

Semiannihilator Small Submodules

Sahira M. Yaseen

Department of Mathematics, College of Science, Baghdad University, Baghdad, Iraq

Abstract: Let R be an associative ring with non-zero identity and M be a left R -module. A submodule N of M is called semiannihilator small (sa-small), if for every submodule L of M with $N+L=M$, then $\text{ann}(L) \ll R$. The properties of sa-small submodules have been studied. The sum of sa-small submodules is studied. Moreover, we introduce the concepts semiannihilator -hollow modules. We give many properties related with this type of modules.

Keywords: semiannihilator small submodule, semiannihilator -hollow modules

1. Introduction

Let R an associative ring with non-zero identity, M a left R -module. A submodule N of M is called small, if for every submodule K of M with $N+K=M$, then $K=M$ [1]. Recently, many authors have been interested in studying different kinds of annihilator small submodules as in [2] and [3], where the authors in [3] introduced the concept of R -annihilator small submodules, that is; a submodule N of an R -module M is called R -annihilator small, if whenever $N+K=M$, where K a submodule of M ; then $\text{ann}_R(K)=0$, where $\text{ann}_R(T) = \{r \in R : r.T=0\}$. This has led us to introduce the concept of semiannihilator small submodules, in way that a submodule N of M is called semiannihilator small (sa-small) in case $\text{ann}(K) \ll R$ where K is a submodule of M whenever $N+K=M$. It is clear that every R -annihilator small submodule is sa-small, but the converse is not true generally as examples can show next.

In this paper we define a subset of M that consists of all semiannihilator small elements by sa and we shall denote the sum of all semiannihilator small submodules of M by $Jsa(M)$, and study its properties. Finally, we shall introduce the concept of semiannihilator -hollow modules as a generalization of hollow modules.

2. Semi Annihilator – Small Submodules

In this section we introduce the concept of semiannihilator-small submodules and give characterizations, properties of this class of submodules.

Definition 2.1 A submodule N of a R -module M is called semiannihilator – small in M (sa-small) if $N+X=M$, X a submodule of M , implies that $\text{ann}X \ll R$, we write $N \ll_{sa} M$ in this case.

Examples and remarks 2.2:-

- 1) sa-small submodule need not be small, For example, consider Z as Z - module, every proper submodule of Z is sa-small. But $\{0\}$ is the only small submodule of Z .
- 2) It is clear that R -a-small submodules are sa-small submodules.
- 3) If M is a faithful R -module then every small submodules are R -annihilator small submodules and then are sa-small submodules.

- 4) Let R be a smpil ring and let M be a R - module .Then every proper submodule of M is sa-small in M .
- 5) There are sa- small submodules that are direct summands as in the $\mathbb{Z}2$ -module $M=\mathbb{Z}2 \oplus \mathbb{Z}2$, where it is clear that $A=\mathbb{Z}2 \oplus (0)$ is a direct summand of M , $M=A \oplus \mathbb{Z}2=A \oplus \langle (1,1) \rangle$ and $\text{ann}(\mathbb{Z}2) \ll \mathbb{Z}2$.

Since the ideal A of R is small in R iff $A \subseteq J(R)$, where $J(R)$ is the jacobson radical of R . The following is a characterizations of sa-small submodules:

Proposition 2.3: A submodule N of a module M is sa- small in M if $N+X=M$, X a submodule of M , implies that $\text{ann}X \ll J(R)$.

Proposition 2.4: Let M be an R -module with submodules $A \subseteq N$. If $N \ll_{sa} M$ then $A \ll_{sa} M$.

Proof: Let X be a submodule of M such that $A+X=M$, since $A \subseteq N$ hence $N+X=M$. N is sa-small in M then $\text{ann}X \ll R$ and hence $A \ll_{sa} M$.

Proposition 2.5: Let M be a R -module with submodules $A \subseteq N$, if $A \ll_{sa} N$ then $A \ll_{sa} M$.

Proof: Let X be any submodule of M such that $A+X=M$, now $N \cap M = N \cap (A+X)$ implies that $N = A + (N \cap X)$ by the modular law. Since $A \ll_{sa} N$, thus $\text{ann}(N \cap X) \ll R$. But $\text{ann}(X) \subseteq \text{ann}(N \cap X)$ then $\text{ann}(X) \ll R$, hence $X \ll_{sa} M$.

Proposition 2.6: Let I be an ideal of commutative ring R and let M be an R - module if IM is a sa-submodule in M , then I is sa- ideal of R .

Proof: Let $R=I+J$, J be an ideal of R . Then $M=RM=(I+J)M=IM+JM$, since IM is a sa-submodule in M , then $\text{ann}(JM) \ll R$, but $\text{ann}J \subseteq JM$, then $\text{ann}J \ll R$ thus I is sa-ideal of R .

Proposition 2.7: let M and N be R - modules and $f: M \rightarrow N$ be an epimorphism. If $H \ll_{sa} N$, then $f^{-1}(H) \ll_{sa} M$.

Proof: Let $M = f^{-1}(H)+X$, $X \subseteq M$. Then $f(M) = f(f^{-1}(H)+X) = f(f^{-1}(H))+f(X)$. Since f is an epimorphism, then $N = H + f(X)$. But $H \ll_{sa} N$, therefor $\text{ann}f(X) \ll R$. Clearly that $\text{ann}X \subseteq \text{ann}f(X)$ and hence $\text{ann}X \ll R$. Thus $f^{-1}(H) \ll_{sa} M$.

Note . Let $f:M \rightarrow N$ be an epimorphism . Then the image of sa-small submodule of M need not be sa-small in N as the following example shows:-

Consider Z_4 and Z as Z - modules and let $\pi :Z \rightarrow Z_4$ be the natural epimorphism , $\{0\} \ll_{sa} Z$. But $\pi (\{0\})=0$ is not s-a-small in Z_4 , where $Z_4=0 + Z_4$ and $\text{ann } Z_4 =4Z$ is not small in R .

Note . The sum of two sa-small submodules of a module M need not be sa-small submodule .For example , In Z as Z -module. Each of $2Z$ and $3Z$ are sa-small submodule of Z .But $Z=3Z+2Z$ is not sa-small in Z where $Z =Z +0$ and $\text{ann}0=Z$ is not small in Z .

We prove the following:

Proposition 2-8:- Let M_1, M_2 be a R - modules . If $N_1 \ll_{sa} M_1$ and $N_2 \ll_{sa} M_2$ thus $N_1 \oplus N_2 \ll_{sa} M_1 \oplus M_2$.

Proof:- Let $p_i: M_1 \oplus M_2 \rightarrow M_i$ be the projection maps .Since $N_1 \ll_{sa} M_1$ and $N_2 \ll_{sa} M_2$, then by prop(2.5), $N_1 \oplus M_2 = P_1^{-1}(N_1) \ll_{sa} M_1 \oplus M_2$. and $M_1 \oplus N_2 = P_2^{-1}(N_2) \ll_{sa} M_1 \oplus M_2$, and $(N_1 \oplus M_2) \cap (M_1 \oplus N_2) = N_1 \oplus N_2 \ll_{sa} M_1 \oplus M_2$.

Theorem 2.9: Let $M = \sum Rm_i$, be an R -module and $d \in M$. Then the following statements are equivalent:

1. $Rd \ll_{sa} M$.
2. $\cap i \in I \text{ann}(m_i - r_i d) \ll R$ for each $r_i \in R$.
3. There exists $j \in I$ such that $rm_j \notin Rrd$ for all $0 \neq r \in J(R)$.

Proof: (1) \Rightarrow (2) For each $i \in I$, $m_i = m_i - rid + rid$ and hence $M = \sum R(m_i - r_i m_i) + Rd$. By (1) we have $Rm \ll_{sa} M$ then $\text{ann}(\sum R(m_i - r_i m_i)) = \cap i \text{ann}(m_i - rid) \ll R$.

(2) \Rightarrow (1) Let L be a submodule of M with $L + Rd = M$. Then for each $i \in I$ $m_i = x_i + rid$, $x_i \in L$ and $x_i \in L$. Let $t \in \text{ann}(L)$, then $tm_i = trid + tx_i$ since $tx_i = 0$ then $t(m_i - r_i d) = 0$ for each $i \in I$, then $t \in \text{ann}(m_i - r_i d)$, hence $t \in \cap i \in I \text{ann}(m_i - r_i d) \ll R$, then $\text{ann}(L) \ll R$.

2) \Rightarrow (3) Let $r \notin J(R)$ and assume that $rm_i \in Rrd$ for all $i \in I$. Then $rm_i = rird = rrid$ for all $i \in I$, so by (1) $r \in \cap i \in I \text{ann}(m_i - rid) \ll R$ which is a contradiction.

(3) \Rightarrow (2) Let $r \in \cap i \in I \text{ann}(m_i - rid)$ and hence $r \in \text{ann}(m_i - rid)$ for all $i \in I$. Thus $rm_i = rrid = rird$ for all $i \in I$, so $rm_i \in Rrd$. By (3) $r \in J(R)$ then $\cap i \in I \text{ann}(m_i - rid) \subseteq J(R)$ so $\cap i \in I \text{ann}(m_i - r_i d) \ll R$.

Theorem 2.10: Let R be a commutative ring, $M = \sum Rm_i$ and N a submodule of M . Then the following statements are equivalent:

1. $N \ll_{sa} M$.
2. $\cap i \in I \text{ann}(m_i - xi) \ll R$ for all $xi \in N$.

Proof: (1) \Rightarrow (2) For each $i \in I$, let $xi \in N$. Then $m_i = m_i - xi + xi$ for each $i \in I$. Then $M = \sum R(m_i - xi) + N$, by (1) $N \ll_{sa} M$, then $\text{ann}(\sum R(m_i - xi)) = \cap i \in I \text{ann}(R(m_i - xi)) \ll R$.

(2) \Rightarrow (1) Let $M = A + N$. Then for each $i \in I$ $m_i = ai + xi$ where $ai \in A$ and $xi \in N$. Hence $ai = m_i - xi$ for each $i \in I$ and

$M = \sum (m_i - xi) + N$. Now, let $t \in \text{ann}(A)$ then $tai = t(m_i - xi)$ for each $i \in I$ and hence $t \in \text{ann}(R(m_i - xi)) \ll R$ by (2), so $\text{ann}(A) \subseteq \text{ann}(R(m_i - xi))$. Thus $N \ll_{sa} M$.

As the same of the definition of Jacobson radical related to small submodules, we will state a definition related to sa-small submodules in the following. But first we need this definition.

Definition 2.11: Let M be an R -module and $a \in M$. An element a in M is semiannihilator small if Ra is semiannihilator small submodule of M . Let $A_{sa} = \{a \in M | Ra \ll_{sa} M\}$.

The set A_{sa} is not a submodule of M . In fact, it is not closed under addition, for example in the \mathbb{Z} -module \mathbb{Z} we have that $4, -3 \in A_{sa}$ but $4 - 3 = 1 \notin A_{sa}$.

We can see by the use of proposition (2.4) that if M is an R -module and $a \in A_{sa}$, then $Ra \subseteq A_{sa}$.

Now we can prove the following:

Remark 2.12:- Let M be a module and N be sa- small submodule of M . , then $N \subseteq A_{sa}$.

Proof:- Let $x \in N$, N is sa-small submodule then Rx is sa-small submodule of M by prop(2.4) .

Thus $x \in A_{sa}$.

Since the sum of sa-small submodules is not sa-small (consider $3Z + 2Z$ in Z) we define.

Definition 2.13: Let M be an R -module. Denote $J_s(M)$ for the sum of all sa- small submodules of M . If M has no sa-small submodule , we write $J_{sa}(M) = M$.

It is clear that $A_{sa} \subseteq J_s(M)$ in every module , but this may not be equality (Z as Z -module) .

Proposition 2.14: Let M be an R -module such that $A_{sa} \neq \emptyset$, then we have the following:

- 1) $J_s(M)$ is a submodule of M and contains every sa- small submodule of M .
- 2) $J_s(M) = \{a_1 + a_2 + \dots + a_n; a_i \in A_{sa} \text{ for each } i, n \geq 1\}$.
- 3) $J_s(M)$ is generated by A_{sa} .

Proof:

1. Let $\{N_\alpha | \alpha \in \Lambda\}$ be the set of all sa- small submodules of M , thus $J_s(M) = \sum N_\alpha$, $\alpha \in \Lambda$. Let $x, y \in J_s(M)$, this means that $x = \sum \alpha x_\alpha$, $\alpha \in \Lambda$ and $y = \sum \alpha y_\alpha$, $\alpha \in \Lambda$ where $x_\alpha, y_\alpha \in N_\alpha$ for each $\alpha \in \Lambda$ and $x_\alpha, y_\alpha \neq 0$ for at most a finite number of $\alpha \in \Lambda$. Then $x + y = \sum (\alpha x_\alpha + \alpha y_\alpha)$ $\alpha \in \Lambda$ such that $\alpha x_\alpha + \alpha y_\alpha \in N_\alpha$ for each $\alpha \in \Lambda$, $x + y \in J_s(M)$. Now, let $r \in R$ and $x \in J_s(M)$, it is clear that $rx \in J_s(M)$. Hence, $J_s(M)$ is a submodule of M .

2. Follows from (1) and $A_{sa} \subseteq J_s(M)$.

3. Since $A_{sa} \subseteq J_s(M)$, then $\langle A_{sa} \rangle \subseteq J_s(M)$. Clearly, $J_s(M) \subseteq \langle A_{sa} \rangle$. Then $J_s(M)$ is generated by A_{sa} .

Proposition 2.15: Let M be an R -module such that $A_{sa} \neq \emptyset$. Then the following statements are equivalent:

1. A_{sa} is closed under addition; that is, a finite sum of sa-small elements is sa-small.
2. $J_s(M) = A_{sa}$.

Proof:

(1) \Rightarrow (2) Let $a_1 + a_2 + \dots + a_n \in J_s(M)$, $a_i \in A_i$ $i=1, \dots, n$, A_i is sa-small in M for each $i=1, \dots, n$. then $Ra_i \ll_{sa} M$ by proposition (2.4). Hence $a_i \in A_{sa}$ for each $i=1, \dots, n$, by the assumption in (1) we get that $a_1 + \dots + a_n \in A_{sa}$. thus $J_s(M) \subseteq A_{sa}$ and hence $J_s(M) = A_{sa}$.
 (2) \Rightarrow (1) Let $x, y \in A_{sa}$, since $A_{sa} \subseteq J_s(M)$ then $x, y \in J_s(M)$ and by using proposition (2.14) we have $x+y \in J_s(M)$. Hence, $x+y \in A_{sa}$, but A_{sa} is closed under addition. We can prove that a finite sum of sa-small elements is sa-small by the use of induction.

Proposition 2.16: Let M be an R -module such that $A_{sa} \neq \phi$. consider the following statements:

- 1) $J_s(M)$ is an sa-small submodule of M .
- 2) If K and L are sa-small submodules of M , then $K+L$ is an sa-small submodule of M .
- 3) A_{sa} is closed under addition; that is, sum of sa-small elements of M is sa-small.
- 4) $J_s(M) = A_{sa}$.

Then (1) \Rightarrow (2) \Rightarrow (3) \Leftrightarrow (4). If M is finitely generated, then (1) \Leftrightarrow (2).

Proof:

(1) \Rightarrow (2) Let K, L be sa-small in M , then $K+L \subseteq J_s(M)$ which is sa-small by assumption. Thus by using proposition (2.4) we get $K+L \ll_{sa} M$.

(2) \Rightarrow (3) Let $x, y \in A_{sa}$, then Rx, Ry are sa-small in M , and hence by (2) $Rx+Ry$ is sa-small in M . But $R(x+y) \subseteq Rx+Ry$ and by using proposition (2.4) we get $R(x+y) \ll_{sa} M$. Hence, $x+y \in A_{sa}$.

(3) \Leftrightarrow (4) By proposition (2.14).

Now, let M be finitely generated to prove (2) \Rightarrow (1). Consider $\{m_1, 2, \dots, m_n\}$ to be the set of generators of M . Let X be a submodule of M such that $J_s(M)+X=M$, then $mi=ai+xi$ such that $ai \in J_s(M)$ and $xi \in X$ for each $i=1, \dots, n$. Thus $\sum Rmini=1 = \sum Raini=1 + \sum Rxini=1$ and hence $M = \sum Raini=1 + X$. Now, since $ai \in J_s(M)$ and since (2) \Rightarrow (3) \Leftrightarrow (4) we get $J_s(M) = A_{sa}$; that is, $ai \in A_{sa}$ and hence $Rai \ll_{sa} M$ thus $\text{ann}(X) \ll R$ implies that $J_s(M) \ll_{sa} M$.

3. Semiannihilator-hollow module

In this section we introduce the concept of semiannihilator-hollow modules and give basic properties of this module.

Definition 3.1: A nontrivial R -module M is called semiannihilator -hollow (sa-hollow) if every proper submodule of M is sa-small in M .

Examples and Remarks 3.2 :

- (1) Z as Z - module is sa-hollow module but it is not hollow.
- (2) Z_6 and Z_4 as Z - module are not sa-hollow modules.

The epimorphic image of sa-hollow module need not be sa-hollow as the following example shows:-

Consider Z and Z_4 as Z - modules and $\pi: Z \rightarrow Z_4$ let be the natural epimorphism. Z as Z - module is sa-hollow module, $\{0\} \ll_{sa} Z$. But $\pi(\{0\}) = \{0\}$ is not sa-small in Z_4 , where $Z_4 = 0 + Z_4$ and $\text{ann } Z_4 = 4Z$ not small in Z .

Proposition 3.3:- Let M and N be two R - modules and $f: M \rightarrow N$ be an epimorphism. If N is sa-hollow module then M is sa-hollow module.

Proof:- Let K be submodule of M , then $f(K)$ is submodule of N and since N is sa-hollow module, $f(K)$ is sa-small submodule, then $f^{-1}(f(K)) \ll_{sa} M$ by (Proposition 2.6), $f^{-1}(f(K)) = K + \ker f$, $K \leq K + \ker f$ then by (Proposition 2.4), K is sa-small submodule of M .

Corollary 3.4: Let M be R - module, K be submodule of M . If M/K is sa-hollow module then M is sa-hollow module.

A submodule N of R -module M is called fully invariant submodule of M if $f(N) \subseteq N$, for every $f \in \text{Hom}(M, M)$. A module M is called duo module if every submodule of M is fully invariant [3].

Proposition 3.5: Let $M = M_1 \oplus M_2$ be duo module. If M_1 and M_2 are sa-hollow modules, then M is sa-hollow module.

Proof: Let M_1 and M_2 be sa-hollow modules, and $N_1 \oplus N_2$ be a proper Submodule of $M_1 \oplus M_2$

$N_1 \leq M_1$ and $N_2 \leq M_2$, then N_1 and N_2 are sa-small submodules of M_1, M_2 respectively thus by (Proposition 2.7) $N_1 \oplus N_2$ is sa-small submodule of M .

Corollary 3.6: Let $M = M_1 \oplus M_2$ be an R - module such that $R = \text{ann}(M_1) + \text{ann}(M_2)$. If M_1 and M_2 are sa-hollow, then so is M .

We call a ring R is sa-hollow, if R is sa-hollow R -module. A module M is called multiplication, if for every submodule N of M there exists an ideal I of R such that $N = IM = (N:M)M$ [4].

Proposition 3.7 : Let M be a multiplication R -module. If M is sa-hollow then R is a sa-hollow ring.

Proof: Suppose that M is sa-hollow. Let I be an ideal of R . Then IM is a submodule of M and hence IM is sa-small (Prop. 2.6). Then I is sa-small ideal of R and hence R is sa-hollow.

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