Semiannihilator Small Submodules

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Abstract: Let R be an associative ring with non-zero identity and M be a left R-module. A submodule N of M is called semiannihilator small (sa-small), if for every submodule L of M with N+L=M, then ann(L) ≪ R. The properties of sa-small submodules have been studied. The sum of sa-small submodules is studied. Moreover, we introduce the concepts semiannihilator -hollow modules. We give many properties related with this type of modules.

Keywords: semiannihilator small submodule, semiannihilator -hollow modules

1. Introduction

Let R be an associative ring with non-zero identity, M a left R-module. A submodule N of M is called small, if for every submodule K of M with N+K=M, then K=M [1]. Recently, many authors have been interested in studying different kinds of annihilator small submodules as in [2] and [3], where the authors in [3] introduced the concept of R-annihilator small submodules, that is; a submodule N of an R-module M is called R-annihilator small, if whenever N+K=M, where K a submodule of M; then ann(K)=0, where annk (T) ={r R: r.T=0}. This has led us to introduce the concept of semiannihilator small submodules, in way that a submodule N of M is called semiannihilator small (sa-small) in case ann(K) ≪ R where K is a submodule of M whenever N+K=M. It is clear that every R-annihilator small submodule is sa-small, but the converse is not true generally as examples can show next.

In this paper we define a subset of M that consists of all semiannihilator small elements by sa and we shall denote the sum of all semiannihilator small submodules of M by fsa(M), and study its properties. Finally, we shall introduce the concept of semiannihilator -hollow modules as a generalization of hollow modules.

2. Semi Annihilator – Small Submodules

In this section we introduce the concept of semiannihilator-small submodules and give characterizations, properties of this class of submodules.

Definition 2.1 A submodule N of a R-module M is called semiannihilator – small in M(sa-small) if N+X=M, X a submodule of M ,implies that annX ≪R, we write N ≪sa M in this case .

Examples and remarks2.2:-

1) sa-small submodule need not be small, For example , consider Z as Z- module , every proper submodule of Z is sa-small. But {0} is the only small submodule of Z.
2) It is clear that R-a-small submodules are sa-small submodules.
3) If M is a faithful R-module then every small submodules are R-annihilator small submodules and then are sa-small submodules.

4) Let R be a smpl ring and let M be a R- module .Then every proper submodule of M is sa-small in M.

5) There are sa- small submodules that are direct summands as in the Z2-module M=Z2 ⊕ Z2, where it is clear that A=Z2(0) is a direct summand of M , M=A ⊕ Z2=A ⊕ <(1,1)> and ann(Z2) ≪ Z2.

Since the ideal A of R is small in R iff A⊆ J(R ),where J(R) is the jacobson radical of R . The following is a characterizations of sa-small submodules:

Proposition 2.3: A submodule N of a module M is sa-small in M if N+X=M, X a submodule of M ,implies that annX ⊆ J (R).

Proposition 2.4: Let M be an R-module with submodules A⊆N. If N ≪sa M then A ≪sa M.

Proof: Let X be a submodule of M such that A+X=M, since A⊆N hence N+X=M. N is sa-small in M then annX ≪ R and hence A ≪sa M.

Proposition 2.5: Let M be a R-module with submodules A⊆N, if A ≪sa N then A ≪sa M.

Proof: Let X be any submodule of M such that A+X=M, now N∩M=N∩ (A+X) implies that N=A+(N∩X) by the modular law. Since A ≪sa N, thus ann(N∩X) ≪ R . But ann(X)⊆ann(N∩X) then ann(X) ≪ R, hence X ≪sa M.

Proposition 2.6: Let I be an ideal of commutative ring R and let M be an R- module . Then M is a sa-module in M, then I is sa- ideal of R.

Proof: Let R=I+J, J be an ideal of R. Then M=R= (I+J)=IM+JM, since IM is a sa-submodule in M then ann(IM) ≪ R, but annJ ≲ JM , then annJ ≪ R thus I is sa-ideal of R.

Proposition 2.7: let M and N be R- modules and f:M→ N be an epimorphism . If H ≲sa N, then f−1(H) ≲sa M.

Proof: Let M = f−1(H)+X , X ⊆M.Then f ( M) =f(f−1(H)+X)=f(f−1(H))+fX. Since f is an epimorphism , then N=H+fX. But H ≲sa N , therefor annf(X) ≪ R. Clearly that annfX ≲ annf(X) and hence annfX ≪ R. Thus f−1(H) ≲sa M.
Note. Let \( f : M \rightarrow N \) be an epimorphism. Then the image of a small submodule of \( M \) need not be small-in-\( M \) as the following example shows:

Consider \( Z_4 \) and \( Z \) as \( Z \)-modules and let \( \pi : Z \rightarrow Z_4 \) be the natural epimorphism \((0) \triangleleft Z \). But \( \pi (\{0\})=0 \) is not small in \( Z_4 \), where \( Z_4=0+Z_4 \) and \( \text{ann} Z_4=4Z \) is not small in \( Z \).

Note. The sum of two small submodules of a module \( M \) need not be small submodule. For example, \( M \rightarrow N \), where \( Z = Z \times Z \), and ann\( Z_4=4Z \) is not small in \( Z \).

We prove the following:

**Proposition 2.8:** Let \( M_1, M_2 \) be a \( R \)-modules. If \( N_1 \triangleleft M_1 \) and \( N_2 \triangleleft M_2 \) then \( N_1 \cap N_2 \triangleleft M_1 \cap M_2 \).

**Proof:** Let \( p_1 : M_1 \rightarrow M_2 \) and \( p_2 : M_2 \rightarrow M_1 \) be the projection maps. Since \( N_1 \triangleleft M_1 \) and \( N_2 \triangleleft M_2 \), then by prop(2.5), \( N_1 \cap M_1 = N_1 \cap M_1 \) and \( M_1 \cap N_2 = M_1 \cap N_2 \) and \( N_1 \cap N_2 \triangleleft M_1 \cap M_2 \).

**Theorem 2.9:** Let \( M=\sum Rm_i \) be an \( R \)-module and \( \text{ann} M \in \mathbb{D} M \). Then the following statements are equivalent:
1. \( \text{ann} M \). 
2. \( \forall i \in \text{ann}(m_i r_i d) \Rightarrow \forall i \in \text{ann}(r_i) \).
3. There exists \( \forall i \in \text{ann}(m_i r_i d) \).

**Proof:**
(1) \( \Rightarrow \) (2) For each \( i \in \text{ann}(m_i r_i d) \) and hence \( M=\sum (m_i r_i d) \). By (1) we have, \( Rm_i \triangleleft M \) and \( \text{ann}(m_i r_i d) \triangleleft R \).

(2) \( \Rightarrow \) (1) Let \( L \subset M \) and \( m_i r_i d \in \text{ann}(m_i r_i d) \).

(3) \( \Rightarrow \) (2) Let \( r \in \text{ann}(m_i r_i d) \) and hence \( \forall i \in \text{ann}(m_i r_i d) \).

**Theorem 2.10:** Let \( R \) be a commutative ring, \( M=\sum Rm_i \in R \) and \( N \) a submodule of \( M \). Then the following statements are equivalent:
1. \( N \triangleleft M \).
2. \( \forall i \in \text{ann}(m_i) \Rightarrow \forall i \in \text{ann}(m_i) \).

**Proof:**
(1) \( \Rightarrow \) (2) For each \( i \in \text{ann}(m_i) \), \( \forall i \in \text{ann}(m_i) \). 

(2) \( \Rightarrow \) (1) Let \( N \triangleleft M \) and \( m_i \in \text{ann}(m_i) \). Hence \( m_i \in \text{ann}(m_i) \).

**Definition 2.11:** Let \( M \) be an \( R \)-module and \( a \in M \). An element \( a \) in \( M \) is semiannihilator small if \( Ra \) is semiannihilator small submodule of \( M \) and \( a \in M \).

The set \( \text{ann} M \) is not a submodule of \( M \). In fact, it is not closed under addition, for example in the \( \mathbb{Z} \)-module \( \mathbb{Z} \) we have that \( 4, -3 \in \text{ann} M \) but \( 4-3=1 \in \text{ann} M \).

We can see by the use of proposition (2.4) that if \( M \) is an \( R \)-module and \( a \in M \), then \( Ra \triangleleft M \).

Now we can prove the following:

**Remark 2.12:** Let \( M \) be a module and \( N \) be a small submodule of \( M \), then \( N \in \text{ann} M \).

**Proposition 2.14:** Let \( M \) be an \( R \)-module such that \( A \neq \phi \), then the following hold:
1. \( J(M) \) is a submodule of \( M \) and contains every small submodule of \( M \).
2. \( J(M) = \{a_1 + a_2 + \cdots + a_n | a_i \in A, i \geq 1\} \).
3. \( J(M) \) is generated by \( A \).

**Proof:**
1. Let \( \{Na | a \in A \} \) be the set of all small submodules of \( M \), then \( J(M) = \sum Na \). Let \( x \in J(M) \), this means that \( x = x_1a \), \( a \in A \) and \( y = y_1a \) for each \( a \in A \).

2. Follows from (1) and \( A \neq \phi \).
3. Since \( A \neq \phi \), \( J(M) \). Clearly, \( J(M) \).

**Proposition 2.15:** Let \( M \) be an \( R \)-module such that \( A \neq \phi \). Then the following statements are equivalent:
1. $A_{sa}$ is closed under addition; that is, a finite sum of sa-small elements is sa-small.

2. $fs(M)=A_{sa}$

**Proof:**

(1) $\Rightarrow$ (2) Let $a_1+a_2+\cdots+a_n\in fsa(M)$, $a_i\in\mathcal{A}$ for each $i=1,\ldots,n$. Hence $\mathcal{A}\subseteq A_{sa}$ by assumption in (1). From the basic properties of this module, the sum of $sa$-small elements is $sa$-small in $M$, then $K+L$ is a submodule of $M$ and since $N$ is $sa$-small submodule, then $f^{-1}(f(k))\subseteq a_{sa}$ $M$. Thus by (2) we have $x+y\in fsa(M)$.

Proposition 2.16: Let $M$ be an $R$-module such that $A_{sa}\neq\phi$. Consider the following statements:

1. $fs(M)$ is a $sa$-small submodule of $M$.
2. If $K$ and $L$ are $sa$-small submodules of $M$, then $K+L$ is a $sa$-small submodule of $M$.
3. $A_{sa}$ is closed under addition; that is, a finite sum of $sa$-small elements is $sa$-small.
4. $fs(M)=A_{sa}$.

Proof: Let $K$ and $L$ be $sa$-small submodules of $M$. Then $K+L\subseteq fsa(M)$ which is $a_{sa}$ small by the use of induction.

**Corollary 3.4:** Let $M$ be $R$-module, $K$ be submodule of $M$. If $M/K$ is $sa$-small module, then $M$ is $sa$-small module. A submodule $N$ of $R$-module $M$ is called fully invariant submodule of $M$ if $f(N)\subseteq N$, for every $f\in \text{Hom}(M,M)$. A module $M$ is called a module if $M$ is fully invariant.

**Proposition 3.5:** Let $M=M_1\oplus M_2$ be a $sa$-small module. If $M_1$ and $M_2$ are $sa$-small submodules, then $M$ is $sa$-small module.

**References**


