Semiannihilator Small Submodules

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Abstract: ***Let R be an associative ring with non-zero identity and M be a left R-module. A submodule N of M is called semiannihilator small (sa-small), if for every submodule L of M with N+L=M, then ann(L)≪R. The properties of sa-small submodules have been studied. The sum of sa-small submodules is studied. Moreover,we introduce the concepts semiannihilator -hollow modules. We give many properties related with this type of modules.***

Keywords: semiannihilator small submodule**,** semiannihilator -hollow modules

1. Introduction

Let R an associative ring with non-zero identity, M a left R-module. A submodule N of M is called small, if for every submodule K of M with N+K=M, then K=M [1]. Recently, many authors have been interested in studying different kinds of annihilator small submodules as in [2] and [3], where the authors in [3] introduced the concept of R-annihilator small submodules, that is; a submodule N of an R-module M is called R-annihilator small, if whenever N+K=M, where K a submodule of M; then ann(K)=0, where ann_R (T) ={r R: r.T=0}. This has led us to introduce the concept of semiannihilator small submodules, in way that a submodule N of M is called semiannihilator small (sa-small) in case ann(K)≪ R where K is a submodule of M whenever N+K=M. It is clear that every R-annihilator small submodule is sa-small, but the converse is not true generally as examples can show next.

In this paper we define a subset of M that consists of all semiannihilator small elements by $_{sa}$ and we shall denote the sum of all semiannihilator small submodules of M by $Jsa(M)$, and study its properties. Finally, we shall introduce the concept of semiannihilator -hollow modules as a generalization of hollow modules.

2. Semi Annihilator − Small Submodules

In this section we introduce the concept of semiannihilator-small submodules and give characterizations,properties of this class of submodules.

Definition 2.1 A submodule N of a R-module M is called semiannihilator – small in M(sa-small)
if N+X=M , X a submodule of M ,implies that annX ≪R, we write N $\ll_{sa}$ M in this case .

Examples and remarks2.2:-

1) sa-small submodule need not be small, For example , consider Z as Z- module , every proper submodule of Z is sa-small. But{0} is the only small submodule of Z.
2) It is clear that R-a-small submodules are sa-small submodules.
3) If M is a faithful R-module then every small submodules are R-annihilator small submodules and then are sa-small submodules.
4) Let R be a smpil ring and let M be a R- module .Then every proper submodule of M is sa-small in M.
5) There are sa- small submodules that are direct summands as in the ℤ2-module M=ℤ2⊕ℤ2, where it is clear that A=ℤ2⊕(0) is a direct summand of M, $M=A\oplus\mathbb{Z}2=A\oplus<(\bar{1},\bar{1})>$ and ann(ℤ2) ≪ ℤ2 .

Since the ideal A of R is small in R iff A⊆ J(R),where J(R) is the jacobson radical of R . The following is a characterizations of sa-small submodules:

Proposition 2.3: A submodule N of a module M is sa- small in M if N+X=M, X a submodule of M ,implies that annX ≪J (R).

Proposition 2.4: Let M be an R-module with submodules A⊆N. If N $\ll_{sa}$ M then A $\ll_{sa}$ M.

Proof: Let X be a submodule of M such that A+X=M, since A⊆N hence N+X=M. N is sa-small in M then annX ≪R and hence A $\ll_{sa}$ M.

Proposition 2.5: Let M be a R-module with submodules A⊆N, if A $\ll_{sa}$ N then A $\ll_{sa}$ M.

Proof: Let X be any submodule of M such that A+X=M, now N∩M=N∩ (A+X) implies that N=A+(N∩X) by the modular law. Since A $\ll_{sa}$ N, thus ann(N∩X) ≪R . But ann(X)⊆ann(N∩X) then ann(X) ≪R, hence X $\ll_{sa}$ M.

Proposition 2.6: Let I be an ideal of commutative ring R and let M be an R- module if IM is a sa-submodule in M ,then I is sa- ideal of R.

Proof: Let R=I+J, J be an ideal of R. Then M=RM=(I+J)M=IM+JM, since IM is a sa-submodule in M ,then ann(JM) ≪R, but annJ ≤ JM ,then annJ≪R thus I is sa-ideal of R.

Proposition 2.7: let M and N be R- modules and f:M→ N be an epimorphism . If H $\ll_{sa}$ N, then f^{-1}(H) $\ll_{sa}$ M.

Proof: Let M = f^{-1}(H)+X , X ⊆M.Then f(M) =f(f^{-1}(H)+X))= f(f^{-1}(H))+f(X) .Since f is an epimorphism , then N=H +f(X).But H $\ll_{sa}$N , therefor annf(X) ≪R. Clearly that annX⊆ annf(X) and hence annX≪R. Thus f^{-1}(H) $\ll_{sa}$ M.

Note . Let f:M → N be an epimorphism . Then the image of sa-small submodule of M need not be sa-small in N as the following example shows:-

Consider Z_4 and Z as Z – modules and let π :Z→ Z_4 be the natural epimorphism ,{0} $\ll_{sa}$ Z.But π ({0})=0 is not s-a-small in Z_4 , where Z_4=0 + Z_4 and ann Z_4 =4Z is not small in R.

Note . The sum of two sa-small submodules of a module M need not be sa-small submodule .For example , In Z as Z-module. Each of 2Z and 3Z are sa-small submodule of Z .But Z=3Z+2Z is not sa-small in Z where Z =Z +0 and ann0=Z is not small in Z .

We prove the following:

Proposition 2-8:- Let M_1 ,M_2 be a R- modules . If $N_1 \ll_{sa} M_1$ and $N_2 \ll_{sa} M_2$ thus $N_1 \oplus N_2 \ll_{sa} M_1 \oplus M_2$.

Proof:- Let p_i: $M_1 \oplus M_2 \to M_1$ be the projection maps .Since $N_1 \ll_{sa} M_1$ and $N_2 \ll_{sa} M_2$, then by prop(2.5), $N_1 \oplus M_2 = P_1^{-1}(N_1) \ll_{sa} M_1 \oplus M_2$.and $M_1 \oplus N_2 = P_2^{-1}(N_2) \ll_{sa} M_1 \oplus M_2$, and $(N_1 \oplus M_2) \cap (M_1 \oplus N_2) = N_1 \oplus N_2 \ll_{sa} M_1 \oplus M_2$.

Theorem 2.9: Let M=ΣRmi, be an R-module and d∈M. Then the following statements are equivalent:

1. Rd $\ll_{sa}$ M.
2. $\cap i \in I$ann $(m_i - r_i\ d) \ll R$ $for\ each\ ri \in R$.
3. There exists j∈I such that $rmj \notin Rr$d for all 0 $\neq r \notin$J(R).

Proof: (1)⟹(2) For each i∈I, $mi = mi - rid + rid$ and hence M=$\Sigma R(m_i - r_i\ m_i) + R$d. By (1) we have . Rm $\ll_{sa}$ M then ann($\Sigma R(m_i - r_i\ m_i)$)=$\cap_i$ann$(mi - rid) \ll$R.

(2)⟹(1) Let L be a submodule of M with L+Rd=M. Then for each i∈I $mi = x_i + rid$,$\in R\ and\ xi \in$L . Let t∈ann(L), then $tmi = trid$+t x_i since t x_i =0 then t$(m_i - r_i$ d)=0 for each i∈I , then

t∈ ann$(m_i - r_i$ d) ,hence t∈ $\cap i \in I$ann $(m_i - r_i$ d) ≪R , then ann(L) ≪R .

2)⟹(3) Let $r \notin$ J(R) and assume that $rmi \in Rr$d for all i∈ I. Then $rmi = rird = rrid$ for all $i \in I$, so by (1) $r \in \cap i \in I$ann$(mi - rid) \ll$R which is a contradiction.

(3)⟹(2) Let $r \in \cap i \in I$ann$(mi - rid)$ and hence $r \in$ann$(mi - rid)$ for all $i \in I$. Thus $rmi = rrid = rird$ for all $i \in I$, so $rmi \in Rr$d. By (3) $r \in$ J(R) then $\cap i \in I$ann$(mi - rid) \subseteq$ J(R) so $\cap i \in I$ann $(m_i - r_i$ d) ≪R.

Theorem 2.10: Let R be a commutative ring, M=$\Sigma Rmi i \in I$ and N a submodule of M. Then the following statements are equivalent:

1. N $\ll_{sa}$ M.
2. $\cap i \in I$ann$(mi - xi) \ll$R for all $xi \in$N.

Proof: (1)⟹(2) For each $i \in I$, let xi∈N. Then $mi = mi - xi + xi$ for each $i \in I$. Then $M = \Sigma R(mi - xi) i \in I$+N, by (1) N $\ll_{sa}$ M, then ann$(\Sigma R(mi - xi))$=$\cap i \in I$ann$(R(mi - xi)) \ll$R.

(2)⟹(1) Let M=A+N. Then for each $i \in I$ $mi = ai + xi$ where $ai \in A\ and$ xi∈N. Hence $ai = mi - xi$ for each $i \in I$ and $M = \Sigma(mi - ki) i \in I$+N. Now, let $t \in$ann(A) then $tai = t(mi - xi)$ for each $i \in I$ and hence $t \in$ann$(R(mi - xi)) \ll$R by (2), so ann$(A) \subseteq$ ann$(R(mi - xi))$. Thus N $\ll_{sa}$ M.

As the same of the definition of Jacobson radical related to small submodules, we will state a definition related to sa-small submodules in the following.But first we need this definition.

Definition 2.11: Let M be an R-module and $a \in M$. an element a in M is semiannihilator small if Ra is semiannihilator small submodule of M. let A_{sa}={$a \in M | Ra \ll_{sa} M$}.

The set $_{sa}$ is not a submodule of M. In fact, it is not closed under addition, for example in the $\mathbb{Z}-module\ \mathbb{Z}$ we have that 4,-3 ∈ A_{sa} but 4-3=1∉ A_{sa}.

We can see by the use of proposition (2.4) that if M is an R-module and $a \in A_{sa}$, then $Ra \subseteq A_{sa}$.

Now we can prove the following:

Remark 2.12:- Let M be a module and N be sa- small submodule of M. , then N⊆ A_{sa}.

Proof:- Let x∈N ,N is sa-small submodule then Rx is sa-small submodule of M by prop(2.4) .

Thus x ∈ A_{sa}.

Since the sum of sa-small submodules is not sa-small (consider 3Z+2Z in Z)we define.

Definition 2.13: Let M be an R-module. Denote $Js(M)$ for the sum of all sa- small submodules of M.. If M has no sa-small submodule , we write $Jsa(M)$ =M .

It is clear that $A_{sa} \subseteq Js(M)$ in every module , but this may not be equality (Z as Z-module) .

Proposition 2.14: Let M be an R-module such that $A_{sa} \neq \phi$, then we have the following:

1) $Js(M)$ is a submodule of M and contains every sa- small submodule of M.
2) $Js(M)$ ={$a1 + a2 + \cdots + an; ai \in A_{sa}\ for\ each\ i,n \geq 1$}.
3) $Js(M)$ is generated by A_{sa}.

Proof:

1. Let {$N\alpha | \alpha \in \Lambda$} be the set of all sa- small submodules of M, thus $Js(M) = \Sigma N\alpha$, α∈Λ. Let $x,y \in Jsa(M)$, this means that $x = \Sigma x\alpha$, α∈Λ $and\ y = \Sigma y\alpha, \alpha \in \Lambda\ where\ x\alpha, y\alpha \in N\alpha\ for\ each$ $\alpha \in \Lambda\ and\ x\alpha, y\alpha \neq 0$ for at most a finite number of α∈Λ. Then $x + y = \Sigma(x\alpha + y\alpha)$ α∈Λ such that $x\alpha + y\alpha \in N\lambda$ for each α∈Λ, $x + y \in Jsa(M)$. Now, let $r \in R\ and\ x \in Js(M)$. it is clear that $rx \in Ja(M)$. Hence, $Js(M)$ is a submodule of M.

2. Follows from (1) and $A_{sa} \subseteq Js(M)$.

3. Since $A_{sa} \subseteq Js(M)$, then $< A_{sa} > \subseteq Jsa(M)$. Clearly, $Js(M) \subseteq < A_{sa} >$.Then $Jsa(M)$ is generated by A_{sa}.

Proposition 2.15: Let M be an R-module such that $A_{sa} \neq \phi$. Then the following statements are equivalent:

1. A_{sa} is closed under addition; that is, a finite sum of sa-small elements is sa-small.
2. $Js(M)= A_{sa}$.

Proof:
(1)⟹(2) Let $a1+a2+\cdots+an\in Jsa(M)$, $ai\in Ai$ $i=1,\ldots,n$, Ai is s-a-small in M for each $i=1,\ldots,n$. then $Rai \ll_{sa}$ M by proposition (2.4). Hence $ai\in A_{sa}$ for each $i=1,\ldots,n$, by the assumption in (1) we get that $a1+\cdots+an\in A_{sa}$. thus $Jsa(M)\subseteq A_{sa}$ and hence $Jsa(M)= A_{sa}$.
(2)⟹(1) Let $x,y \in A_{sa}$, since $A_{sa} \subseteq Jsa(M)$ then $x,y\in Jsa(M)$and by using proposition (2.14) we have $x+y\in Jsa(M)$. Hence, $x+y\in A_{sa}$,but A_{sa} is closed under addition. We can prove that a finite sum of sa- small elements is sa-small by the use of induction.

Proposition 2.16: Let M be an R-module such that $A_{sa} \neq \phi$. consider the following statements:
1) $Js(M)$is an sa-small submodule of M.
2) If K and L are sa-small submodules of M, then K+L is an sa-small submodule of M.
3) A_{sa} is closed under addition; that is, sum of sa-small elements of M is sa- small.
4) $Js(M)= A_{sa}$.

Then (1)⟹(2)⟹(3)⟺(4) . If M is finitely generated, then (1)⟺(2).

Proof:
(1)⟹(2) Let K,L be sa-small in M, then K+L⊆$Jsa(M)$ which is sa-small by assumption. Thus by using proposition (2.4) we get K+L $\ll_{sa}$ M.

(2)⟹(3) Let x, $y \in A_{sa}$, then Rx, Ry are sa-small in M, and hence by (2) $Rx+Ry$ is sa- small in M. But $R(x+y)\subseteq Rx+Ry$ and by using proposition (2.4) we get $R(x+y) \ll_{sa}$ M. Hence, $x+y \in A_{sa}$.

(3)⟺(4) By proposition (2.14).

Now, let M be finitely generated to prove (2)⟹(1). Consider $\{m1,2,\ldots,mn\}$ to be the set of generators of M. Let X be a submodule of M such that $Js(M)$+X=M, then $mi=ai+xi$ such that $ai\in Jsa(M)$ and $xi\in$ X for each i=1,…,n. Thus $\Sigma Rmini=1=\Sigma Raini=1+\Sigma Rxini=1$ and hence M=$\Sigma Raini=1+X$. Now, since $ai\in Jsa(M)$ and since (2) ⟹(3)⟺(4) we get $Jsa(M)= A_{sa}$; that is, $ai\in A_{sa}$ and hence $Rai \ll_{sa}$ M thus ann(X) ≪R implies that $Jsa(M) \ll_{sa}$M.

3. Semiannihilator-hollow module

In this section we introduce the concept of semiannihilator-hollow modules and give basic properties of this module.

Definition 3.1: A nontrivial R-module M is called semiannihilator -hollow(sa-hollow) if every proper submodule of M is sa-small in M.

Examples and Remarks 3.2 :
(1) Z as Z- module is sa-hollow module but it is not hollow.
(2) Z_6 and Z_4 as Z- module are not sa- hollow modules.

The epimorphic image of sa-hollow module need not be sa-hollow as the following example shows:-

Consider Z and Z_4 as Z – modules and π:Z→ Z_4 let be the natural epimorphism . Z as Z- module is sa-hollow module,{0} $\ll_{sa}$ Z . But π ({0})=0 is not sa-small in Z_4 , where Z_4=0 + Z_4 and ann Z_4 =4Z not small in Z.

Proposition 3.3:-Let M and N be two R- modules and f:M→ N be an epimorphism .If N is sa-hollow module then M is sa-hollow module.

Proof:-Let K be submodule of M ,then f(K) is submodule of N and since N is sa-hollow module , f(K) is sa-small submodule, then f^{-1}(f(k)) $\ll_{sa}$ M by (Proposition 2.6), f^{-1}(f(k))=K+kerf , K ≤ K+kerf then by(Proposition 2.4), K is sa-small submodule of M.

Corollary 3.4: Let M be R- module, K be submodule of M . If M/K is sa-hollow module then M is sa-hollow module.
A submodule N of R-module M is called fully invariant submodule of M if f(N)⊆N, for every f ∈Hom(M,M) .A module M is called duo module if every submodule of M is fully invariant[3].

Proposition 3.5: Let M=M_1⊕M_2 be duo module .If M_1 and M_2 are sa-hollow modules, then M is sa-hollow module.

Proof: Let $M_1$and M_2 be sa-hollow modules,and N_1⊕N_2be a proper Submodule of M_1⊕M_2

N_1 ≤ $M_1$and N_2≤ M_2 , then $N_1$and N_2 are s-a-small submodules of M_1,M_2 respectivly thus by(Proposition 2.7) N_1⊕N_2 is sa-small submodule of M.

Corollary3.6: Let M= M_1⊕M_2 be an R- module such that R=ann(M_1)+ann(M_2).If M_1 and M_2 are be sa-hollow, then so is M.

We call a ring R is sa-hollow, if R is sa-hollow R-module. A module M is called multiplication , if for every submodule N of M there exists an ideal I of R such that N = IM = (N:M)M [4].

Proposition 3.7 : Let M be a multiplication R-module. If M is sa-hollow then R is a sa-hollow ring.

Proof: Suppose that M is sa-hollow. Let I be an ideal of R. Then IM is a submodule of M and hence IM is sa-small(Prop.2.6). Then I is sa-small ideal of R and hence R is sa-hollow.

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