)}(u_{i,n}, v_{i,n}) \geq G^{(2)}(u'_{i,n}, v'_{i,n})$$

Lemma 3.2 let $w_{i,n}$ be function defined in $\overline{\Lambda_p}$ such that

$$L[w_{i,n}] +\; C_{i,n} w_{i,n} \geq 0 \;(i,n) \;\in\; \Lambda_p$$
$$B[w_{i,n}] \geq 0(i,n) \;\in\; S_p$$
$$w_{i,n} \geq 0 \; i \in\; \Lambda_p$$

Where $C_{i,n} \geq 0$. then $w_{i,n} \geq 0\,, i \in\; \overline{\Lambda_p}$.

Lemma 3.3 let $(\, f^{(1)}, f^{(2)})$ and $(\, g^{(1)}, g^{(2)})$be quasimonotone nondecreasing $C^1 - \mathit{functions}$ in $S_{i,n}$.
Then the sequence $\{\; \bar{u}_{i,n}^{(m)}, \; \bar{v}_{i,n}^{(m)}\}$ and $\{\underline{u}_{i,n}^{(m)}, \underline{v}_{i,n}^{(m)}\}$obtained from the iteration process (3.3) with initial iterations$(\; \bar{u}_{i,n}^{(m)}, \; \bar{u}_{i,n}^{(m)}) = (\tilde{u}_{i,n}\,, \tilde{v}_{i,n}\,)$ and $(\; \underline{u}_{i,n}^{(m)}, \underline{v}_{i,n}^{(m)}) = (\hat{u}_{i,n}, \hat{v}_{i,n})$ respctively posses the monotone property

$$(\; \underline{u}_{i,n}^{(m-1)}, \underline{v}_{i,n}^{(m-1)}) \leq (\; \underline{u}_{i,n}^{(m)}, \underline{v}_{i,n}^{(m)}) \leq (\; \bar{u}_{i,n}^{(m)}, \; \bar{u}_{i,n}^{(m)})$$
$$\leq (\; \bar{u}_{i,n}^{(m-1)}, \bar{v}_{i,n}^{(m-1)})\,, (i,n) \in \overline{\Lambda_p} \quad (3.5)$$

Where $m = \; 1,2, \ldots$

Proof .

Let $w_{i,n} \;=\; \bar{u}_{i,n}^{(0)} \;-\; \bar{u}_{i,n}^{(1)} = \tilde{u}_{i,n} \;-\; \bar{u}_{i,n}^{(1)}$

$z_{i,n} \;=\; \bar{v}_{i,n}^{(0)} \;-\; \bar{v}_{i,n}^{(1)} = \tilde{v}_{i,n} \;-\; \bar{v}_{i,n}^{(1)}$

Then by (2.9),(2.10),(3.2), iteration process (3.3 and (3.4))

$$\mathcal{L}^{(1)}[\, w_{i,n}\,] = \mathcal{L}^{(1)}[\; \tilde{u}_{i,n}] - F^{(1)}(\; \bar{u}_{i,n}^{(0)}, \; \bar{v}_{i,n}^{(0)})$$
$$= k_n^{-1}(\tilde{u}_{i,n} - \tilde{u}_{i,n-1}) - L^{(1)}[\tilde{u}_{i,n}] - f^{(1)}(\; \bar{u}_{i,n}^{(0)}, \; \bar{v}_{i,n}^{(0)}) \geq 0$$

$$\mathcal{B}^{(1)}[\, w_{i,n}\,] = \mathcal{B}^{(1)}[\; \tilde{u}_{i,n}] - G^{(1)}(\; \bar{u}_{i,n}^{(0)}, \; \bar{v}_{i,n}^{(0)})$$
$$= B^{(1)}[\; \tilde{u}_{i,n}] - g^{(1)}(\; \bar{u}_{i,n}^{(0)}, \; \bar{v}_{i,n}^{(0)}) \geq 0$$

By (2.11) and iteration process (3.3)

$$w_{i,0} \;=\; \bar{u}_{i,n}^{(0)} \;-\; \bar{u}_{i,n}^{(1)} \;= \tilde{u}_{i,0} - \; \psi_i^{(1)} \geq 0$$

By lemma 3.2, $w_{i,n} \geq 0$. This gives $\; \bar{u}_{i,n}^{(0)} \;\geq\; \bar{u}_{i,n}^{(1)}$.

Now by (2.9),(2.10),(3.2),iteration process (3.3) and (3.4),

$$\mathcal{L}^{(2)}[\, z_{i,n}\,] = \mathcal{L}^{(2)}[\; \tilde{v}_{i,n}] - F^{(2)}(\; \bar{u}_{i,n}^{(1)}, \; \bar{v}_{i,n}^{(0)})$$
$$= k_n^{-1}(\tilde{v}_{i,n} - \tilde{v}_{i,n-1}) - L^{(2)}[\tilde{v}_{i,n}] - f^{(2)}(\; \bar{u}_{i,n}^{(0)}, \; \bar{v}_{i,n}^{(0)})$$
$$= k_n^{-1}(\tilde{v}_{i,n} - \tilde{v}_{i,n-1}) - L^{(2)}[\tilde{v}_{i,n}] - f^{(2)}(\; \bar{u}_{i,n}^{(0)}, \; \bar{v}_{i,n}^{(0)})$$
$$+ f^{(2)}(\; \bar{u}_{i,n}^{(0)}, \; \bar{v}_{i,n}^{(0)}) - f^{(2)}(\; \bar{u}_{i,n}^{(0)}, \; \bar{v}_{i,n}^{(0)})$$
$$\geq f^{(2)}(\; \bar{u}_{i,n}^{(0)}, \; \bar{v}_{i,n}^{(0)}) - f^{(2)}(\; \bar{u}_{i,n}^{(0)}, \; \bar{v}_{i,n}^{(0)})$$
$$\geq 0 \;(\because f^{(2)} \; \mathit{is\ quasimonotone\ decreasing\ and}\; \bar{u}_{i,n}^{(0)} \geq \; \bar{u}_{i,n}^{(1)}\,)$$

$$\mathcal{B}^{(2)}[\, z_{i,n}\,] = \mathcal{B}^{(2)}[\; \tilde{v}_{i,n}] - G^{(2)}(\; \bar{u}_{i,n}^{(1)}, \; \bar{v}_{i,n}^{(0)})$$
$$= B^{(2)}[\; \tilde{v}_{i,n}] - g^{(2)}(\; \bar{u}_{i,n}^{(1)}, \; \bar{v}_{i,n}^{(0)})$$
$$= B^{(2)}[\; \tilde{v}_{i,n}] - g^{(2)}(\; \bar{u}_{i,n}^{(1)}, \; \bar{v}_{i,n}^{(0)}) + g^{(2)}(\; \bar{u}_{i,n}^{(1)}, \; \bar{v}_{i,n}^{(0)})$$
$$- \; g^{(2)}(\; \bar{u}_{i,n}^{(1)}, \; \bar{v}_{i,n}^{(0)})$$

$$\geq g^{(2)}(\bar{u}_{i,n}^{(0)},\ \bar{v}_{i,n}^{(0)}) - g^{(2)}(\bar{u}_{i,n}^{(1)},\ \bar{v}_{i,n}^{(0)})$$
$$\geq 0\ (\because g^{(2)}\ is\ quasimonotone\ decreasing\ and\ \bar{u}_{i,n}^{(0)} \geq \bar{u}_{i,n}^{(1)})$$

By (2.11) and iteration process (3.3)

$$z_{i,0} = \bar{v}_{i,n}^{(0)} - \bar{v}_{i,n}^{(1)} = \tilde{v}_{i,0} - \psi_i^{(2)} \geq 0$$

It follows from lemma 3.2 that $z_{i,n} \geq 0\ in\ \overline{\Lambda_p}, which\ gives\ \bar{v}_{i,n}^{(0)} \geq \bar{v}_{i,n}^{(1)}$. thus we get $(\bar{u}_{i,n}^{(1)},\ \bar{v}_{i,n}^{(1)}) \leq (\bar{u}_{i,n}^{(0)},\ \bar{v}_{i,n}^{(0)})$

Similarly letting $w_{i,n} = \underline{u}_{i,n}^{(0)} - \underline{u}_{i,n}^{(0)} = \underline{u}_{i,n}^{(1)} - \hat{u}_{i,n}$

$$z_{i,n} = \underline{v}_{i,n}^{(0)} - \underline{v}_{i,n}^{(0)} = \underline{v}_{i,n}^{(1)} - \hat{v}_{i,n}$$

We obtain $(\underline{u}_{i,n}^{(1)}, \underline{v}_{i,n}^{(1)}) \geq (\underline{u}_{i,n}^{(0)}, \underline{v}_{i,n}^{(0)})$.

Now let

$$w_{i,n}^{(1)} = \bar{u}_{i,n}^{(1)} - \underline{u}_{i,n}^{(1)}$$
$$z_{i,n}^{(1)} = \bar{v}_{i,n}^{(1)} - \underline{v}_{i,n}^{(1)}$$

Then by iteration process (3.3) and lemma 3.1

$$\mathcal{L}^{(1)}[\,w_{i,n}^{(1)}\,] = \mathcal{L}^{(1)}[\,\bar{u}_{i,n}^{(1)}] - \mathcal{L}^{(1)}[\,\underline{u}_{i,n}^{(1)}]$$
$$= F^{(1)}(\bar{u}_{i,n}^{(0)},\ \bar{v}_{i,n}^{(0)}) - F^{(1)}(\underline{u}_{i,n}^{(1)}, \underline{v}_{i,n}^{(1)}) \geq 0$$
$$\mathcal{B}^{(1)}[\,w_{i,n}^{(1)}\,] = \mathcal{B}^{(1)}[\,\bar{u}_{i,n}^{(1)}] - \mathcal{B}^{(1)}[\,\underline{u}_{i,n}^{(1)}]$$
$$= G^{(1)}(\bar{u}_{i,n}^{(0)},\ \bar{v}_{i,n}^{(0)}) - G^{(1)}(\underline{u}_{i,n}^{(1)}, \underline{v}_{i,n}^{(1)}) \geq 0$$

By iteration process (3.3) $w_{i,0}^{(1)} = \bar{u}_{i,0}^{(1)} - \underline{u}_{i,0}^{(1)} = \psi_i^{(1)} - \psi_i^{(1)} = 0$

From lemma 3.2, it shows that

$$w_{i,n}^{(1)} \geq 0\ on\ \overline{\Lambda_p}. It\ leads\ to$$
$$\bar{u}_{i,n}^{(1)} \geq \underline{u}_{i,n}^{(1)}$$

Again by iteration process (3.3) and lemma 3.1,

$$\mathcal{L}^{(2)}[\,z_{i,n}^{(1)}\,] = \mathcal{L}^{(2)}[\,\bar{v}_{i,n}^{(1)}] - \mathcal{L}^{(2)}[\,\underline{v}_{i,n}^{(1)}]$$
$$= F^{(2)}(\bar{u}_{i,n}^{(1)},\ \bar{v}_{i,n}^{(0)}) - F^{(2)}(\underline{u}_{i,n}^{(1)}, \underline{v}_{i,n}^{(0)}) \geq 0$$
$$\mathcal{B}^{(2)}[\,z_{i,n}^{(1)}\,] = \mathcal{B}^{(2)}[\,\bar{v}_{i,n}^{(1)}] - \mathcal{B}^{(2)}[\,\underline{u}_{i,n}^{(1)}]$$
$$= G^{(2)}(\bar{u}_{i,n}^{(1)},\ \bar{v}_{i,n}^{(0)}) - G^{(2)}(\underline{u}_{i,n}^{(1)}, \underline{v}_{i,n}^{(1)}) \geq 0$$

By iteration process (3.3)

$$z_{i,0}^{(1)} = \bar{v}_{i,0}^{(1)} - \underline{v}_{i,0}^{(1)} = \psi_i^{(2)} - \psi_i^{(2)} = 0$$

Using Lemma 3.2we get $z_{i,n}^{(1)} \geq 0\ on\ \overline{\Lambda_p}$,which gives $\bar{v}_{i,n}^{(1)} \geq \underline{u}_{i,n}^{(1)}$

Thus we get $(\bar{u}_{i,n}^{(1)},\ \bar{v}_{i,n}^{(1)}) \geq (\underline{u}_{i,n}^{(1)}, \underline{v}_{i,n}^{(1)})$

The above conclusion show that $(\underline{u}_{i,n}^{(0)}, \underline{v}_{i,n}^{(0)}) \leq (\underline{u}_{i,n}^{(1)}, \underline{v}_{i,n}^{(1)}) \leq (\bar{u}_{i,n}^{(1)},\ \bar{v}_{i,n}^{(1)}) \leq (\bar{u}_{i,n}^{(0)},\ \bar{v}_{i,n}^{(0)})$

Assume that (3.5) holds for some integer m .suppose that $w_{i,n}^{(m)} = \bar{u}_{i,n}^{(m)} - \underline{u}_{i,n}^{(m+1)}$ and $z_{i,n}^{(m)} = \bar{v}_{i,n}^{(m)} - \underline{v}_{i,n}^{(m+1)}$.

Then by iteration process (3.3) and lemma 3.1

$$\mathcal{L}^{(1)}[\,w_{i,n}^{(m)}\,] = F^{(1)}(\bar{u}_{i,n}^{(m-1)},\ \bar{v}_{i,n}^{(m-1)}) - F^{(1)}(\underline{u}_{i,n}^{(m)}, \underline{v}_{i,n}^{(m)}) \geq 0$$
$$\mathcal{B}^{(1)}[\,w_{i,n}^{(m)}\,] = G^{(1)}(\bar{u}_{i,n}^{(m-1)},\ \bar{v}_{i,n}^{(m-1)}) - G^{(1)}(\underline{u}_{i,n}^{(m)}, \underline{v}_{i,n}^{(m)}) \geq 0$$
$$w_{i,0}^{(m)} = \bar{u}_{i,0}^{m} - \underline{u}_{i,0}^{(m-1)} = \psi_i^{(1)} - \psi_i^{(1)} = 0$$

By lemma 3.2, it leads ti the conclusion $w_{i,0}^{(m)} \geq 0,\ \bar{u}_{i,n}^{(m)} \geq \bar{u}_{i,n}^{(m+1)}$

$$\mathcal{L}^{(2)}[\,z_{i,n}^{(m)}\,] = F^{(2)}(\bar{u}_{i,n}^{(m)},\ \bar{v}_{i,n}^{(m-1)}) - F^{(2)}(\bar{u}_{i,n}^{(m+1)},\ \bar{v}_{i,n}^{(m)}) \geq 0$$
$$\mathcal{B}^{(2)}[\,z_{i,n}^{(m)}\,] = G^{(2)}(\bar{u}_{i,n}^{(m)},\ \bar{v}_{i,n}^{(m-1)}) - G^{(2)}(\bar{u}_{i,n}^{(m+1)},\ \bar{v}_{i,n}^{(m)}) \geq 0$$
$$z_{i,0}^{(m)} = \bar{u}_{i,0}^{(m)} - \bar{u}_{i,0}^{(m-1)} = \psi_i^{(2)} - \psi_i^{(2)} = 0$$

Again by Lemma 3.2, $z_{i,n}^{(m)} \geq 0\ i.e.\ \bar{v}_{i,n}^{(m)} \geq \bar{v}_{i,n}^{(m+1)}$

So $(\bar{u}_{i,n}^{(m+1)},\ \bar{v}_{i,n}^{(m+1)}) \leq (\bar{u}_{i,n}^{(m)},\ \bar{v}_{i,n}^{(m)})$

A similar argument gives $(\underline{u}_{i,n}^{(m+1)}, \underline{v}_{i,n}^{(m+1)}) \geq (\underline{u}_{i,n}^{(m)}, \underline{v}_{i,n}^{(m)})$

By induction principal the result (3.5) follows for all m. this complete the proof. The results in Lemma 3.3 shows that

$$lim(\bar{u}_{i,n}^{(m)},\ \bar{v}_{i,n}^{(m)}) = (\bar{u}_{i,n}, \bar{u}_{i,n}) \quad as \quad m \to \infty, \qquad (3.6)$$
$$lim(\underline{u}_{i,n}^{(m)}, \underline{v}_{i,n}^{(m)}) = (\underline{u}_{i,n}, \underline{v}_{i,n})\ as\ m \to \infty,$$

Exits in $\overline{\Lambda_p}$ .Taking $m \to \infty$ in the iteration process (3.3) implies that both $(\bar{u}_{i,n}, \bar{u}_{i,n})$

And $(\underline{u}_{i,n}, \underline{v}_{i,n})$. Are solutions of the discrete IBVP(2.4) – (2.6). These two solutions may not be equal. They are equal if for $(u_{i,n}, v_{i,n}), (u'_{i,n}, v'_{i,n})$ in the sector $S_{i,n}$ with $(u_{i,n}, v_{i,n}) \geq (u'_{i,n}, v'_{i,n})$; functions $(f^{(1)}, f^{(2)})$ and $(g^{(1)}, g^{(2)})$satisfy the folloeing one sides Lipchitz conditions:

$$f^{(1)}(u_{i,n}, v_{i,n}) - f^{(1)}(u'_{i,n}, v_{i,n}) \leq \gamma_{i,n}^{(11)}(u_{i,n} - u'_{i,n}) + \gamma_{i,n}^{(11)}(u_{i,n} - u'_{i,n})$$
$$f^{(2)}(u_{i,n}, v_{i,n}) - f^{(2)}(u_{i,n}, v'_{i,n}) \leq \gamma_{i,n}^{(21)}(u_{i,n} - u'_{i,n}) + \gamma_{i,n}^{(22)}(u_{i,n} - u'_{i,n})$$

$$g^{(1)}(u_{i,n}, v_{i,n}) - g^{(1)}(u'_{i,n}, v_{i,n}) \leq \sigma_{i,n}^{(11)}(u_{i,n} - u'_{i,n}) + \sigma_{i,n}^{(12)}(u_{i,n} - u'_{i,n})$$
$$g^{(2)}(u_{i,n}, v_{i,n}) - g^{(2)}(u_{i,n}, v'_{i,n}) \leq \sigma_{i,n}^{(21)}(u_{i,n} - u'_{i,n}) + \sigma_{i,n}^{(22)}(u_{i,n} - u'_{i,n})$$

Where the function $\gamma_{i,n}^{(11)},\ \gamma_{i,n}^{(21)},\ \sigma_{i,n}^{(11)},\ \sigma_{i,n}^{(21)}$ for $l = 1{,}2$ are given by

$$\gamma^{(11)} = max\left\{\frac{\partial}{\partial u}f^{(1)}\,; (i,n) \in \overline{\Lambda_p}, (u_{i,n}, v_{i,n}) \in S_{i,n}\right\}$$
$$\gamma^{(12)} = max\left\{\left|\frac{\partial}{\partial v}f^{(1)}\right|; (i,n) \in \overline{\Lambda_p}, (u_{i,n}, v_{i,n}) \in S_{i,n}\right\}$$
$$\gamma^{(21)} = max\left\{\left|\frac{\partial}{\partial u}f^{(2)}\right|; (i,n) \in \overline{\Lambda_p}, (u_{i,n}, v_{i,n}) \in S_{i,n}\right\}$$
$$\gamma^{(22)} = max\left\{\frac{\partial}{\partial v}f^{(2)}\,; (i,n) \in \overline{\Lambda_p}, (u_{i,n}, v_{i,n}) \in S_{i,n}\right\}$$
$$\sigma^{(11)} = max\left\{\frac{\partial}{\partial u}g^{(1)}\,; (i,n) \in S_p\,, (u_{i,n}, v_{i,n}) \in S_{i,n}\right\}$$
$$\sigma^{(12)} = max\left\{\left|\frac{\partial}{\partial v}g^{(1)}\right|; (i,n) \in S_p\,, (u_{i,n}, v_{i,n}) \in S_{i,n}\right\}$$
$$\sigma^{(21)} = max\left\{\left|\frac{\partial}{\partial u}g^{(2)}\right|; (i,n) \in S_p\,, (u_{i,n}, v_{i,n}) \in S_{i,n}\right\}$$
$$\sigma^{(22)} = max\left\{\frac{\partial}{\partial u}g^{(2)}\,; (i,n) \in S_p\,, (u_{i,n}, v_{i,n}) \in S_{i,n}\right\}$$

From (3.8) it is clear that $\gamma^{(12)} \geq 0, \gamma^{(12)} \geq 0, \sigma^{(12)} \geq 0, \sigma^{(21)} \geq 0.$

The conditions in(3.1) and (3.7) leads to the following results.

4. Existence-comparison and uniqueness results

In this section, we prove existence-comparison and uniqueness of the solution for the discrete IBVP $(2.4)-(2.6)$

Theorem 4.1. Suppose that both $(f^{(1)}, f^{(2)})$ *and* $(g^{(1)}, g^{(2)})$ are quasimontone non- decreasing C^1- functions in the sector $S_{i,n}$ and satisfies Lipschitz conditions$(3.1),(3.7)$.

Suppose pairs of functions $(\tilde{u}_{i,n}, \tilde{v}_{i,n})$ and $(\hat{u}_{i,n}, \hat{v}_{i,n})$ are ordered upper-lower solutions of the IBYP $(2.4)-(2.6)$. Then the sequences $\{\overline{u}_{i,n}^{(m)}, \overline{v}_{i,n}^{(m)}\}$, $\{\underline{u}_{i,n}^{(m)}, \underline{v}_{i,n}^{(m)}\}$ Obtained from the iteration process (3.3) with initial iterations $(\overline{u}_{i,n}^{(0)}, \overline{v}_{i,n}^{(0)}) = (\tilde{u}_{i,n}, \tilde{v}_{i,n})$ and $(\overline{u}_{i,n}^{(0)}, \overline{v}_{i,n}^{(0)}) = (\hat{u}_{i,n}, \hat{v}_{i,n})$ converge monotonically from above and below respectively to the solutions $(\bar{u}_{i,n}, \bar{v}_{i,n})$ and $(\underline{u}_{i,n}, \underline{v}_{i,n})$ of the discrete IBVP $(2.4)-(2.6)$. Moreover

$$(\hat{u}_{i,n}, \hat{v}_{i,n}) \le (\underline{u}_{i,n}, \underline{v}_{i,n}) \le (\bar{u}_{i,n}, \bar{v}_{i,n}) \le (\tilde{u}_{i,n}, \tilde{v}_{i,n}), (i,n) \in \overline{\Lambda_p} \quad (4.1)$$

If, in addition

$$k_n^{-1} > max\{\gamma^{(11)} + \gamma^{(21)} + \sigma^{(11)} + \sigma^{(21)}. \gamma^{(12)} + \gamma^{(22)} + \sigma^{(12)} + \sigma^{(22)}\} \quad (4.2)$$

then $(\bar{u}_{i,n}, \bar{v}_{i,n}) = (\underline{u}_{i,n}, \underline{v}_{i,n})$, *is unique solution of the discrete initial boundary value problem* (2.4) – (2.6) *in* $S_{i.n}$.

Proof . It is seen from the Lemma 3.3, that the limits $(\bar{u}_{i,n}, \bar{v}_{i,n})$ and $(\underline{u}_{i,n}, \underline{v}_{i,n})$ in (3.6) are solutions of the discrete IBVP. $(2.4)-(2.6)$. Clearly they satisfy the relation (4.1)

Now we show that $(\bar{u}_{i,n}, \bar{v}_{i,n}) = (\underline{u}_{i,n}, \underline{v}_{i,n})$ in $\overline{\Lambda_p}$. Let $k_n = k$ be constant time step for n =1,2,…, n_1 where $1 \le n_1 \le N$ and let α be a fixed number such that

$$\gamma^{(11)} + \gamma^{(21)} + \sigma^{(21)} + \sigma^{(21)} < \alpha < k^{-1} \quad (4.3)$$

and $\gamma^{(12)} + \gamma^{(22)} + \sigma^{(12)} + \sigma^{(22)} < \alpha < k^{-1}$

Clearly, existence of such a α follows from [4.2].

Let

$$U_{i,n} = (1-ak)^{t_n/k}(\bar{u}_{i,n}, - \underline{u}_{i,n}) \quad (4.4)$$

$$V_{i,n} = (1-ak)^{t_n/k}(\bar{v}_{i,n}, - \underline{v}_{i,n})$$

for $n = 1,2,\dots,n_1$.

Then observe that $U_{i,n} \ge 0, V_{i,n} \ge 0$ *and*

$$(\bar{u}_{i,n}, - \underline{u}_{i,n}) - (\bar{u}_{i,n-1} - \underline{u}_{i,n-1}) = (1-ak)^{t_n/k}[U_{i,n} - (1-ak)U_{i,n-1}]$$

$$(\bar{v}_{i,n}, - \underline{v}_{i,n}) - (\bar{v}_{i,n1,} - \underline{v}_{i,n-1}) = (1-ak)^{t_n/k}[V_{i,n} - (1-ak)V_{i,n-1}] \quad (4.5)$$

By $(2.4),(3.7)$ *amd* $k_n = k$, *we have*

$$k_n^{-1}[(\bar{u}_{i,n}, - \underline{u}_{i,n}) - (\bar{u}_{i,n\,1} - \underline{u}_{i,n-1})] - L^{(1)}[\bar{u}_{i,n}, - \underline{u}_{i,n}] = f^{(1)}(\bar{u}_{i,n}, \bar{v}_{i,n}) - f^{(1)}(\underline{u}_{i,n}, \underline{v}_{i,n}) \quad (4.6)$$

$$\le \gamma^{(11)}(\bar{u}_{i,n}, - \underline{u}_{i,n}) + \gamma^{(12)}(\bar{v}_{i,n1,} - \underline{v}_{i,n-1})$$

and

$$k_n^{-1}[(\bar{v}_{i,n}, - \underline{v}_{i,n}) - (\bar{v}_{i,n1,} - \underline{v}_{i,n-1})] - L^{(2)}(\bar{v}_{i,n}, - \underline{v}_{i,n}) = f^{(2)}(\bar{u}_{i,n}, \bar{v}_{i,n}) - f^{(2)}(\underline{u}_{i,n}, \underline{v}_{i,n})$$

$$\le \gamma^{(21)}(\bar{u}_{i,n}, - \underline{u}_{i,n}) + \gamma^{(22)}(\bar{v}_{i,n1,} - \underline{v}_{i,n-1}) \quad (4.7)$$

Multiplying (4.6) *and* (4.7) by $(1-ak)t_n/k$ and using (4.4) and (4.5) , it gives

$$k^{-1}[U_{i,n} - (1-ak)U_{i,n-1}] - L^{(1)}[U_{i,n}] \le \gamma^{(11)}U_{i,n} + \gamma^{(12)}V_{i,n} \quad (4.8)$$

$$k^{-1}[V_{i,n} - (1-ak)V_{i,n-1}] - L^{(2)}[V_{i,n}] \le \gamma^{(21)}U_{i,n} + \gamma^{(22)}V_{i,n} \quad (4.9)$$

Also

$$B^{(1)}[U_{i,n}] = (1-ak)^{t_n/k}B^{(1)}[\bar{u}_{i,n}, - \underline{u}_{i,n}]$$

$$= (1-ak)^{t_n/k}(B^{(1)}[\bar{u}_{i,n},] - B^{(1)}[\underline{u}_{i,n}])$$

$$= (1-ak)^{t_n/k}[g^{(1)}(\bar{u}_{i,n}, \bar{v}_{i,n}) - g^{(1)}(\underline{u}_{i,n}, \underline{v}_{i,n})]$$

$$\le (1-ak)^{t_n/k}[\sigma^{(11)}(\bar{u}_{i,n} - \underline{u}_{i,n}) + \sigma^{(12)}(\bar{v}_{i,n} - \underline{v}_{i,n})]$$

Thus

$$B^{(1)}[U_{i,n}] \le \sigma^{(11)}U_{i,n} + \sigma^{(12)}V_{i,n} \quad (4.10)$$

On similar line, we get

$$B^{(2)}[V_{i,n}] \le \sigma^{(21)}U_{i,n} + \sigma^{(22)}V_{i,n} \quad (4.11)$$

Clearly

$$U_{i,0} = (1-ak)^{t_n/k}[\bar{u}_{i,0} - \underline{u}_{i,0}] = (1-ak)^{t_n/k}[\psi_i^{(1)} - \psi_i^{(1)}] = 0$$

$$V_{i,0} = (1-ak)^{t_n/k}[\bar{v}_{i,0} - \underline{v}_{i,0}] = (1-ak)^{t_n/k}[\psi_i^{(2)} - \psi_i^{(2)}] = 0 \quad (4.12)$$

Let (i',n') *and* $(i",n")$ be in $\overline{\Lambda}_p = \{(i,n); i \in \overline{\Omega}_p, n =$ *1,2,…,n1* such that

$$U_{i',n'} = Max\{U_{i,n;}(i,n); i \in \overline{\Lambda}_p\} = ||U||_1$$

$$V_{i",n"} = Max\{V_{i,n;}(i,n); i \in \overline{\Lambda}_p\} = ||V||_1 \quad (4.13)$$

By initial condition $(4.12); i', i" \in \overline{\Omega}_p$ and $n' \neq 0, n'' \neq 0$ *unless* $||U||_1 = ||V||_1 = 0$

Using n_1 as the initial time step and considering $k_n = k$ *for* $n = n_1 + 1, \dots, n_2$ where $n_1 + 1 \le n_2 \le$ N

The same reasoning gives $(\bar{u}_{i,n}, \bar{v}_{i,n}) = (\underline{u}_{i,n}, \underline{v}_{i,n})$ *for* $n = n_1 + 1_{n1} + 2, \dots, n_2.$

Continuing the same procedure leads to $(\bar{u}_{i,n}, \bar{v}_{i,n}) = (\underline{u}_{i,n}, \underline{v}_{i,n})$ for n= 1,2 , . . .,N.

Next to prove uniqueness, let $(u_{i,n}^* v_{i,n}^*)$ be any other solution of the discrete problem$(2.4)-(2.6)$ in $S_{i,n}$. Lemma 3.3 implies that $(\bar{u}_{i,n}, \bar{v}_{i,n}), (\underline{u}_{i,n}, \underline{v}_{i,n})$ given in (3.6) are solutions of the discrete problem$(2.4)-(2.6)$, then considering $(\tilde{u}_{i,n}, \tilde{v}_{i,n})$ and $(u^*_{i,n} v^*_{i,n})$ as pair of ordered upper- lower solutions, we have

$$(\tilde{u}_{i,n}, \tilde{v}_{i,n}) \geq (u^*_{i,n} v^*_{i,n})$$

Next considering $(u^*_{i,n} v^*_{i,n}), (\hat{u}_{i,n}, \hat{v}_{i,n})$ as pair of ordered upper- lower solutions, we have

$$(u^*_{i,n} v^*_{i,n}) \geq (\hat{u}_{i,n}, \hat{v}_{i,n})$$

This shows that the solution $(u^*_{i,n} v^*_{i,n})$ in $S_{i,n}$,satisfy

$$(\underline{u}_{i,n}, \underline{v}_{i,n}) \leq (u^*_{i,n} v^*_{i,n}) \leq (\bar{u}_{i,n}, \bar{v}_{i,n})$$

Since $(\underline{u}_{i,n}, \underline{v}_{i,n}) = (\bar{u}_{i,n}, \bar{v}_{i,n})$ this implies that $(\underline{u}_{i,n}, \underline{v}_{i,n}) = (\bar{u}_{i,n}, \bar{v}_{i,n}) = (u^*_{i,n} v^*_{i,n})$This completes the proof .

5. Conclusions

Discrete initial boundary value problems are studied by applying the method of upper lower solutions. We developed the results of existence comparison and uniqueness of solutions **.**We conclude that these results also develop by the integro parabolic initial boundary value problems.

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