Semiannihilator Small Submodules

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Abstract: Let \( R \) be an associative ring with non-zero identity and \( M \) be a left \( R \)-module. A submodule \( N \) of \( M \) is called semiannihilator small (sa-small), if for every submodule \( L \) of \( M \) with \( N+L=M \), then \( \text{ann}(L) \leq R \). The properties of sa-small submodules have been studied. The sum of sa-small submodules is studied. Moreover, we introduce the concepts semiannihilator -hollow modules. We give many properties related with this type of modules.

Keywords: semiannihilator small submodule, semiannihilator -hollow modules

1. Introduction

Let \( R \) be an associative ring with non-zero identity, \( M \) a left \( R \)-module. A submodule \( N \) of \( M \) is called small, if for every submodule \( K \) of \( M \) with \( N+K=M \) and \( K \neq M \), then \( \text{ann}(K) \neq R \). Recently, many authors have been interested in studying different kinds of annihilator small submodules as in \([2]\) and \([3]\), where the authors in \([3]\) introduced the concept of semiannihilator small submodules, that is; a submodule \( N \) of an \( R \)-module \( M \) is called R-annihilator small, if whenever \( N+K=M \), where \( K \) a submodule of \( M \), then \( \text{ann}(K)=0 \), where \( \text{ann}(T)\equiv\{r\in R : rT=0\} \). This has led us to introduce the concept of semiannihilator small submodules in way that a submodule \( N \) of \( M \) is called semiannihilator small (sa-small) in case \( \text{ann}(M)\subseteq R \) where \( K \) a submodule of \( M \) whenever \( N+K=M \). It is clear that every \( R \)-annihilator small submodule is sa-small, but the converse is not true generally as examples can show next.

In this paper we define a subset of \( M \) that consists of all semiannihilator small elements by \( sa \) and we shall denote the sum of all semiannihilator small submodules of \( M \) by \( f_{sa}(M) \), and study its properties. Finally, we shall introduce the concept of semiannihilator -hollow modules as a generalization of hollow modules.

2. Semi Annihilator – Small Submodules

In this section we introduce the concept of semiannihilator-small submodules and give characterizations, properties of this class of submodules.

Definition 2.1 A submodule \( N \) of a \( R \)-module \( M \) is called semiannihilator – small in \( M \)(sa-small) if \( N+X=M \) and \( X \) a submodule of \( M \), implies that \( \text{ann}(X) \leq R \), we write \( N \ll_{sa} M \) in this case .

Examples and remarks 2.2:-

1) sa-small submodule need not be small, For example , consider \( Z \) as \( Z \)- module , every proper submodule of \( Z \) is sa-small. But \( \{0\} \) is the only small submodule of \( Z \).

2) It is clear that \( R \)-a small submodules are sa-small submodules.

3) If \( M \) is a faithful \( R \)-module then every small submodules are \( R \)-annihilator small submodules and then are sa-small submodules.

4) Let \( R \) be a simpl ring and let \( M \) be a \( R \)- module .Then every proper submodule of \( M \) is sa-small in \( M \).

5) There are sa- small submodules that are direct summands as in the \( Z2 \)-module \( M=Z2\oplus Z2 \), where it is clear that \( A=Z2\oplus(0) \) is a direct summand of \( M \), \( M=A\oplus Z2=A\oplus (1,1) \) and \( \text{ann}(Z2) \ll Z2 \).

Since the ideal \( A \) of \( R \) is small in \( R \), if \( A\subseteq J(R) \), where \( J(R) \) is the jacobson radical of \( R \) . The following is a characterizations of sa-small submodules:

- **Proposition 2.3**: A submodule \( N \) of a module \( M \) is sa-small in \( M \) if \( N+X=M \), \( X \) a submodule of \( M \), implies that \( \text{ann}(X) \ll J(R) \).

- **Proposition 2.4**: Let \( M \) be an \( R \)-module with submodules \( A\subseteq N \). If \( N \ll_{sa} M \) then \( A \ll_{sa} M \).

- **Proof**: Let \( X \) be a submodule of \( M \) such that \( A+X=M \), since \( A\subseteq N \) hence \( N+X=M \). \( N \) is sa-small in \( M \) then \( \text{ann}(X) \ll R \) and hence \( A \ll_{sa} M \).

- **Proposition 2.5**: Let \( M \) be a \( R \)-module with submodules \( A\subseteq N \), if \( A \ll N \) then \( A \ll_{sa} M \).

- **Proof**: Let \( X \) be any submodule of \( M \) such that \( A+X=M \), now \( A\subseteq N \Rightarrow (A+X) \subseteq N \) implies that \( N\subseteq A+(N\cap X) \) by the modular law. Since \( A \ll N \), thus \( \text{ann}(N\cap X) \ll R \). But \( \text{ann}(X)\subseteq\text{ann}(N\cap X) \) then \( \text{ann}(X) \ll R \), hence \( X \ll_{sa} M \).

- **Proposition 2.6**: Let \( I \) be an ideal of commutative ring \( R \) and let \( M \) be an \( R \)-module if \( IM \) is a sa-submodule in \( M \), then \( I \) is sa-ideal of \( R \).

- **Proof**: Let \( R=I+J \), \( J \) be an ideal of \( R \). Then \( M=IM=IM+JM \), since \( IM \) is a sa-submodule in \( M \), then \( \text{ann}(JM) \ll R \), but \( \text{ann}(J)\subseteq\text{ann}(IM) \) then \( \text{ann}(J) \ll R \) hence \( I \ll_{sa} M \).

- **Proposition 2.7**: let \( M \) and \( N \) be \( R \)-modules and \( f:M\rightarrow N \) be an epimorphism . If \( H \ll_{sa} N \), then \( f^{-1}(H) \ll_{sa} M \).

- **Proof**: Let \( M = f^{-1}(H)+X \), \( X \subseteq M \).Then \( f(M) = f(f^{-1}(H)+X) = f(f^{-1}(H))+f(X) \). Since \( f \) is an epimorphism, then \( N=H+f(X) \). Hence \( H \ll_{sa} N \), therefor \( \text{ann}(H) \ll R \), clearly that \( \text{ann}(X)\subseteq\text{ann}(f(X)) \) and hence \( \text{ann}(X) \ll R \). Thus \( f^{-1}(H) \ll_{sa} M \).
Note. Let $f: M \to N$ be an epimorphism. Then the image of sa-small submodule of $M$ need not be sa-small in $N$ as the following example shows:

Consider $Z_4$ and $Z$ as $Z-$modules and let $\pi : Z \to Z_4$ be the natural epimorphism, $\{0\} \ll Z$. But $x (\{0\}) = 0$ is not s-a-small in $Z_4$, where $Z_4 \cong \mathbb{Z}$ and $\text{ann} Z_4 = 4 \mathbb{Z}$ is not small in $R$.

Note. The sum of two sa-small submodules of a module $M$ need not be sa-small submodule. For example, $\mathbb{Z}$ in $\mathbb{Z}$ as $\mathbb{Z}$-module. Each of $2 \mathbb{Z}$ and $3 \mathbb{Z}$ are sa-small submodule of $\mathbb{Z}$. But $Z = 3 \mathbb{Z} + 2 \mathbb{Z}$ is not sa-small in $\mathbb{Z}$ where $Z = \mathbb{Z} + 0$ and $\text{ann} 0 = \mathbb{Z}$ is not small in $\mathbb{Z}$.

We prove the following:

**Proposition 2.8:** Let $M_1, M_2$ be a $R$-modules. If $N_1 \ll_{sa} M_1$ and $N_2 \ll_{sa} M_2$ then $N_1 \oplus N_2 \ll_{sa} M_1 \oplus M_2$.

**Proof:** Let $p_1: M_1 \oplus M_2 \to M_1$ be the projection maps. Since $N_1 \ll_{sa} M_1$ and $N_2 \ll_{sa} M_2$, then by (2.5), $N_1 \oplus M_2 = P_1^{-1}(N_1) \ll_{sa} M_1 \oplus M_2$, and $M_1 \oplus N_2 = P_2^{-1}(N_2) \ll_{sa} M_1 \oplus M_2$, and $(N_1 \oplus M_2) \cap (M_1 \oplus N_2) = \{N_1 \oplus N_2 \ll_{sa} M_1 \oplus M_2 \}$.

**Theorem 2.9:** Let $M=\Sigma Rm_i$ be an $R$-module and $d \in M$. Then the following statements are equivalent:

1. $d \ll_{sa} M$.
2. $\cap \{i \in \mathbb{L} : (m_i - r_i, d) \ll \mathbb{L} \}$
3. There exists $e \in \mathbb{L}$ such that $rm \mathbb{E} Rd$ for all $0 \neq r \in \mathbb{L}(R)$.

**Proof:** (1) $\Rightarrow$ (2) For each $i \in \mathbb{L}$, $m_i = m_i - r_i + d$ and hence $M=\Sigma \mathbb{E}(m_i - r_i, m_i) + R$. By (1) we have $Rm \ll_{sa} M$ then $\text{ann}(M(m_i - r_i, m_i)) = \cap \text{ann}(m_i - r_i)) \ll \mathbb{L}$. (2) $\Rightarrow$ (1) Let $L$ be a submodule of $M + Rd$. Then for each $i \in \mathbb{L}$, $m_i = x_i + r_i \mathbb{E} R$ and $x_i \mathbb{E} L$. Let $\text{ann}(L)$, then $m_i = x_i + r_i \mathbb{E} L$. Hence, $x_i = 0$, then $t(m_i - r_i, d) = 0$ for each $i \in \mathbb{L}$, then $t \text{ann}(m_i - r_i, d)$ hence $t \in \cap \{i \in \mathbb{L} : (m_i - r_i, d) \ll \mathbb{L} \}$, then $Rm \ll_{sa} M$.

(3) $\Rightarrow$ (2) Let $r \in \cap \{i \in \mathbb{L} : (m_i - r_i, d) \ll \mathbb{L} \}$ and hence $r \text{ann}(m_i - r_i, d)$ for all $i \in \mathbb{L}$. Thus $m_i = m_i + d$ for all $i \in \mathbb{L}$, so $m_i \in Rd$. By (3) $r \in \mathbb{L}(R)$ then $t \in \text{ann}(m_i - r_i, d) \subseteq \mathbb{L}(R)$ so $t \text{ann}(m_i - r_i, d) \ll \mathbb{L}(R)$.

**Theorem 2.10:** Let $R$ be a commutative ring, $M=\Sigma Rm_i \in \mathbb{L}$ and $N$ a submodule of $M$. Then the following statements are equivalent:

1. $N \ll_{sa} M$.
2. $\cap \{i \in \mathbb{L} : (m_i - x_i, d) \ll \mathbb{L} \}$

**Proof:** (1) $\Rightarrow$ (2) For each $i \in \mathbb{L}$, let $x_i \in \mathbb{N}$. Then $m_i = m_i - x_i + x_i$ for each $i \in \mathbb{L}$. Then $M=\Sigma (m_i - x_i) \mathbb{E} \mathbb{L} + N$. Hence, $ai \in N$ and $x_i \mathbb{E} \mathbb{L}$. Hence $ai = m_i - x_i$ for each $i \in \mathbb{L}$ and $M=\Sigma (m_i - k_i) \mathbb{E} \mathbb{L} + N$. Now, let $t \in \text{ann}(A)$ then $t = t(m_i - x_i)$ for each $i \in \mathbb{L}$ and hence $t \in \text{ann}(R(m_i - x_i)) \ll \mathbb{L}$. Thus $N \ll_{sa} M$.

As the same of the definition of Jacobson radical related to small submodules, we will state a definition related to sa-small submodules in the following. But first we need this definition.

**Definition 2.11:** Let $M$ be an $R$-module and $a \in \mathbb{M}$. An element $a$ in $M$ is semiannihilator small if $Ra$ is semiannihilator small submodule of $M$. Let $A_{sa} = \{a \in \mathbb{M} \mid Ra \ll_{sa} M\}$.

We can see by the use of proposition (2.4) that if $M$ is an $R$-module and $a \in A_{sa}$ then $Ra \ll_{sa} M$.

Now we can prove the following:

**Remark 2.12:** Let $M$ be a module and $N$ be sa-small submodule of $M$, then $N \subseteq A_{sa}$.

**Proof:** Let $x \in N$, then $x \mathbb{E} \mathbb{L}$ is sa-small submodule then $Rx$ is sa-small submodule of $M$ by (2.4). Thus $x \in A_{sa}$.

Since the sum of all sa-small submodules of $M$ is not sa-small (consider $3 \mathbb{Z} + 2 \mathbb{Z}$ in $\mathbb{Z}$).

**Definition 2.13:** Let $M$ be an $R$-module. Denote $Js(M)$ for the sum of all sa-small submodules of $M$. If $M$ has no sa-small submodule, we write $Js(M) = M$.

It is clear that $A_{sa} \subseteq Js(M)$ in every module, but this may not be equality (Z as Z-module).

**Proposition 2.14:** Let $M$ be an $R$-module such that $A_{sa} \neq \phi$, then we have the following:

1. $Js(M)$ is a submodule of $M$ and contains every sa-small submodule of $M$.
2. $Js(M) = \{a_1 + a_2 + \cdots + a_n \mid a_i \in A_{sa} \text{ for each } i, n \geq 1\}$.
3. $Js(M)$ is generated by $A_{sa}$.

**Proof:**

1. Let $\{N_\alpha \mid \alpha \in \Lambda\}$ be the set of all sa-small submodules of $M$, thus $Js(M) = \Sigma_\Lambda N_\alpha, a \in \Lambda$. Let $x, y \in Js(M)$, this means that $x = x_\alpha a_\alpha$ and $y = y_\alpha a_\alpha$ where $x_\alpha y_\alpha \in N_\alpha$ for each $a_\alpha \in a_\alpha$ and $x_\alpha y_\alpha \neq 0$ for at most a finite number of $a_\alpha$. Then $x + y = (x_\alpha + y_\alpha) a_\alpha$ such that $x_\alpha + y_\alpha \in N_\alpha$ for each $a_\alpha$, $x + y \in Js(M)$. Now, let $r \in R$ and $x \in Js(M)$, it is clear that $r x \in Js(M)$. Hence, $Js(M)$ is a submodule of $M$.

2. Follows from (1) and $A_{sa} \subseteq Js(M)$.

3. Since $A_{sa} \subseteq Js(M)$, then $A_{sa} > \subseteq Js(M)$. Clearly, $Js(M) \subseteq A_{sa}$. Then $Js(M)$ is generated by $A_{sa}$.

**Proposition 2.15:** Let $M$ be an $R$-module such that $A_{sa} \neq \phi$. Then the following statements are equivalent:
1. A\textsubscript{sa} is closed under addition; that is, a finite sum of sa-small elements is sa-small.

2. J\textsubscript{sa}(M) = A\textsubscript{sa}.

**Proof:**

(1) \implies (2) Let a\textsubscript{1} + a\textsubscript{2} + \cdots + a\textsubscript{n} \in J\textsubscript{sa}(M), a\textsubscript{i} \in A\textsubscript{sa} for each i = 1, \ldots, n, then R\textsubscript{ai} \ll A\textsubscript{sa} by proposition (2.4). Hence a\textsubscript{i} \in A\textsubscript{sa} for each i = 1, \ldots, n, by the assumption in (1) we get that a\textsubscript{1} + \cdots + a\textsubscript{n} \in A\textsubscript{sa}. Thus J\textsubscript{sa}(M) \subseteq A\textsubscript{sa} and hence J\textsubscript{sa}(M) = A\textsubscript{sa}.

(2) \implies (1) Let x, y \in A\textsubscript{sa} since A\textsubscript{sa} \subseteq J\textsubscript{sa}(M) then x, y \in J\textsubscript{sa}(M) and by using proposition (2.14) we have x + y \in J\textsubscript{sa}(M). Hence, x \in A\textsubscript{sa} , but A\textsubscript{sa} is closed under addition. We can prove that a finite sum of sa-small elements is sa-small by the use of induction.

**Proposition 2.16:** Let M be an R-module such that A\textsubscript{sa} \neq \phi. Then, the following statements hold:

1. \text{J}\textsubscript{sa}(M) is a sa-small submodule of M.
2. If K and L are sa-small submodules of M, then K + L is an sa-small submodule of M.
3. A\textsubscript{sa} is closed under addition; that is, sum of sa-small elements of M is sa-small.
4. \text{J}\textsubscript{sa}(M) = A\textsubscript{sa}.

**Proof:**

(1) \implies (2) Let K, L be sa-small in M, then K + L \subseteq \text{J}\textsubscript{sa}(M) which is sa-small by assumption. Thus by using proposition (2.4) we get K + L \ll M.

(2) \implies (3) Let x, y \in A\textsubscript{sa}, then Rx, Ry are sa-small in M, and hence by (2) Rx + Ry is sa-small in M. But R(x + y) = Rx + Ry and using proposition (2.4) we get R(x + y) \ll M. Hence, x + y \in A\textsubscript{sa}.

(3) \implies (4) By proposition (2.14).

Now, let M be finitely generated. We prove \(2 \implies 1\). Consider \{m\textsubscript{1}, Z\textsubscript{2}, \ldots, m\textsubscript{n}\} to be the set of generators of M. Let X be a submodule of M such that \text{J}\textsubscript{sa}(M) + X = M, then m\textsubscript{i} = a\textsubscript{i} + \text{X} such that a\textsubscript{i} \in J\textsubscript{sa}(M) and \text{X} \subseteq X for each i = 1, \ldots, n. Thus \Sigma R\textsubscript{m\textsubscript{i}} = \Sigma R\textsubscript{a\textsubscript{i}} = 1 + \Sigma R\textsubscript{X} \textsubscript{i} = 1 and hence M = \Sigma R\textsubscript{X} + 1 + X. Now, since a\textsubscript{i} \in J\textsubscript{sa}(M) and since (2) \implies (3) \implies (4) we get J\textsubscript{sa}(M) = A\textsubscript{sa}. That is, a\textsubscript{i} \in A\textsubscript{sa} and hence R\textsubscript{a\textsubscript{i}} \ll M thus that \text{ann}(X) \subseteq \text{X} implies that J\textsubscript{sa}(M) \ll M.

**3. Semiannihilator-hollow module**

In this section we introduce the concept of semiannihilator-hollow modules and give basic properties of this module.

**Definition 3.1:** A nontrivial R-module M is called semiannihilator-hollow module if every proper submodule of M is sa-small in M.

**Examples and Remarks 3.2:**

1. Z as Z-module is sa-hollow module but it is not hollow.
2. Z\textsubscript{a} and Z\textsubscript{d} as Z-module are not sa-hollow modules.

The epimorphic image of sa-hollow module need not be sa-hollow as the following example shows:-

Consider Z and Z\textsubscript{d} as Z-modules and \pi: Z \rightarrow Z\textsubscript{d} be let be the natural epimorphism . Z as Z-module is sa-hollow module,\{0\} \ll Z\textsubscript{d}. But \pi(\{0\})=\{0\} is not sa-small in Z\textsubscript{d} , where Z\textsubscript{d} = Z + Z\textsubscript{d} and ann Z\textsubscript{d} = 4Z not small in Z.

**Proposition 3.3:** Let M and N be two R-modules and f:M \rightarrow N be an epimorphism. If N is sa-hollow module then M is sa-hollow module.

**Proof:** Let K be submodule of M, then f(K) is submodule of N and since N is sa-hollow module, f(K) is sa-small submodule, then f^{-1}(f(K)) \ll M by (Proposition 2.6), f^{-1}(f(K)) = K + ker f, K \subseteq K + ker f then by (Proposition 2.4), K is sa-small submodule of M.

**Corollary 3.4:** Let M be an R-module, K be submodule of M. If M/K is sa-hollow module then M is sa-hollow module. A submodule N of R-module M is called fully invariant submodule of M if f(N) \subseteq N, for every f \in \text{Hom}(M,M) A module M is called duo module if every submodule of M is fully invariant.

**Proposition 3.5:** Let M = M\textsubscript{1} \oplus M\textsubscript{2} be duo module. If M\textsubscript{1} and M\textsubscript{2} are sa-hollow modules, then M is sa-hollow module.

**Proof:** Let M\textsubscript{1} and M\textsubscript{2} be sa-hollow modules, and N\textsubscript{1} \oplus N\textsubscript{2} be a proper Submodule of M\textsubscript{1} \oplus M\textsubscript{2}.

N\textsubscript{1} \subseteq M\textsubscript{1}\text{and N2} \subseteq M\textsubscript{2} , then N\textsubscript{1} and N\textsubscript{2} are sa-small submodules of M\textsubscript{1} and M\textsubscript{2} respectively thus by (Proposition 2.7) N\textsubscript{1} \oplus N\textsubscript{2} is sa-small submodule of M.

**Corollary 3.6:** Let M = M\textsubscript{1} \oplus M\textsubscript{2} be an R-module such that M = \text{ann}(M\textsubscript{1}) + \text{ann}(M\textsubscript{2}). If M\textsubscript{1} and M\textsubscript{2} are sa-hollow, then so is M.

We call a ring R is sa-hollow, if R is sa-hollow R-module. A module M is called multiplication module, if for every submodule N of M there exists an ideal I of R such that N = IM = (N:M)M [4].

**Proposition 3.7:** Let M be a multiplication R-module. If M is sa-hollow then R is a sa-hollow ring.

**Proof:** Suppose that M is sa-hollow. Let I be an ideal of R. Then IM is a submodule of M and hence IM is sa-small(Prop.2.6). Then I is sa-small ideal of R and hence R is sa-hollow.

**References**

