Generalization of Rough Set Theory Using a Finite Number of a Finite d. g.'s

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Abstract: ***This paper is concerned with introducing and studying the new approximation operators based on a finite family of d. g. 'swhich are the core concept in this paper. In addition, we study generalization of some Pawlak's concepts and we offer generalize the definition of accuracy measure of approximations by using a finite family of d. g. 's.***

Keywords: Digraph, Mixed degree set, *n*-lower approximation, *n*-upper approximation and *n*-accuracy measure.

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1.Introduction and preliminaries

Rough set theory was developed by Pawlak in 1982 [8], since then it has been widely applied in many fields, such as machine learning data mining and pattern recognition and the original rough set theory was developed framework of set theory algebra and logic. We relied on proposition in [21] and we built on some of the results in [1], [3], [4], [6], [7], [10], [11], [12], [13], [14], [15], [17], [18], [19], [20] and [22].

A directed graph (d. g.) [16] is pair $D = (V(D), E(D))$ where $V(D)$ is a non-empty set (called vertex set) and $E(D)$ of ordered pairs of elements of $V(D)$ (called edge set). An edge of the from (ϖ, ϖ) is called a loop. If $\varpi \in V(D)$, the out-degree of ϖ is $|\{u \in V(D) : (\varpi, u) \in E(D)\}|$ and in-degree of ϖ is $|\{u \in V(D) : (u, \varpi) \in E(D)\}|$. A subd. g. of a d. g. D is a d. g. each of whose vertices belong to $V(D)$ and each of whose edges belong to $E(D)$. An empty d. g. [2] if the vertices set and edge set is empty. The out-degree set of ϖ is denoted by $OD(\varpi)$ and defined by: $OD(\varpi) = \{u \in V(D) : (\varpi, u) \in E(D)\}$ and in-degree set of ϖ is denoted by $ID(\varpi)$ and defined by: $ID(\varpi) = \{u \in V(D) : (u, \varpi) \in E(D)\}$.Let $D = (V(D), E(D))$ be a d. g. The mixed degree system of a vertex $\varpi \in V(D)$ is denoted by $MDS(\varpi)$ and defined by: $MDS(\varpi) = \{ODS(\varpi), IDS(\varpi)\}$. Let $D = (V(D), E(D))$ be a d. g. the mixed degree of a vertex $\varpi \in V(D)$ is denoted by $MD(\varpi)$ such that $MD(\varpi) \in MDS(\varpi)$.The lower and upper approximations of H using mixed degree systems are denoted by $L_m(V(H))$ and $U_m(V(H))$ and defined by $L_m(V(H)) = \{\varpi \in V(D)$; for some $MD(\varpi) \subseteq V(H)\}$ and $U_m(V(H)) = \{\varpi \in V(D)$; for all $MD(\varpi) \cap V(H) \neq \phi\}$[21].

2. New approximation operators based on a finite family of d. g. 's

In this section, some of their definitions and propositions about new approximation operators on a family of d. g. 's are studied and we gave examples in the case of properties that are not true in general.

Definition 2.1. Let $\boldsymbol{D} = \{D_i;\ i = 1, 2, 3, \dots, n\}$ be a finite family of arbitrary non-empty finite d. g. 's. The *n*-lower and *n*-upper approximations of $H \subseteq D$ according to $\boldsymbol{D}$ are denoted by $L_n(V(H))$ and $U_n(V(H))$, respectively and defined by:

$$L_n(V(H)) = \cup_{i=1}^{n} L_{mi}(V(H)),\quad U_n(V(H)) = \cap_{i=1}^{n} U_{mi}(V(H)).$$

Definition 2.2. Let $\boldsymbol{D} = \{D_i;\ i = 1, 2, 3, \dots, n\}$ be a finite family of arbitrary non-empty finite d. g. 's. The *n*-boundary, *n*-positive and *n*-negative of $H \subseteq D$ according to $\boldsymbol{D}$ are denoted by $Bd_n(V(H))$, $BOS_n(V(H))$, respectively and $NEG_n(V(H))$ anddefined by:

$$Bd_n(V(H)) = U_n(V(H)) - L_n(V(H)),$$

$$BOS_n(V(H)) = L_n(V(H)),$$

$$NEG_n(V(H)) = V(D) - U_n(V(H)).$$

Example 2.3. Let $\boldsymbol{D} = \{D_i;\ i = 1, 2, 3\}$ be three d. g. 's defined as:$V(D) = V(D_1) = V(D_2) = V(D_3) = \{\varpi_1, \varpi_2, \varpi_3, \varpi_4, \varpi_5\}$, $E(D_1) = \{(\varpi_1, \varpi_1), (\varpi_1, \varpi_2), (\varpi_1, \varpi_5), (\varpi_2, \varpi_3), (\varpi_2, \varpi_4), (\varpi_3, \varpi_1), (\varpi_3, \varpi_3), (\varpi_5, \varpi_2), (\varpi_5, \varpi_4), (\varpi_5, \varpi_5)\}$, $E(D_2) = \{(\varpi_1, \varpi_1), (\varpi_1, \varpi_5), (\varpi_2, \varpi_3), (\varpi_2, \varpi_4), (\varpi_3, \varpi_1), (\varpi_3, \varpi_3), (\varpi_5, \varpi_2), (\varpi_5, \varpi_4), (\varpi_5, \varpi_5)\}$ and $E(D_3) = \{(\varpi_1, \varpi_4), (\varpi_1, \varpi_5), (\varpi_2, \varpi_2), (\varpi_2, \varpi_3), (\varpi_3, \varpi_1), (\varpi_3, \varpi_3), (\varpi_3, \varpi_5), (\varpi_4, \varpi_1), (\varpi_4, \varpi_4)\}$.

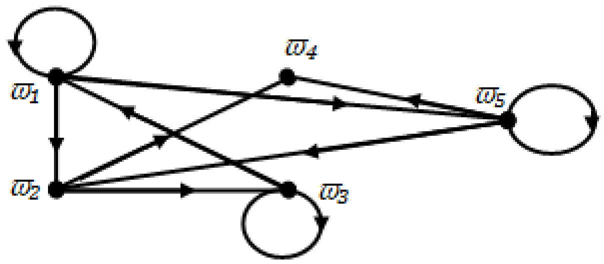

Figure 2.1: d. g. D_1.

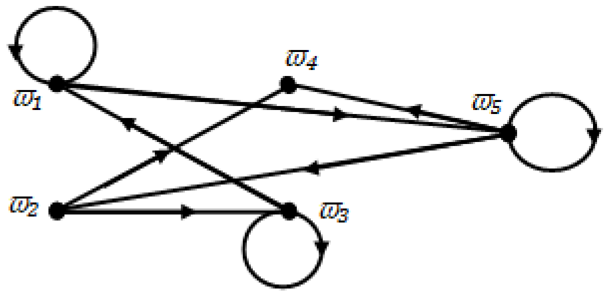

Figure 2.2: d. g. D_2.

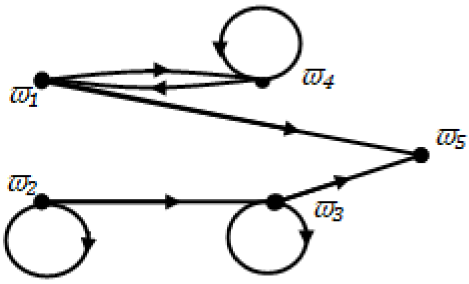

Figure 2.3: d. g. D_3.

The mixed degree systems based on D_1 are given by:

$MD_{m_1}(\varpi_1) = \{\{\varpi_1, \varpi_2, \varpi_5\}, \{\varpi_1, \varpi_3\}\}$, $MD_{m_1}(\varpi_2) = \{\{\varpi_3, \varpi_4\}, \{\varpi_1, \varpi_5\}\}$,$MD_{m_1}(\varpi_3) = \{\{\varpi_1, \varpi_3\}, \{\varpi_2, \varpi_3\}\}$,$MD_{m_1}(\varpi_4) = \{\phi, \{\varpi_2, \varpi_5\}\}$ and$MD_{m_1}(\varpi_5) = \{\{\varpi_2, \varpi_4, \varpi_5\}, \{\varpi_1, \varpi_5\}\}$.

The mixed degree systems based on D_2 are given by:

$MD_{m_2}(\varpi_1) = \{\{\varpi_1, \varpi_5\}, \{\varpi_1, \varpi_3\}\}$,$MD_{m_2}(\varpi_2) = \{\{\varpi_3, \varpi_4\}, \{\varpi_5\}\}$,$MD_{m_2}(\varpi_3) = \{\{\varpi_1, \varpi_3\}, \{\varpi_2, \varpi_3\}\}$,$MD_{m_2}(\varpi_4) = \{\phi, \{\varpi_2, \varpi_5\}\}$ and $MD_{m_2}(\varpi_5) = \{\{\varpi_2, \varpi_4, \varpi_5\}, \{\varpi_1, \varpi_5\}\}$.

The mixed degree systems based on D_3 are given by:

$MD_{m_3}(\varpi_1) = \{\{\varpi_4, \varpi_5\}, \{\varpi_4\}\}$,$MD_{m_3}(\varpi_2) = \{\{\varpi_2, \varpi_3\}, \{\varpi_2\}\}$,$MD_{m_3}(\varpi_3) = \{\{\varpi_3, \varpi_5\}, \{\varpi_2, \varpi_3\}\}$, $MD_{m_3}(\varpi_4) = \{\{\varpi_1, \varpi_4\}\}$ and $MD_{m_3}(\varpi_5) = \{\phi, \{\varpi_1, \varpi_3\}\}$.

The lower approximation, for all $H \subseteq D$, are given in the table:

$V(H)$	$L_{m_1}(V(H))$	$L_{m_2}(V(H))$	$L_{m_3}(V(H))$	$L_n(V(H))$
$\{\varpi_1\}$	$\{\varpi_4\}$	$\{\varpi_4\}$	$\{\varpi_5\}$	$\{\varpi_4, \varpi_5\}$
$\{\varpi_2\}$	$\{\varpi_4\}$	$\{\varpi_4\}$	$\{\varpi_2,\varpi_5\}$	$\{\varpi_2,\varpi_4, \varpi_5\}$
$\{\varpi_3\}$	$\{\varpi_4\}$	$\{\varpi_4\}$	$\{\varpi_5\}$	$\{\varpi_4, \varpi_5\}$
$\{\varpi_4\}$	$\{\varpi_4\}$	$\{\varpi_4\}$	$\{\varpi_1,\varpi_5\}$	$\{\varpi_1,\varpi_4, \varpi_5\}$
$\{\varpi_5\}$	$\{\varpi_4\}$	$\{\varpi_2,\varpi_4\}$	$\{\varpi_5\}$	$\{\varpi_2,\varpi_4, \varpi_5\}$
$\{\varpi_1,\varpi_2\}$	$\{\varpi_4\}$	$\{\varpi_4\}$	$\{\varpi_2,\varpi_5\}$	$\{\varpi_2,\varpi_4, \varpi_5\}$
$\{\varpi_1,\varpi_3\}$	$\{\varpi_1,\varpi_3, \varpi_4\}$	$\{\varpi_1,\varpi_3, \varpi_4\}$	$\{\varpi_5\}$	$\{\varpi_1,\varpi_3, \varpi_4, \varpi_5\}$
$\{\varpi_1,\varpi_4\}$	$\{\varpi_4\}$	$\{\varpi_4\}$	$\{\varpi_1,\varpi_4, \varpi_5\}$	$\{\varpi_2,\varpi_4, \varpi_5\}$
$\{\varpi_1,\varpi_5\}$	$\{\varpi_2,\varpi_4, \varpi_5\}$	$\{\varpi_1,\varpi_2, \varpi_4, \varpi_5\}$	$\{\varpi_5\}$	$\{\varpi_1,\varpi_2, \varpi_4, \varpi_5\}$
$\{\varpi_2,\varpi_3\}$	$\{\varpi_3, \varpi_4\}$	$\{\varpi_3, \varpi_4\}$	$\{\varpi_2,\varpi_3, \varpi_5\}$	$\{\varpi_2, \varpi_3, \varpi_4, \varpi_5\}$
$\{\varpi_2,\varpi_4\}$	$\{\varpi_4\}$	$\{\varpi_4\}$	$\{\varpi_1,\varpi_2, \varpi_5\}$	$\{\varpi_1,\varpi_2, \varpi_4, \varpi_5\}$
$\{\varpi_2,\varpi_5\}$	$\{\varpi_4\}$	$\{\varpi_2,\varpi_4\}$	$\{\varpi_2,\varpi_5\}$	$\{\varpi_2,\varpi_4, \varpi_5\}$
$\{\varpi_3, \varpi_4\}$	$\{\varpi_2,\varpi_4\}$	$\{\varpi_2,\varpi_4\}$	$\{\varpi_1,\varpi_5\}$	$\{\varpi_1,\varpi_2, \varpi_4, \varpi_5\}$
$\{\varpi_3, \varpi_5\}$	$\{\varpi_4\}$	$\{\varpi_2,\varpi_4\}$	$\{\varpi_3, \varpi_5\}$	$\{\varpi_2, \varpi_3, \varpi_4, \varpi_5\}$
$\{\varpi_4, \varpi_5\}$	$\{\varpi_4\}$	$\{\varpi_2,\varpi_4\}$	$\{\varpi_1,\varpi_5\}$	$\{\varpi_1,\varpi_2, \varpi_4, \varpi_5\}$
$\{\varpi_1, \varpi_2, \varpi_3\}$	$\{\varpi_1,\varpi_3, \varpi_4\}$	$\{\varpi_2,\varpi_3, \varpi_4\}$	$\{\varpi_2,\varpi_3, \varpi_5\}$	$V(D)$
$\{\varpi_1,\varpi_2, \varpi_4\}$	$\{\varpi_4\}$	$\{\varpi_4\}$	$\{\varpi_1,\varpi_2, \varpi_4, \varpi_5\}$	$\{\varpi_1,\varpi_2, \varpi_4, \varpi_5\}$
$\{\varpi_1,\varpi_2, \varpi_5\}$	$\{\varpi_1,\varpi_2, \varpi_4, \varpi_5\}$	$\{\varpi_1,\varpi_2, \varpi_4, \varpi_5\}$	$\{\varpi_2,\varpi_5\}$	$\{\varpi_1,\varpi_2, \varpi_4, \varpi_5\}$
$\{\varpi_1,\varpi_3, \varpi_4\}$	$\{\varpi_1,\varpi_2, \varpi_3, \varpi_4\}$	$\{\varpi_1,\varpi_2, \varpi_3, \varpi_4\}$	$\{\varpi_1,\varpi_4, \varpi_5\}$	$V(D)$
$\{\varpi_1,\varpi_3, \varpi_5\}$	$V(D)$	$V(D)$	$\{\varpi_3, \varpi_5\}$	$V(D)$
$\{\varpi_1,\varpi_4, \varpi_5\}$	$\{\varpi_2,\varpi_4, \varpi_5\}$	$\{\varpi_1,\varpi_2, \varpi_4, \varpi_5\}$	$\{\varpi_1,\varpi_4, \varpi_5\}$	$\{\varpi_1,\varpi_2, \varpi_4, \varpi_5\}$
$\{\varpi_2,\varpi_3, \varpi_4\}$	$\{\varpi_2,\varpi_3, \varpi_4\}$	$\{\varpi_2,\varpi_3, \varpi_4\}$	$\{\varpi_1, \varpi_2, \varpi_3, \varpi_5\}$	$V(D)$
$\{\varpi_2,\varpi_3, \varpi_5\}$	$\{\varpi_3, \varpi_4\}$	$\{\varpi_2,\varpi_3, \varpi_4\}$	$\{\varpi_2,\varpi_3, \varpi_5\}$	$\{\varpi_2, \varpi_3, \varpi_4, \varpi_5\}$
$\{\varpi_2,\varpi_4, \varpi_5\}$	$\{\varpi_4, \varpi_5\}$	$\{\varpi_2,\varpi_4, \varpi_5\}$	$\{\varpi_1,\varpi_2, \varpi_5\}$	$\{\varpi_1,\varpi_2, \varpi_4, \varpi_5\}$
$\{\varpi_3, \varpi_4, \varpi_5\}$	$\{\varpi_2,\varpi_4\}$	$\{\varpi_2,\varpi_4\}$	$\{\varpi_1,\varpi_3, \varpi_5\}$	$V(D)$
$\{\varpi_1,\varpi_2, \varpi_3, \varpi_4\}$	$\{\varpi_1,\varpi_2, \varpi_3, \varpi_4\}$	$\{\varpi_1,\varpi_2, \varpi_3, \varpi_4\}$	$V(D)$	$V(D)$
$\{\varpi_1, \varpi_2, \varpi_3, \varpi_5\}$	$V(D)$	$V(D)$	$\{\varpi_2,\varpi_3, \varpi_5\}$	$V(D)$
$\{\varpi_1,\varpi_2, \varpi_4, \varpi_5\}$	$\{\varpi_1,\varpi_2, \varpi_4, \varpi_5\}$	$\{\varpi_1,\varpi_2, \varpi_4, \varpi_5\}$	$\{\varpi_1,\varpi_2, \varpi_4, \varpi_5\}$	$\{\varpi_1,\varpi_2, \varpi_4, \varpi_5\}$
$\{\varpi_1,\varpi_3, \varpi_4, \varpi_5\}$	$V(D)$	$V(D)$	$\{\varpi_1,\varpi_3, \varpi_4, \varpi_5\}$	$V(D)$
$\{\varpi_2, \varpi_3, \varpi_4, \varpi_5\}$	$\{\varpi_2, \varpi_3, \varpi_4, \varpi_5\}$	$\{\varpi_2, \varpi_3, \varpi_4, \varpi_5\}$	$\{\varpi_1, \varpi_2, \varpi_3, \varpi_5\}$	$V(D)$
$V(D)$	$V(D)$	$V(D)$	$V(D)$	$V(D)$
ϕ	$\{\varpi_4\}$	$\{\varpi_4\}$	$\{\varpi_5\}$	$\{\varpi_4, \varpi_5\}$

The upper approximation, for all $H \subseteq D$, are given in the table:

$V(H)$	$U_{m_1}(V(H))$	$U_{m_2}(V(H))$	$U_{m_3}(V(H))$	$U_n(V(H))$
$\{\varpi_1\}$	$\{\varpi_1\}$	$\{\varpi_1\}$	$\{\varpi_4\}$	ϕ
$\{\varpi_2\}$	ϕ	ϕ	$\{\varpi_2\}$	ϕ
$\{\varpi_3\}$	$\{\varpi_3\}$	$\{\varpi_3\}$	$\{\varpi_3\}$	$\{\varpi_3\}$
$\{\varpi_4\}$	ϕ	ϕ	$\{\varpi_1,\varpi_4\}$	ϕ
$\{\varpi_5\}$	$\{\varpi_5\}$	$\{\varpi_5\}$	ϕ	ϕ
$\{\varpi_1,\varpi_2\}$	$\{\varpi_1,\varpi_3, \varpi_5\}$	$\{\varpi_1,\varpi_3, \varpi_5\}$	$\{\varpi_2,\varpi_4\}$	ϕ
$\{\varpi_1,\varpi_3\}$	$\{\varpi_1, \varpi_2, \varpi_3\}$	$\{\varpi_1,\varpi_3\}$	$\{\varpi_3, \varpi_4\}$	$\{\varpi_3\}$
$\{\varpi_1,\varpi_4\}$	$\{\varpi_1,\varpi_2, \varpi_5\}$	$\{\varpi_1,\varpi_5\}$	$\{\varpi_1,\varpi_4\}$	$\{\varpi_1\}$
$\{\varpi_1,\varpi_5\}$	$\{\varpi_1,\varpi_5\}$	$\{\varpi_1,\varpi_5\}$	$\{\varpi_4\}$	ϕ
$\{\varpi_2,\varpi_3\}$	$\{\varpi_1,\varpi_3\}$	$\{\varpi_3\}$	$\{\varpi_2,\varpi_3\}$	$\{\varpi_3\}$
$\{\varpi_2,\varpi_4\}$	ϕ	ϕ	$\{\varpi_1,\varpi_2, \varpi_4\}$	ϕ
$\{\varpi_2,\varpi_5\}$	$\{\varpi_5\}$	$\{\varpi_5\}$	$\{\varpi_2,\varpi_3\}$	ϕ

$\{\varpi_3, \varpi_4\}$	$\{\varpi_3\}$	$\{\varpi_3\}$	$\{\varpi_1, \varpi_3, \varpi_4\}$	$\{\varpi_3\}$
$\{\varpi_3, \varpi_5\}$	$\{\varpi_1, \varpi_2, \varpi_3, \varpi_5\}$	$\{\varpi_1, \varpi_2, \varpi_3, \varpi_5\}$	$\{\varpi_3\}$	$\{\varpi_3\}$
$\{\varpi_4, \varpi_5\}$	$\{\varpi_2, \varpi_5\}$	$\{\varpi_2, \varpi_5\}$	$\{\varpi_1, \varpi_4\}$	ϕ
$\{\varpi_1, \varpi_2, \varpi_3\}$	$\{\varpi_1, \varpi_2, \varpi_3, \varpi_5\}$	$\{\varpi_1, \varpi_3, \varpi_5\}$	$\{\varpi_2, \varpi_3, \varpi_4\}$	$\{\varpi_3\}$
$\{\varpi_1, \varpi_2, \varpi_4\}$	$\{\varpi_1, \varpi_2, \varpi_3, \varpi_5\}$	$\{\varpi_1, \varpi_3, \varpi_5\}$	$\{\varpi_1, \varpi_2, \varpi_4\}$	$\{\varpi_1\}$
$\{\varpi_1, \varpi_2, \varpi_5\}$	$\{\varpi_1, \varpi_3, \varpi_5\}$	$\{\varpi_1, \varpi_3, \varpi_5\}$	$\{\varpi_2, \varpi_3, \varpi_4\}$	$\{\varpi_3\}$
$\{\varpi_1, \varpi_3, \varpi_4\}$	$\{\varpi_1, \varpi_2, \varpi_3, \varpi_5\}$	$\{\varpi_1, \varpi_3, \varpi_5\}$	$\{\varpi_1, \varpi_3, \varpi_4\}$	$\{\varpi_1, \varpi_3\}$
$\{\varpi_1, \varpi_3, \varpi_5\}$	$\{\varpi_1, \varpi_2, \varpi_3, \varpi_5\}$	$\{\varpi_1, \varpi_2, \varpi_3, \varpi_5\}$	$\{\varpi_3, \varpi_4\}$	$\{\varpi_3\}$
$\{\varpi_1, \varpi_4, \varpi_5\}$	$\{\varpi_1, \varpi_2, \varpi_5\}$	$\{\varpi_1, \varpi_2, \varpi_5\}$	$\{\varpi_1, \varpi_4\}$	$\{\varpi_1\}$
$\{\varpi_2, \varpi_3, \varpi_4\}$	$\{\varpi_1, \varpi_3\}$	$\{\varpi_3\}$	$\{\varpi_1, \varpi_2, \varpi_3, \varpi_4\}$	$\{\varpi_3\}$
$\{\varpi_2, \varpi_3, \varpi_5\}$	$\{\varpi_1, \varpi_2, \varpi_3, \varpi_5\}$	$\{\varpi_1, \varpi_2, \varpi_3, \varpi_5\}$	$\{\varpi_2, \varpi_3\}$	$\{\varpi_2, \varpi_3\}$
$\{\varpi_2, \varpi_4, \varpi_5\}$	$\{\varpi_2, \varpi_5\}$	$\{\varpi_2, \varpi_5\}$	$\{\varpi_1, \varpi_2, \varpi_3, \varpi_4\}$	$\{\varpi_2\}$
$\{\varpi_3, \varpi_4, \varpi_5\}$	$\{\varpi_1, \varpi_2, \varpi_3, \varpi_5\}$	$\{\varpi_1, \varpi_2, \varpi_3, \varpi_5\}$	$\{\varpi_1, \varpi_3, \varpi_4\}$	$\{\varpi_1, \varpi_3\}$
$\{\varpi_1, \varpi_2, \varpi_3, \varpi_4\}$	$\{\varpi_1, \varpi_2, \varpi_3, \varpi_5\}$	$\{\varpi_1, \varpi_3, \varpi_5\}$	$\{\varpi_1, \varpi_2, \varpi_3, \varpi_4\}$	$\{\varpi_1, \varpi_3\}$
$\{\varpi_1, \varpi_2, \varpi_3, \varpi_5\}$	$\{\varpi_1, \varpi_2, \varpi_3, \varpi_5\}$	$\{\varpi_1, \varpi_2, \varpi_3, \varpi_5\}$	$\{\varpi_2, \varpi_3, \varpi_4\}$	$\{\varpi_2, \varpi_3\}$
$\{\varpi_1, \varpi_2, \varpi_4, \varpi_5\}$	$\{\varpi_1, \varpi_2, \varpi_3, \varpi_5\}$	$\{\varpi_1, \varpi_2, \varpi_3, \varpi_5\}$	$\{\varpi_1, \varpi_2, \varpi_3, \varpi_4\}$	$\{\varpi_1, \varpi_2, \varpi_3\}$
$\{\varpi_1, \varpi_3, \varpi_4, \varpi_5\}$	$\{\varpi_1, \varpi_2, \varpi_3, \varpi_5\}$	$\{\varpi_1, \varpi_2, \varpi_3, \varpi_5\}$	$\{\varpi_1, \varpi_3, \varpi_4\}$	$\{\varpi_1, \varpi_3\}$
$\{\varpi_2, \varpi_3, \varpi_4, \varpi_5\}$	$\{\varpi_1, \varpi_2, \varpi_3, \varpi_5\}$	$\{\varpi_1, \varpi_2, \varpi_3, \varpi_5\}$	$\{\varpi_1, \varpi_2, \varpi_3, \varpi_4\}$	$\{\varpi_1, \varpi_2, \varpi_3\}$
$V(D)$	$\{\varpi_1, \varpi_2, \varpi_3, \varpi_5\}$	$\{\varpi_1, \varpi_2, \varpi_3, \varpi_5\}$	$\{\varpi_1, \varpi_2, \varpi_3, \varpi_4\}$	$\{\varpi_1, \varpi_2, \varpi_3\}$
ϕ	ϕ	ϕ	ϕ	ϕ

The boundary according to D, for all $H \subseteq D$, are given in the table:

$V(H)$	$Bd_{m_1}(V(H))$	$Bd_{m_2}(V(H))$	$Bd_{m_3}(V(H))$	$Bd_n(V(H))$
$\{\varpi_1\}$	$\{\varpi_1\}$	$\{\varpi_1\}$	$\{\varpi_4\}$	ϕ
$\{\varpi_2\}$	ϕ	ϕ	ϕ	ϕ
$\{\varpi_3\}$	$\{\varpi_3\}$	$\{\varpi_3\}$	$\{\varpi_3\}$	$\{\varpi_3\}$
$\{\varpi_4\}$	ϕ	ϕ	$\{\varpi_4\}$	ϕ
$\{\varpi_5\}$	$\{\varpi_5\}$	$\{\varpi_5\}$	ϕ	ϕ
$\{\varpi_1, \varpi_2\}$	$\{\varpi_1, \varpi_3, \varpi_5\}$	$\{\varpi_1, \varpi_3, \varpi_5\}$	$\{\varpi_4\}$	ϕ
$\{\varpi_1, \varpi_3\}$	$\{\varpi_2\}$	ϕ	$\{\varpi_3, \varpi_4\}$	ϕ
$\{\varpi_1, \varpi_4\}$	$\{\varpi_1, \varpi_2, \varpi_5\}$	$\{\varpi_1, \varpi_5\}$	ϕ	ϕ
$\{\varpi_1, \varpi_5\}$	$\{\varpi_1\}$	ϕ	$\{\varpi_4\}$	ϕ
$\{\varpi_2, \varpi_3\}$	$\{\varpi_1\}$	ϕ	ϕ	ϕ
$\{\varpi_2, \varpi_4\}$	ϕ	ϕ	$\{\varpi_4\}$	ϕ
$\{\varpi_2, \varpi_5\}$	$\{\varpi_5\}$	$\{\varpi_5\}$	$\{\varpi_3\}$	ϕ
$\{\varpi_3, \varpi_4\}$	$\{\varpi_5\}$	$\{\varpi_3\}$	$\{\varpi_3, \varpi_4\}$	$\{\varpi_3\}$
$\{\varpi_3, \varpi_5\}$	$\{\varpi_1, \varpi_2, \varpi_3, \varpi_5\}$	$\{\varpi_1, \varpi_3, \varpi_5\}$	ϕ	ϕ
$\{\varpi_4, \varpi_5\}$	$\{\varpi_2, \varpi_5\}$	$\{\varpi_5\}$	$\{\varpi_4\}$	ϕ
$\{\varpi_1, \varpi_2, \varpi_3\}$	$\{\varpi_2, \varpi_5\}$	$\{\varpi_5\}$	$\{\varpi_4\}$	ϕ
$\{\varpi_1, \varpi_2, \varpi_4\}$	$\{\varpi_1, \varpi_2, \varpi_3, \varpi_5\}$	$\{\varpi_1, \varpi_3, \varpi_5\}$	ϕ	ϕ
$\{\varpi_1, \varpi_2, \varpi_5\}$	$\{\varpi_3\}$	$\{\varpi_3\}$	$\{\varpi_3, \varpi_4\}$	$\{\varpi_3\}$
$\{\varpi_1, \varpi_3, \varpi_4\}$	$\{\varpi_5\}$	$\{\varpi_5\}$	$\{\varpi_3\}$	ϕ
$\{\varpi_1, \varpi_3, \varpi_5\}$	ϕ	ϕ	$\{\varpi_4\}$	ϕ
$\{\varpi_1, \varpi_4, \varpi_5\}$	$\{\varpi_1\}$	ϕ	ϕ	ϕ
$\{\varpi_2, \varpi_3, \varpi_4\}$	$\{\varpi_1\}$	ϕ	$\{\varpi_4\}$	ϕ
$\{\varpi_2, \varpi_3, \varpi_5\}$	$\{\varpi_1, \varpi_2, \varpi_5\}$	$\{\varpi_1, \varpi_5\}$	ϕ	ϕ
$\{\varpi_2, \varpi_4, \varpi_5\}$	$\{\varpi_2\}$	ϕ	$\{\varpi_3, \varpi_4\}$	ϕ
$\{\varpi_3, \varpi_4, \varpi_5\}$	$\{\varpi_1, \varpi_3, \varpi_5\}$	$\{\varpi_1, \varpi_3, \varpi_5\}$	$\{\varpi_4\}$	ϕ
$\{\varpi_1, \varpi_2, \varpi_3, \varpi_4\}$	$\{\varpi_5\}$	$\{\varpi_5\}$	ϕ	ϕ
$\{\varpi_1, \varpi_2, \varpi_3, \varpi_5\}$	ϕ	ϕ	$\{\varpi_1, \varpi_4\}$	ϕ
$\{\varpi_1, \varpi_2, \varpi_4, \varpi_5\}$	$\{\varpi_3\}$	$\{\varpi_3\}$	$\{\varpi_3\}$	$\{\varpi_3\}$
$\{\varpi_1, \varpi_3, \varpi_4, \varpi_5\}$	ϕ	ϕ	ϕ	ϕ
$\{\varpi_2, \varpi_3, \varpi_4, \varpi_5\}$	$\{\varpi_1\}$	$\{\varpi_1\}$	$\{\varpi_4\}$	ϕ
$V(D)$	ϕ	ϕ	ϕ	ϕ
ϕ	ϕ	ϕ	ϕ	ϕ

Proposition 2.4. Let $\boldsymbol{D} = \{D_i;\, i = 1, 2, 3, \ldots, n\}$ be a finite family of arbitrary non-empty finite d. g. 's, then the following hold for every $H, K \subseteq D$:

(L_2) $L_n(V(D)) = V(D)$,

(L_4) If $V(H) \subseteq V(K)$, then $L_n(V(H)) \subseteq L_n(V(K))$,

(L_5) $L_n(V(H) \cap V(K)) \subseteq L_n(V(H)) \cap L_n(V(K))$,

(L_6) $L_n(V(H) \cup V(K)) \supseteq L_n(V(H)) \cup L_n(V(K))$,

(L_7) $L_n(V(H)) = V(D) - [U_n(V(D) - V(H))]$,

(U_3) $U_n(\phi) = \phi$,

(U_4) If $V(H) \subseteq V(K)$, then $U_n(V(H)) \subseteq U_n(V(K))$,

(U_5) $U_n(V(H) \cap V(K)) \subseteq U_n(V(H)) \cap U_n(V(K))$,

(U_6) $U_n(V(H) \cup V(K)) \supseteq U_n(V(H)) \cup U_n(V(K))$ and

(U_7) $U_n(V(H)) = V(D) - [L_n(V(D) - V(H))]$.

Proof.

(L_2) By (L_2) in Proposition (2.3.1) in [21], we have
$L_{mi}(V(D)) = V(D) \forall i = 1, 2, \dots, n$
$\Rightarrow \bigcup_{i=1}^{n} L_{mi}(V(D)) = V(D)$
$\Rightarrow L_n(V(D)) = V(D)$.
(L_4) Let $H \subseteq K$, then by (L_4) in Proposition (2.3.1) in [21], we get $L_{mi}(V(H)) \subseteq L_{mi}(V(K))\ \forall\ i = 1, 2, \dots, n$
$\Rightarrow \bigcup_{i=1}^{n} L_{mi}(V(H)) \subseteq \bigcup_{i=1}^{n} L_{mi}(V(K))$
$\Rightarrow L_n(V(H)) \subseteq L_n(V(K))$.
(L_5) Let $V(H) \cap V(K) \subseteq V(H)$ and $V(H) \cap V(K) \subseteq V(K)$, then by Proposition (2.3.1) in [21], we have that
$L_{mi}(V(H) \cap V(K)) \subseteq L_{mi}(V(H)) \wedge L_{mi}(V(H) \cap V(K)) \subseteq L_{mi}(V(K))$
$\Rightarrow \bigcup_{i=1}^{n} L_{mi}(V(H) \cap V(K)) \subseteq \bigcup_{i=1}^{n} L_{mi}(V(H)) \wedge \bigcup_{i=1}^{n} L_{mi}(V(H) \cap V(K)) \subseteq \bigcup_{i=1}^{n} L_{mi}(V(K))$
$\Rightarrow L_n(V(H) \cap V(K)) \subseteq L_n(V(H)) \wedge L_n(V(H) \cap V(K)) \subseteq L_n(V(K))$.
$\Rightarrow L_n(V(H) \cap V(K)) \subseteq L_n(V(H)) \cap L_n(V(K))$.
(L_6) Let $V(H) \subseteq V(H) \cup V(K)$ or $V(K) \subseteq V(H) \cup V(K)$, then by Proposition (2.3.1) in [21], we have that
$L_{mi}(V(H)) \subseteq L_{mi}(V(H) \cup V(K)) \vee L_{mi}(V(K)) \subseteq L_{mi}(V(H) \cup V(K))$
$\Rightarrow \bigcup_{i=1}^{n} L_{mi}(V(H)) \subseteq \bigcup_{i=1}^{n} L_{mi}(V(H) \cup V(K)) \vee \bigcup_{i=1}^{n} L_{mi}(V(K)) \subseteq \bigcup_{i=1}^{n} L_{mi}(V(H) \cup V(K))$
$\Rightarrow L_n(V(H)) \subseteq L_n(V(H) \cup V(K)) \vee L_n(V(K)) \subseteq L_n(V(H) \cup V(K))$.
$\Rightarrow L_n(V(H)) \cup L_n(V(K)) \subseteq L_n(V(H) \cup V(K))$.
(L_7) Let $\varpi \in L_n(V(H)) \Leftrightarrow \varpi \in \bigcup_{i=1}^{n} L_{mi} V(H)$
$\Leftrightarrow \varpi \in L_{mi} V(H)$
$\Leftrightarrow$ by (L_7) in Proposition (2.3.1) in [21], we have $\varpi \notin U_n[V(D) - V(H)]$
$\Leftrightarrow \varpi \in V(D) - [U_n(V(D) - V(H))]$
$\Leftrightarrow L_n(V(H)) = V(D) - [U_n(V(D) - V(H))]$.
(U_3) By (U_3) in Proposition (2.3.1) in [21], we have $U_{mi}(\phi) = \phi \forall i = 1, 2, \dots, n$
$\Rightarrow \bigcap_{i=1}^{n} U_{mi}(\phi) = \phi$.
$\Rightarrow U_n(\phi) = \phi$.
(U_4) Let $H \subseteq K$, then by (U_4) in proposition (2.3.1) in [21], we have $U_{mi}(V(H)) \subseteq U_{mi}(V(K))\ \forall\ i = 1, 2, \dots, n$
$\Rightarrow \bigcap_{i=1}^{n} U_{mi}(V(H)) \subseteq \bigcap_{i=1}^{n} U_{mi}(V(K))$
$\Rightarrow U_n(V(H)) \subseteq L_n(V(K))$.
(U_5) Let $V(H) \cap V(K) \subseteq V(H)$ and $V(H) \cap V(K) \subseteq V(K)$, then by Proposition (2.3.1) in [21], we have that
$U_{mi}(V(H) \cap V(K)) \subseteq U_{mi}(V(H)) \wedge U_{mi}(V(H) \cap V(K)) \subseteq U_{mi}(V(K))$
$\Rightarrow \bigcap_{i=1}^{n} U_{mi}(V(H) \cap V(K)) \subseteq \bigcap_{i=1}^{n} U_{mi}(V(H)) \wedge \bigcap_{i=1}^{n} U_{mi}(V(H) \cap V(K)) \subseteq \bigcap_{i=1}^{n} U_{mi}(V(K))$
$\Rightarrow U_n(V(H) \cap V(K)) \subseteq U_n(V(H)) \wedge U_n(V(H) \cap V(K)) \subseteq U_n(V(K))$.
$\Rightarrow U_n(V(H) \cap V(K)) \subseteq U_n(V(H)) \cap U_n(V(K))$.
(U_6) Let $V(H) \subseteq V(H) \cup V(K)$ or $V(K) \subseteq V(H) \cup V(K)$, then by Proposition (2.3.1) in [21], we have that
$U_{mi}(V(H)) \subseteq U_{mi}(V(H) \cup V(K)) \vee U_{mi}(V(K)) \subseteq U_{mi}(V(H) \cup V(K))$
$\Rightarrow \bigcap_{i=1}^{n} U_{mi}(V(H)) \subseteq \bigcap_{i=1}^{n} U_{mi}(V(H) \cup V(K)) \vee \bigcap_{i=1}^{n} U_{mi}(V(K)) \subseteq \bigcap_{i=1}^{n} U_{mi}(V(H) \cup V(K))$
$\Rightarrow U_n(V(H)) \subseteq U_n(V(H) \cup V(K)) \vee U_n(V(K)) \subseteq U_n(V(H) \cup V(K))$.
$\Rightarrow U_n(V(H)) \cup U_n(V(K)) \subseteq U_n(V(H) \cup V(K))$.
(U_7) By substituting $V(D) - V(H)$ for $V(H)$ in (L_7) we have $U_n(V(H)) = V(D) - [L_n(V(D) - V(H))]$.

Remark 2.5. Let $D = \{D_i;\ i = 1, 2, 3, \dots, n\}$ be a finite family of arbitrary non-empty finite d. g. 's and $H, K \subseteq D$, then the following are not true in general:
(L_1) $L_n(V(H)) \subseteq V(H)$,
(L_3) $L_n(\phi) = \phi$,
(L_8) $L_n(V(H)) = L_n(L_n(V(H))$,
(L_9) $L_n(V(H)) = U_n(L_n(V(H))$,
(L_{10}) $V(H) \subseteq L_n(U_n(V(H))$,
(L_{11}) $L_n(V(H)) \subseteq L_n(L_n(V(H))$,
(L_{12}) $L_n(V(H) \cap V(K)) = L_n(V(H)) \cap L_n(V(K))$,
(U_1) $V(H) \subseteq U_n(V(H))$,
(U_2) $U_n(V(D)) = V(D)$,
(U_8) $U_n(V(H)) = U_n(U_n(V(H))$,
(U_9) $U_n(V(H)) = L_n(U_n(V(H))$,
(U_{10}) $V(H) \supseteq U_n(L_n(V(H))$,
(U_{11}) $U_n(V(H)) \supseteq U_n(U_n(V(H))$,
(U_{12}) $U_n(V(H) \cup V(K)) = U_n(V(H)) \cup U_n(V(K))$ and
(LU) $L_n(V(H)) \subseteq U_n(V(H))$.

The following two examples illustrate the previous remark.

Example 2.6.
(L_1) Let $H = (V(H), E(H))$: $V(H) = \{\varpi_2\}$, $E(H) = \{(\varpi_2, \varpi_2)\}$, then $L_n(V(H)) = \{\varpi_2, \varpi_4, \varpi_5\}$. Therefore, $L_n(V(H)) \nsubseteq V(H)$.
(L_3) Let $H = (V(H), E(H))$: $V(H) = \phi$, $E(H) = \phi$, then $L_n(V(H)) = \{\varpi_4, \varpi_5\}$. Therefore, $L_n(\phi) \neq \phi$.
(L_8) Let $H = (V(H), E(H))$: $V(H) = \{\varpi_3\}$, $E(H) = \{(\varpi_3, \varpi_3)\}$, then $L_n(V(H)) = \{\varpi_4, \varpi_5\}$, $L_n(L_n(V(H))) = \{\varpi_1, \varpi_2, \varpi_4, \varpi_5\}$. Therefore, $L_n(V(H)) \neq L_n(L_n(V(H))$.
(L_9) Let $H = (V(H), E(H))$: $V(H) = \{\varpi_1, \varpi_4, \varpi_5\}$, $E(H) = \{(\varpi_1, \varpi_1), (\varpi_1, \varpi_4), (\varpi_1, \varpi_5), (\varpi_4, \varpi_1), (\varpi_4, \varpi_4), (\varpi_5, \varpi_4), (\varpi_5, \varpi_5)\}$, then $L_n(V(H)) = \{\varpi_1, \varpi_2, \varpi_4, \varpi_5\}$, $U_n(L_n(V(H)) = \{\varpi_1, \varpi_2, \varpi_3\}$. Therefore, $L_n(V(H)) \neq U_n(L_n(V(H))$.
(L_{10}) Let $H = (V(H), E(H))$: $V(H) = \{\varpi_3\}$, $E(H) = \{(\varpi_3, \varpi_3)\}$, then $U_n(V(H)) = \{\varpi_3\}$, $L_n(U_n(V(H)) = \{\varpi_4, \varpi_5\}$. Therefore, $V(H) \nsubseteq L_n(U_n(V(H))$.
(L_{11}) In Example (2.7). Let $H = (V(H), E(H))$: $V(H) = \{\varpi_2, \varpi_3\}$, $E(H) = \{(\varpi_2, \varpi_2), (\varpi_2, \varpi_3), (\varpi_3, \varpi_3)\}$, then $L_n(V(H)) = \{\varpi_3, \varpi_4\}$, $L_n(L_n(V(H))) = \{\varpi_2, \varpi_4\}$. Therefore, $L_n(V(H)) \nsubseteq L_n(L_n(V(H))$.
(L_{12}) Let $H = (V(H), E(H))$: $V(H) = \{\varpi_1, \varpi_4\}$, $E(H) = \{(\varpi_1, \varpi_1), (\varpi_1, \varpi_4), (\varpi_4, \varpi_1), (\varpi_4, \varpi_4)\}$ and $K = (V(K), E(K))$: $V(K) = \{\varpi_1, \varpi_5\}$, $E(K) = \{(\varpi_1, \varpi_1), (\varpi_1, \varpi_5), (\varpi_5, \varpi_5)\}$, then $L_n(V(H)) = \{\varpi_1, \varpi_4, \varpi_5\}$, $L_n(V(K)) = \{\varpi_1, \varpi_2, \varpi_4, \varpi_5\}$. $H \cap K = (V(H \cap K), E(H \cap K))$: $V(H \cap K) = \{\varpi_1\}$, then $L_n(V(H) \cap V(K)) = \{\varpi_4, \varpi_5\}$. So, $L_n(V(H) \cap V(K)) \neq L_n(V(H)) \cap L_n(V(K))$.
(U_1) Let $H = (V(H), E(H))$: $V(H) = \{\varpi_2, \varpi_5\}$, $E(H) = \{(\varpi_2, \varpi_2), (\varpi_5, \varpi_2), (\varpi_5, \varpi_5)\}$, then $U_n(V(H)) = \phi$. Therefore, $V(H) \nsubseteq U_n(V(H))$.
(U_2) $L_n(V(D)) = \{\varpi_1, \varpi_2, \varpi_3\} \neq V(D)$.
(U_8) Let $H = (V(H), E(H))$: $V(H) = \{\varpi_1, \varpi_3, \varpi_4\}$, $E(H) = \{(\varpi_1, \varpi_1), (\varpi_1, \varpi_4), (\varpi_3, \varpi_1), (\varpi_3, \varpi_3), (\varpi_4, \varpi_1), (\varpi_4, \varpi_4)\}$, then $U_n(V(H)) = \{\varpi_1, \varpi_3\}$, $U_n(U_n(V(H))) = \{\varpi_3\}$. Therefore, $U_n(V(H)) \neq U_n(U_n(V(H))$.
(U_9) Let $H = (V(H), E(H))$: $V(H) = \{\varpi_2, \varpi_3, \varpi_5\}$, $E(H) = \{(\varpi_2, \varpi_2), (\varpi_2, \varpi_3), (\varpi_3, \varpi_3), (\varpi_3, \varpi_5), (\varpi_5, \varpi_2), (\varpi_5, \varpi_5)\}$, then $U_n(V(H)) = \{\varpi_2, \varpi_3\}$, $L_n(U_n(V(H)) = \{\varpi_2, \varpi_3, \varpi_4, \varpi_5\}$. Therefore, $U_n(V(H)) \neq L_n(U_n(V(H))$.

(U_{10}) Let$H$ = $(V(H), E(H))$: $V(H) = \{\varpi_5\}$, $E(H) = \{(\varpi_5, \varpi_5)\}$, then $L_n(V(H)) = \{\varpi_2, \varpi_4, \varpi_5\}$, $U_n(L_n(V(H)) = \{\varpi_2\}$. Therefore, $V(H) \not\supseteq U_n(L_n(V(H))$.

(U_{11}) In Example (2.7). Let$H$ = $(V(H), E(H))$: $V(H) = \{\varpi_1, \varpi_2\}$, $E(H) = \{(\varpi_1, \varpi_1), (\varpi_1, \varpi_2), (\varpi_2, \varpi_2)\}$, then $U_n(V(H)) = \{\varpi_1, \varpi_2, \varpi_3\}$, $U_n(U_n(V(H))) = \{\varpi_1, \varpi_3, \varpi_5\}$. Therefore, $U_n(V(H)) \not\subseteq U_n(U_n(V(H))$.

(U_{12}) Let$H$ = $(V(H), E(H))$: $V(H) = \{\varpi_3, \varpi_4\}$ and$E(H) = \{(\varpi_3, \varpi_3), (\varpi_4, \varpi_4)\}$, $K = (V(K), E(K))$: $V(K) = \{\varpi_3, \varpi_5\}$, $E(K) = \{(\varpi_3, \varpi_3), (\varpi_3, \varpi_5), (\varpi_5, \varpi_5)\}$, then $U_n(V(H)) = \{\varpi_3\}$, $U_n(V(K)) = \{\varpi_3\}$. $H \cup K = (V(H \cup K), E(H \cup K))$: $V(H \cup K) = \{\varpi_3, \varpi_4, \varpi_5\}$ then $U_n(V(H) \cup V(K)) = \{\varpi_1, \varpi_3\}$. So, $U_n(V(H) \cup V(K)) \neq U_n(V(H)) \cup U_n(V(K))$.

(LU) Let $H = (V(H), E(H))$: $V(H) = \{\varpi_2, \varpi_3\}$, $E(H) = \{(\varpi_2, \varpi_2), (\varpi_2, \varpi_3), (\varpi_3, \varpi_3)\}$, then $L_n(V(H)) = \{\varpi_2, \varpi_3, \varpi_4, \varpi_5\}$, $U_n(V(H)) = \{\varpi_3\}$. Therefore, $L_n(V(H)) \not\subseteq U_n(V(H))$.

Example 2.7. Let $\boldsymbol{D} = \{D_i;\ i = 1, 2, 3\}$ be three d. g. 's defined as: $V(D) = V(D_1) = V(D_2) = V(D_3) = \{\varpi_1, \varpi_2, \varpi_3, \varpi_4, \varpi_5\}$, $E(D_1) = \{(\varpi_1, \varpi_1), (\varpi_1, \varpi_2), (\varpi_1, \varpi_5), (\varpi_2, \varpi_3), (\varpi_2, \varpi_4), (\varpi_2, \varpi_5), (\varpi_3, \varpi_1), (\varpi_3, \varpi_3), (\varpi_5, \varpi_2), (\varpi_5, \varpi_4), (\varpi_5, \varpi_5)\}$, $E(D_2) = \{(\varpi_1, \varpi_1), (\varpi_1, \varpi_2), (\varpi_1, \varpi_5), (\varpi_2, \varpi_3), (\varpi_2, \varpi_4), (\varpi_3, \varpi_1), (\varpi_3, \varpi_3), (\varpi_5, \varpi_2), (\varpi_5, \varpi_4), (\varpi_5, \varpi_5)\}$ and $E(D_3) = \{(\varpi_1, \varpi_1), (\varpi_1, \varpi_5), (\varpi_2, \varpi_3), (\varpi_2, \varpi_4), (\varpi_3, \varpi_1), (\varpi_3, \varpi_3), (\varpi_3, \varpi_5), (\varpi_5, \varpi_2), (\varpi_5, \varpi_4), (\varpi_5, \varpi_5)\}$.

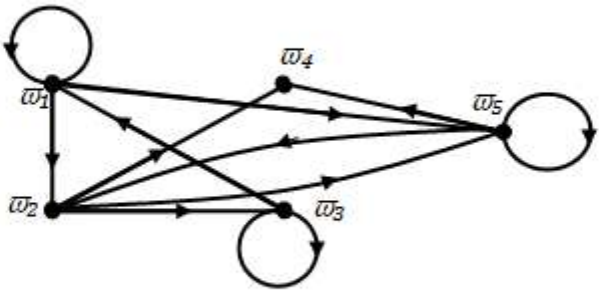

Figure 2.4: d. g. D_1.

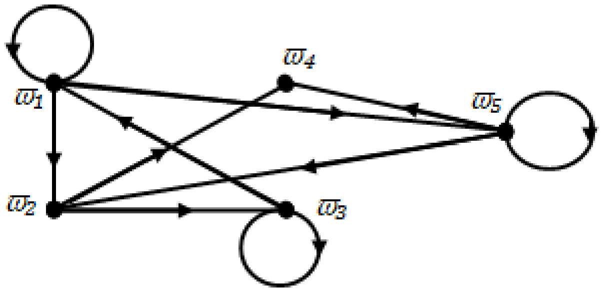

Figure 2.5: d. g. D_2.

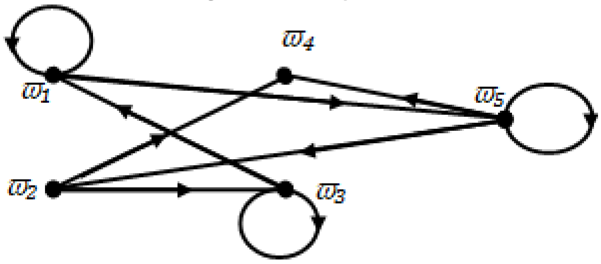

Figure 2.6: d. g. D_3.

The mixed degree systems based on D_1 are given by: $MD_{m_1}(\varpi_1) = \{\{\varpi_1, \varpi_2, \varpi_5\}, \{\varpi_1, \varpi_3\}\}$,$MD_{m_1}(\varpi_2) = \{\{\varpi_3, \varpi_4, \varpi_5\}, \{\varpi_1, \varpi_5\}\}$, $MD_{m_1}(\varpi_3) = \{\{\varpi_1, \varpi_3\}, \{\varpi_2, \varpi_3\}\}$,$MD_{m_1}(\varpi_4) = \{\phi, \{\varpi_2, \varpi_5\}\}$ and$MD_{m_1}(\varpi_5) = \{\{\varpi_2, \varpi_4, \varpi_5\}, \{\varpi_1, \varpi_2, \varpi_5\}\}$.

The mixed degree systems based on D_2 are given by: $MD_{m_2}(\varpi_1) = \{\{\varpi_1, \varpi_2, \varpi_5\}, \{\varpi_1, \varpi_3\}\}$,$MD_{m_2}(\varpi_2) = \{\{\varpi_3, \varpi_4\}, \{\varpi_1, \varpi_5\}\}$, $MD_{m_2}(\varpi_3) = \{\{\varpi_1, \varpi_3\}, \{\varpi_2, \varpi_3\}\}$, $MD_{m_2}(\varpi_4) = \{\phi, \{\varpi_2, \varpi_5\}\}$ and$MD_{m_2}(\varpi_5) = \{\{\varpi_2, \varpi_4, \varpi_5\}, \{\varpi_1, \varpi_5\}\}$.

The mixed degree systems based on D_3 are given by: $MD_{m_3}(\varpi_1) = \{\{\varpi_1, \varpi_5\}, \{\varpi_1, \varpi_3\}\}$, $MD_{m_3}(\varpi_2) = \{\{\varpi_3, \varpi_4\}, \{\varpi_5\}\}$,$MD_{m_3}(\varpi_3) = \{\{\varpi_1, \varpi_3\}, \{\varpi_2, \varpi_3\}\}$,$MD_{m_3}(\varpi_4) = \{\phi, \{\varpi_2, \varpi_5\}\}$ and $MD_{m_3}(\varpi_5) = \{\{\varpi_2, \varpi_4, \varpi_5\}, \{\varpi_1, \varpi_5\}\}$.

The lower approximation, for all $H \subseteq D$, are given in the table:

$V(H)$	$L_{m_1}(V(H))$	$L_{m_2}(V(H))$	$L_{m_3}(V(H))$	$L_n(V(H))$
$\{\varpi_1\}$	$\{\varpi_4\}$	$\{\varpi_4\}$	$\{\varpi_4\}$	$\{\varpi_4\}$
$\{\varpi_2\}$	$\{\varpi_4\}$	$\{\varpi_4\}$	$\{\varpi_4\}$	$\{\varpi_4\}$
$\{\varpi_3\}$	$\{\varpi_4\}$	$\{\varpi_4\}$	$\{\varpi_4\}$	$\{\varpi_4\}$
$\{\varpi_4\}$	$\{\varpi_4\}$	$\{\varpi_4\}$	$\{\varpi_4\}$	$\{\varpi_4\}$
$\{\varpi_5\}$	$\{\varpi_4\}$	$\{\varpi_4\}$	$\{\varpi_2,\varpi_4\}$	$\{\varpi_2,\varpi_4\}$
$\{\varpi_1,\varpi_2\}$	$\{\varpi_4\}$	$\{\varpi_4\}$	$\{\varpi_4\}$	$\{\varpi_4\}$
$\{\varpi_1,\varpi_3\}$	$\{\varpi_1,\varpi_3, \varpi_4\}$	$\{\varpi_1,\varpi_3, \varpi_4\}$	$\{\varpi_1,\varpi_3, \varpi_4\}$	$\{\varpi_1,\varpi_3, \varpi_4\}$
$\{\varpi_1,\varpi_4\}$	$\{\varpi_4\}$	$\{\varpi_4\}$	$\{\varpi_4\}$	$\{\varpi_4\}$
$\{\varpi_1,\varpi_5\}$	$\{\varpi_2,\varpi_4\}$	$\{\varpi_2,\varpi_4, \varpi_5\}$	$\{\varpi_1,\varpi_2, \varpi_4, \varpi_5\}$	$\{\varpi_1,\varpi_2, \varpi_4, \varpi_5\}$
$\{\varpi_2,\varpi_3\}$	$\{\varpi_3, \varpi_4\}$	$\{\varpi_3, \varpi_4\}$	$\{\varpi_3, \varpi_4\}$	$\{\varpi_3, \varpi_4\}$
$\{\varpi_2,\varpi_4\}$	$\{\varpi_4\}$	$\{\varpi_4\}$	$\{\varpi_4\}$	$\{\varpi_4\}$
$\{\varpi_2,\varpi_5\}$	$\{\varpi_4\}$	$\{\varpi_4\}$	$\{\varpi_2,\varpi_4\}$	$\{\varpi_2,\varpi_4\}$
$\{\varpi_3, \varpi_4\}$	$\{\varpi_4\}$	$\{\varpi_2,\varpi_4\}$	$\{\varpi_2,\varpi_4\}$	$\{\varpi_2,\varpi_4\}$
$\{\varpi_3, \varpi_5\}$	$\{\varpi_4\}$	$\{\varpi_4\}$	$\{\varpi_2,\varpi_4\}$	$\{\varpi_2,\varpi_4\}$
$\{\varpi_4, \varpi_5\}$	$\{\varpi_4\}$	$\{\varpi_4\}$	$\{\varpi_2,\varpi_4\}$	$\{\varpi_2,\varpi_4\}$
$\{\varpi_1, \varpi_2, \varpi_3\}$	$\{\varpi_1,\varpi_3, \varpi_4\}$	$\{\varpi_1,\varpi_3, \varpi_4\}$	$\{\varpi_1,\varpi_3, \varpi_4\}$	$\{\varpi_1,\varpi_3, \varpi_4\}$
$\{\varpi_1,\varpi_2, \varpi_4\}$	$\{\varpi_4\}$	$\{\varpi_4\}$	$\{\varpi_4\}$	$\{\varpi_4\}$
$\{\varpi_1,\varpi_2, \varpi_5\}$	$\{\varpi_1,\varpi_2, \varpi_4, \varpi_5\}$	$\{\varpi_1,\varpi_2, \varpi_4, \varpi_5\}$	$\{\varpi_1,\varpi_2, \varpi_4, \varpi_5\}$	$\{\varpi_1,\varpi_2, \varpi_4, \varpi_5\}$
$\{\varpi_1,\varpi_3, \varpi_4\}$	$\{\varpi_1,\varpi_3, \varpi_4\}$	$\{\varpi_1,\varpi_2, \varpi_3, \varpi_4\}$	$\{\varpi_1,\varpi_2, \varpi_3, \varpi_4\}$	$\{\varpi_1,\varpi_2, \varpi_3, \varpi_4\}$
$\{\varpi_1,\varpi_3, \varpi_5\}$	$\{\varpi_1,\varpi_2, \varpi_3, \varpi_4\}$	$V(D)$	$V(D)$	$V(D)$
$\{\varpi_1,\varpi_4, \varpi_5\}$	$\{\varpi_2,\varpi_4\}$	$\{\varpi_2,\varpi_4, \varpi_5\}$	$\{\varpi_1,\varpi_2, \varpi_4, \varpi_5\}$	$\{\varpi_1,\varpi_2, \varpi_4, \varpi_5\}$
$\{\varpi_2,\varpi_3, \varpi_4\}$	$\{\varpi_3, \varpi_4\}$	$\{\varpi_2,\varpi_3, \varpi_4\}$	$\{\varpi_2,\varpi_3, \varpi_4\}$	$\{\varpi_2,\varpi_3, \varpi_4\}$
$\{\varpi_2,\varpi_3, \varpi_5\}$	$\{\varpi_3, \varpi_4\}$	$\{\varpi_3, \varpi_4\}$	$\{\varpi_2,\varpi_3, \varpi_4\}$	$\{\varpi_2,\varpi_3, \varpi_4\}$

$\{\varpi_2,\varpi_4,\varpi_5\}$	$\{\varpi_4,\varpi_5\}$	$\{\varpi_4,\varpi_5\}$	$\{\varpi_2,\varpi_4,\varpi_5\}$	$\{\varpi_2,\varpi_4,\varpi_5\}$
$\{\varpi_3,\varpi_4,\varpi_5\}$	$\{\varpi_2,\varpi_4\}$	$\{\varpi_2,\varpi_4\}$	$\{\varpi_2,\varpi_4\}$	$\{\varpi_2,\varpi_4\}$
$\{\varpi_1,\varpi_2,\varpi_3,\varpi_4\}$	$\{\varpi_1,\varpi_3,\varpi_4\}$	$\{\varpi_1,\varpi_2,\varpi_3,\varpi_4\}$	$\{\varpi_1,\varpi_2,\varpi_3,\varpi_4\}$	$\{\varpi_1,\varpi_2,\varpi_3,\varpi_4\}$
$\{\varpi_1,\varpi_2,\varpi_3,\varpi_5\}$	$V(D)$	$V(D)$	$V(D)$	$V(D)$
$\{\varpi_1,\varpi_2,\varpi_4,\varpi_5\}$	$\{\varpi_1,\varpi_2,\varpi_4,\varpi_5\}$	$\{\varpi_1,\varpi_2,\varpi_4,\varpi_5\}$	$\{\varpi_1,\varpi_2,\varpi_4,\varpi_5\}$	$\{\varpi_1,\varpi_2,\varpi_4,\varpi_5\}$
$\{\varpi_1,\varpi_3,\varpi_4,\varpi_5\}$	$\{\varpi_1,\varpi_2,\varpi_3,\varpi_4\}$	$V(D)$	$V(D)$	$V(D)$
$\{\varpi_2,\varpi_3,\varpi_4,\varpi_5\}$	$\{\varpi_2,\varpi_3,\varpi_4,\varpi_5\}$	$\{\varpi_2,\varpi_3,\varpi_4,\varpi_5\}$	$\{\varpi_2,\varpi_3,\varpi_4,\varpi_5\}$	$\{\varpi_2,\varpi_3,\varpi_4,\varpi_5\}$
$V(D)$	$V(D)$	$V(D)$	$V(D)$	$V(D)$
ϕ	$\{\varpi_4\}$	$\{\varpi_4\}$	$\{\varpi_4\}$	$\{\varpi_4\}$

The upper approximation, for all $H \subseteq D$, are given in the table:

$V(H)$	$U_{m_1}(V(H))$	$U_{m_2}(V(H))$	$U_{m_3}(V(H))$	$U_n(V(H))$
$\{\varpi_1\}$	$\{\varpi_1\}$	$\{\varpi_1\}$	$\{\varpi_1\}$	$\{\varpi_1\}$
$\{\varpi_2\}$	$\{\varpi_5\}$	ϕ	ϕ	ϕ
$\{\varpi_3\}$	$\{\varpi_3\}$	$\{\varpi_3\}$	$\{\varpi_3\}$	$\{\varpi_3\}$
$\{\varpi_4\}$	ϕ	ϕ	ϕ	ϕ
$\{\varpi_5\}$	$\{\varpi_2,\varpi_5\}$	$\{\varpi_5\}$	$\{\varpi_5\}$	$\{\varpi_5\}$
$\{\varpi_1,\varpi_2\}$	$\{\varpi_1,\varpi_3,\varpi_5\}$	$\{\varpi_1,\varpi_3,\varpi_5\}$	$\{\varpi_1,\varpi_3,\varpi_5\}$	$\{\varpi_1,\varpi_3,\varpi_5\}$
$\{\varpi_1,\varpi_3\}$	$\{\varpi_1,\varpi_2,\varpi_3\}$	$\{\varpi_1,\varpi_2,\varpi_3\}$	$\{\varpi_1,\varpi_3\}$	$\{\varpi_1,\varpi_3\}$
$\{\varpi_1,\varpi_4\}$	$\{\varpi_1,\varpi_2,\varpi_5\}$	$\{\varpi_1,\varpi_2,\varpi_5\}$	$\{\varpi_1,\varpi_5\}$	$\{\varpi_1,\varpi_5\}$
$\{\varpi_1,\varpi_5\}$	$\{\varpi_1,\varpi_2,\varpi_5\}$	$\{\varpi_1,\varpi_5\}$	$\{\varpi_1,\varpi_5\}$	$\{\varpi_1,\varpi_5\}$
$\{\varpi_2,\varpi_3\}$	$\{\varpi_1,\varpi_3,\varpi_5\}$	$\{\varpi_1,\varpi_3\}$	$\{\varpi_3\}$	$\{\varpi_3\}$
$\{\varpi_2,\varpi_4\}$	$\{\varpi_5\}$	ϕ	ϕ	ϕ
$\{\varpi_2,\varpi_5\}$	$\{\varpi_2,\varpi_5\}$	$\{\varpi_5\}$	$\{\varpi_5\}$	$\{\varpi_5\}$
$\{\varpi_3,\varpi_4\}$	$\{\varpi_3\}$	$\{\varpi_3\}$	$\{\varpi_3\}$	$\{\varpi_3\}$
$\{\varpi_3,\varpi_5\}$	$\{\varpi_1,\varpi_2,\varpi_3,\varpi_5\}$	$\{\varpi_1,\varpi_2,\varpi_3,\varpi_5\}$	$\{\varpi_1,\varpi_2,\varpi_3,\varpi_5\}$	$\{\varpi_1,\varpi_2,\varpi_3,\varpi_5\}$
$\{\varpi_4,\varpi_5\}$	$\{\varpi_2,\varpi_5\}$	$\{\varpi_2,\varpi_5\}$	$\{\varpi_2,\varpi_5\}$	$\{\varpi_2,\varpi_5\}$
$\{\varpi_1,\varpi_2,\varpi_3\}$	$\{\varpi_1,\varpi_2,\varpi_3,\varpi_5\}$	$\{\varpi_1,\varpi_2,\varpi_3,\varpi_5\}$	$\{\varpi_1,\varpi_3,\varpi_5\}$	$\{\varpi_1,\varpi_3,\varpi_5\}$
$\{\varpi_1,\varpi_2,\varpi_4\}$	$\{\varpi_1,\varpi_2,\varpi_3,\varpi_5\}$	$\{\varpi_1,\varpi_2,\varpi_3,\varpi_5\}$	$\{\varpi_1,\varpi_3,\varpi_5\}$	$\{\varpi_1,\varpi_3,\varpi_5\}$
$\{\varpi_1,\varpi_2,\varpi_5\}$	$\{\varpi_1,\varpi_2,\varpi_3,\varpi_5\}$	$\{\varpi_1,\varpi_3,\varpi_5\}$	$\{\varpi_1,\varpi_3,\varpi_5\}$	$\{\varpi_1,\varpi_3,\varpi_5\}$
$\{\varpi_1,\varpi_3,\varpi_4\}$	$\{\varpi_1,\varpi_2,\varpi_3,\varpi_5\}$	$\{\varpi_1,\varpi_2,\varpi_3,\varpi_5\}$	$\{\varpi_1,\varpi_3,\varpi_5\}$	$\{\varpi_1,\varpi_3,\varpi_5\}$
$\{\varpi_1,\varpi_3,\varpi_5\}$	$\{\varpi_1,\varpi_2,\varpi_3,\varpi_5\}$	$\{\varpi_1,\varpi_2,\varpi_3,\varpi_5\}$	$\{\varpi_1,\varpi_2,\varpi_3,\varpi_5\}$	$\{\varpi_1,\varpi_2,\varpi_3,\varpi_5\}$
$\{\varpi_1,\varpi_4,\varpi_5\}$	$\{\varpi_1,\varpi_2,\varpi_5\}$	$\{\varpi_1,\varpi_2,\varpi_5\}$	$\{\varpi_1,\varpi_2,\varpi_5\}$	$\{\varpi_1,\varpi_2,\varpi_5\}$
$\{\varpi_2,\varpi_3,\varpi_4\}$	$\{\varpi_1,\varpi_3,\varpi_5\}$	$\{\varpi_1,\varpi_3\}$	$\{\varpi_3\}$	$\{\varpi_3\}$
$\{\varpi_2,\varpi_3,\varpi_5\}$	$\{\varpi_1,\varpi_2,\varpi_3,\varpi_5\}$	$\{\varpi_1,\varpi_2,\varpi_3,\varpi_5\}$	$\{\varpi_1,\varpi_2,\varpi_3,\varpi_5\}$	$\{\varpi_1,\varpi_2,\varpi_3,\varpi_5\}$
$\{\varpi_2,\varpi_4,\varpi_5\}$	$\{\varpi_2,\varpi_5\}$	$\{\varpi_2,\varpi_5\}$	$\{\varpi_2,\varpi_5\}$	$\{\varpi_2,\varpi_5\}$
$\{\varpi_2,\varpi_4,\varpi_5\}$	$\{\varpi_1,\varpi_2,\varpi_3,\varpi_5\}$	$\{\varpi_1,\varpi_2,\varpi_3,\varpi_5\}$	$\{\varpi_1,\varpi_2,\varpi_3,\varpi_5\}$	$\{\varpi_1,\varpi_2,\varpi_3,\varpi_5\}$
$\{\varpi_1,\varpi_2,\varpi_3,\varpi_4\}$	$\{\varpi_1,\varpi_2,\varpi_3,\varpi_5\}$	$\{\varpi_1,\varpi_2,\varpi_3,\varpi_5\}$	$\{\varpi_1,\varpi_3,\varpi_5\}$	$\{\varpi_1,\varpi_3,\varpi_5\}$
$\{\varpi_1,\varpi_2,\varpi_3,\varpi_5\}$	$\{\varpi_1,\varpi_2,\varpi_3,\varpi_5\}$	$\{\varpi_1,\varpi_2,\varpi_3,\varpi_5\}$	$\{\varpi_1,\varpi_2,\varpi_3,\varpi_5\}$	$\{\varpi_1,\varpi_2,\varpi_3,\varpi_5\}$
$\{\varpi_1,\varpi_2,\varpi_4,\varpi_5\}$	$\{\varpi_1,\varpi_2,\varpi_3,\varpi_5\}$	$\{\varpi_1,\varpi_2,\varpi_3,\varpi_5\}$	$\{\varpi_1,\varpi_2,\varpi_3,\varpi_5\}$	$\{\varpi_1,\varpi_2,\varpi_3,\varpi_5\}$
$\{\varpi_1,\varpi_3,\varpi_4,\varpi_5\}$	$\{\varpi_1,\varpi_2,\varpi_3,\varpi_5\}$	$\{\varpi_1,\varpi_2,\varpi_3,\varpi_5\}$	$\{\varpi_1,\varpi_2,\varpi_3,\varpi_5\}$	$\{\varpi_1,\varpi_2,\varpi_3,\varpi_5\}$
$\{\varpi_2,\varpi_3,\varpi_4,\varpi_5\}$	$\{\varpi_1,\varpi_2,\varpi_3,\varpi_5\}$	$\{\varpi_1,\varpi_2,\varpi_3,\varpi_5\}$	$\{\varpi_1,\varpi_2,\varpi_3,\varpi_5\}$	$\{\varpi_1,\varpi_2,\varpi_3,\varpi_5\}$
$V(D)$	$\{\varpi_1,\varpi_2,\varpi_3,\varpi_5\}$	$\{\varpi_1,\varpi_2,\varpi_3,\varpi_5\}$	$\{\varpi_1,\varpi_2,\varpi_3,\varpi_5\}$	$\{\varpi_1,\varpi_2,\varpi_3,\varpi_5\}$
ϕ	ϕ	ϕ	ϕ	ϕ

The boundary according to D, for all $H \subseteq D$, are given in the table:

$V(H)$	$Bd_{m_1}(V(H))$	$Bd_{m_2}(V(H))$	$Bd_{m_3}(V(H))$	$Bd_n(V(H))$
$\{\varpi_1\}$	$\{\varpi_1\}$	$\{\varpi_1\}$	$\{\varpi_1\}$	$\{\varpi_1\}$
$\{\varpi_2\}$	$\{\varpi_5\}$	ϕ	ϕ	ϕ
$\{\varpi_3\}$	$\{\varpi_3\}$	$\{\varpi_3\}$	$\{\varpi_3\}$	$\{\varpi_3\}$
$\{\varpi_4\}$	ϕ	ϕ	ϕ	ϕ
$\{\varpi_5\}$	$\{\varpi_2,\varpi_5\}$	$\{\varpi_5\}$	$\{\varpi_5\}$	$\{\varpi_5\}$
$\{\varpi_1,\varpi_2\}$	$\{\varpi_1,\varpi_3,\varpi_5\}$	$\{\varpi_1,\varpi_3,\varpi_5\}$	$\{\varpi_1,\varpi_3,\varpi_5\}$	$\{\varpi_1,\varpi_2,\varpi_3\}$
$\{\varpi_1,\varpi_3\}$	$\{\varpi_2\}$	$\{\varpi_2\}$	ϕ	ϕ
$\{\varpi_1,\varpi_4\}$	$\{\varpi_1,\varpi_2,\varpi_5\}$	$\{\varpi_1,\varpi_2,\varpi_5\}$	$\{\varpi_1,\varpi_5\}$	$\{\varpi_1,\varpi_5\}$
$\{\varpi_1,\varpi_5\}$	$\{\varpi_1,\varpi_5\}$	$\{\varpi_1\}$	ϕ	ϕ
$\{\varpi_2,\varpi_3\}$	$\{\varpi_1,\varpi_5\}$	$\{\varpi_1\}$	ϕ	ϕ
$\{\varpi_2,\varpi_4\}$	$\{\varpi_5\}$	ϕ	ϕ	ϕ
$\{\varpi_2,\varpi_5\}$	$\{\varpi_2,\varpi_5\}$	$\{\varpi_5\}$	$\{\varpi_5\}$	$\{\varpi_5\}$
$\{\varpi_3,\varpi_4\}$	$\{\varpi_3\}$	$\{\varpi_3\}$	$\{\varpi_3\}$	$\{\varpi_3\}$
$\{\varpi_3,\varpi_5\}$	$\{\varpi_1,\varpi_2,\varpi_3,\varpi_5\}$	$\{\varpi_1,\varpi_2,\varpi_3,\varpi_5\}$	$\{\varpi_1,\varpi_3,\varpi_5\}$	$\{\varpi_1,\varpi_3,\varpi_5\}$
$\{\varpi_4,\varpi_5\}$	$\{\varpi_2,\varpi_5\}$	$\{\varpi_2,\varpi_5\}$	$\{\varpi_5\}$	$\{\varpi_5\}$
$\{\varpi_1,\varpi_2,\varpi_3\}$	$\{\varpi_2,\varpi_5\}$	$\{\varpi_2,\varpi_5\}$	$\{\varpi_5\}$	$\{\varpi_5\}$

$\{\varpi_1,\varpi_2,\varpi_4\}$	$\{\varpi_1,\varpi_2,\varpi_3,\varpi_5\}$	$\{\varpi_1,\varpi_2,\varpi_3,\varpi_5\}$	$\{\varpi_1,\varpi_3,\varpi_5\}$	$\{\varpi_1,\varpi_3,\varpi_5\}$
$\{\varpi_1,\varpi_2,\varpi_5\}$	$\{\varpi_3\}$	$\{\varpi_3\}$	$\{\varpi_3\}$	$\{\varpi_3\}$
$\{\varpi_1,\varpi_3,\varpi_4\}$	$\{\varpi_2,\varpi_5\}$	$\{\varpi_5\}$	$\{\varpi_5\}$	$\{\varpi_5\}$
$\{\varpi_1,\varpi_3,\varpi_5\}$	$\{\varpi_5\}$	ϕ	ϕ	ϕ
$\{\varpi_1,\varpi_4,\varpi_5\}$	$\{\varpi_1,\varpi_5\}$	$\{\varpi_1\}$	ϕ	ϕ
$\{\varpi_2,\varpi_3,\varpi_4\}$	$\{\varpi_1,\varpi_5\}$	$\{\varpi_1\}$	ϕ	ϕ
$\{\varpi_2,\varpi_3,\varpi_5\}$	$\{\varpi_1,\varpi_2,\varpi_5\}$	$\{\varpi_1,\varpi_2,\varpi_5\}$	$\{\varpi_1,\varpi_5\}$	$\{\varpi_1,\varpi_5\}$
$\{\varpi_2,\varpi_4,\varpi_5\}$	$\{\varpi_2\}$	$\{\varpi_2\}$	ϕ	ϕ
$\{\varpi_3,\varpi_4,\varpi_5\}$	$\{\varpi_1,\varpi_3,\varpi_5\}$	$\{\varpi_1,\varpi_3,\varpi_5\}$	$\{\varpi_1,\varpi_3,\varpi_5\}$	$\{\varpi_1,\varpi_3,\varpi_5\}$
$\{\varpi_1,\varpi_2,\varpi_3,\varpi_4\}$	$\{\varpi_2,\varpi_5\}$	$\{\varpi_5\}$	$\{\varpi_5\}$	$\{\varpi_5\}$
$\{\varpi_1,\varpi_2,\varpi_3,\varpi_5\}$	ϕ	ϕ	ϕ	ϕ
$\{\varpi_1,\varpi_2,\varpi_4,\varpi_5\}$	$\{\varpi_3\}$	$\{\varpi_3\}$	$\{\varpi_3\}$	$\{\varpi_3\}$
$\{\varpi_1,\varpi_3,\varpi_4,\varpi_5\}$	$\{\varpi_5\}$	ϕ	ϕ	ϕ
$\{\varpi_2,\varpi_3,\varpi_4,\varpi_5\}$	$\{\varpi_1\}$	$\{\varpi_1\}$	$\{\varpi_1\}$	$\{\varpi_1\}$
$V(D)$	ϕ	ϕ	ϕ	ϕ
ϕ	ϕ	ϕ	ϕ	ϕ

3. Generalization of some Pawlak's concepts and definition using a finite number of d.g.'s

In this section, we introduced generalization of some Pawlak's concepts, offer some definition using a finite number d. g. 's and we gave examples to illustrate these definitions.

Definition 3.1. Let $\boldsymbol{D} = \{D_i;\ i = 1, 2, 3, \dots, n\}$ be a finite family of non-empty d. g. 's. A subd. g. $H \subseteq D$ is called:

(a) R-definable (or R-exact) d. g. if $Bd_n(V(H)) = \phi$,

(b) R-rough d. g. if $Bd_n(V(H)) \neq \phi$.

Remark 3.2. On the contrary to the case of classical rough set theory, there exists some subd. g. $H \subseteq D$ for which $Bd_u(V(H)) = \phi$ but $L_n(V(H)) \neq U_n(V(H))$. For example, consider $H = (V(H), E(H))$; $V(H) = \{\varpi_2, \varpi_3, \varpi_5\}$, $E(H) = \{(\varpi_2, \varpi_2), (\varpi_2, \varpi_3), (\varpi_3, \varpi_3), (\varpi_3, \varpi_5), (\varpi_5, \varpi_2), (\varpi_5, \varpi_5)\}$

$L_n(V(H)) = \{\varpi_2, \varpi_3, \varpi_4, \varpi_5\}$,

$U_n(V(H)) = \{\varpi_2, \varpi_3\}$ and

$Bd_n(V(H)) = \phi$.

In classical rough set theory, it is obvious that the intersection, the union and the difference of two definable sets is also definable [5].

Remark 3.3. On the contrary to the case of classical rough set theory, the intersection (union and difference) of two R-definable d. g. 's is not necessarily R-definable as the following example illustrates.

Example 3.4. According to Example (2.3)

(a) H, K are two R-definable but $H \cap K$ not R-definable.

Let $H = (V(H), E(H))$: $V(H) = \{\varpi_1, \varpi_3, \varpi_4\}$, $E(H) = \{(\varpi_1, \varpi_1), (\varpi_1, \varpi_4), (\varpi_3, \varpi_1), (\varpi_3, \varpi_3), (\varpi_4, \varpi_4)\}$ and $K = (V(K), E(K))$: $V(K) = \{\varpi_2, \varpi_3, \varpi_4\}$, $E(K) = \{(\varpi_2, \varpi_2), (\varpi_2, \varpi_3), (\varpi_2, \varpi_4), (\varpi_3, \varpi_3), (\varpi_4, \varpi_4)\}$ are two R-definable. But $V(H) \cap V(K) = \{\varpi_3, \varpi_4\}$ is not R-definable.

(b)

(c) H, K are two R-definable but $H \cup K$ is not R-definable.

Let $H = (V(H), E(H))$: $V(H) = \{\varpi_5\}$, $E(H) = \{(\varpi_5, \varpi_5)\}$ and $K = (V(K), E(K))$: $V(K) = \{\varpi_1, \varpi_2\}$, $E(K) = \{(\varpi_1, \varpi_1), (\varpi_1, \varpi_2), (\varpi_2, \varpi_2)\}$ are two R-definable. But $V(H) \cup V(K) = \{\varpi_1, \varpi_2, \varpi_5\}$ is not R-definable.

(d) H, K are two R-definable but $H - K$ is not R-definable.

Let $H = (V(H), E(H))$: $V(H) = \{\varpi_1, \varpi_3, \varpi_4\}$, $E(H) = \{(\varpi_1, \varpi_1), (\varpi_1, \varpi_4), (\varpi_3, \varpi_1), (\varpi_3, \varpi_3), (\varpi_4, \varpi_1), (\varpi_4, \varpi_4)\}$ and $K = (V(K), E(K))$: $V(K) = \{\varpi_1, \varpi_4\}$, $E(K) = \{(\varpi_1, \varpi_1), (\varpi_1, \varpi_4), (\varpi_4, \varpi_1), (\varpi_4, \varpi_4)\}$ are two R-definable. But $V(H) - V(K) = \{\varpi_3\}$ is not R-definable.

Now, we are going to generalize the definition of accuracy measure of approximations by using a finite family of arbitrary d. g.'s.

Definition 3.5. Let $\boldsymbol{D} = \{D_i;\ i = 1, 2, 3, \dots, n\}$ be a finite family of non-empty arbitrary d. g. 's. The n-accuracy measure of the approximations $\eta_n(V(H))$ of $H \subseteq D$, is defined as:

$$\eta_n(V(H)) = 1 - \frac{|Bdn(V(H))|}{|V(D)|}.$$

Using the accuracy of the approximations $\eta_n(V(H))$. Another definition of R-rough and R-exact graphs is introduced as follows:

Definition 3.6. Let $\boldsymbol{D} = \{D_i;\ i = 1, 2, 3, \dots, n\}$ be a finite family of non-empty d. g. 's. A subd. g. $H \subseteq D$ is called:

(a) R-definable (or R-exact) graph if $\eta_n(V(H)) = 1$,

(b) R-rough graph if $0 \leq \eta_n(V(H)) < 1$.

Example 3.7. According to Example (2.3), we have the following table

Table 3.1: $\eta_{m_1}(V(H))$, $\eta_{m_2}(V(H))$, $\eta_{m_3}(V(H))$ and $\eta_n(V(H))$ for all $H \subseteq D$.

$V(H)$	$\eta_{m_1}(V(H))$	$\eta_{m_2}(V(H))$	$\eta_{m_3}(V(H))$	$\eta_n(V(H))$
$\{\varpi_1\}$	4/5	4/5	4/5	1
$\{\varpi_2\}$	1	1	1	1
$\{\varpi_3\}$	4/5	4/5	4/5	4/5
$\{\varpi_4\}$	1	1	4/5	1
$\{\varpi_5\}$	4/5	4/5	1	1
$\{\varpi_1,\varpi_2\}$	2/5	2/5	4/5	1

$\{\varpi_1, \varpi_3\}$	4/5	1	3/5	1
$\{\varpi_1, \varpi_4\}$	2/5	3/5	1	1
$\{\varpi_1, \varpi_5\}$	4/5	1	4/5	1
$\{\varpi_2, \varpi_3\}$	4/5	1	1	1
$\{\varpi_2, \varpi_4\}$	1	1	4/5	1
$\{\varpi_2, \varpi_5\}$	4/5	4/5	4/5	1
$\{\varpi_3, \varpi_4\}$	4/5	4/5	3/5	4/5
$\{\varpi_3, \varpi_5\}$	1/5	2/5	1	1
$\{\varpi_4, \varpi_5\}$	3/5	4/5	4/5	1
$\{\varpi_1, \varpi_2, \varpi_3\}$	3/5	4/5	4/5	1
$\{\varpi_1, \varpi_2, \varpi_4\}$	1/5	2/5	1	1
$\{\varpi_1, \varpi_2, \varpi_5\}$	4/5	4/5	3/5	4/5
$\{\varpi_1, \varpi_3, \varpi_4\}$	4/5	4/5	4/5	1
$\{\varpi_1, \varpi_3, \varpi_5\}$	1	1	4/5	1
$\{\varpi_1, \varpi_4, \varpi_5\}$	4/5	1	1	1
$\{\varpi_2, \varpi_3, \varpi_4\}$	4/5	1	4/5	1
$\{\varpi_2, \varpi_3, \varpi_5\}$	2/5	3/5	1	1
$\{\varpi_2, \varpi_4, \varpi_5\}$	4/5	1	4/5	1
$\{\varpi_3, \varpi_4, \varpi_5\}$	2/5	2/5	4/5	1
$\{\varpi_1, \varpi_2, \varpi_3, \varpi_4\}$	4/5	4/5	1	1
$\{\varpi_1, \varpi_2, \varpi_3, \varpi_5\}$	1	1	3/5	1
$\{\varpi_1, \varpi_2, \varpi_4, \varpi_5\}$	4/5	4/5	4/5	4/5
$\{\varpi_1, \varpi_3, \varpi_4, \varpi_5\}$	1	1	1	1
$\{\varpi_2, \varpi_3, \varpi_4, \varpi_5\}$	4/5	4/5	4/5	1
$V(D)$	1	1	1	1
ϕ	1	1	1	1

In the above table, for instance, we see that the degree of exactness of the subd. g. $H = (V(H), E(H))$: $V(H) = \{\varpi_3, \varpi_4, \varpi_5\}$, $E(H) = \{(\varpi_3, \varpi_3), (\varpi_3, \varpi_5), (\varpi_4, \varpi_4), (\varpi_5, \varpi_4), (\varpi_5, \varpi_5)\}$ by using D_1 equals to 40% and by using D_2 equals to 40% and by using D_3 equals to 80%. But when we use $\boldsymbol{D} = \{D_i;\ i = 1, 2, 3\}$, the degree of exactness of the subd. g. H equals to 100%

Proposition 3.8. Let $\boldsymbol{D} = \{D_i;\ i = 1, 2, 3, \dots, n\}$ be a finite family of arbitrary non-empty d. g. and $H \subseteq D$, then the following statement are true:

(a) $L_{mi}(V(H)) \subseteq L_n(V(H))$,

(b) $U_n(V(H)) \subseteq U_{mi}(V(H))$,

(c) $Bd_n(V(H)) \subseteq Bd_{mi}(V(H))$.

Proof. The proof (a) and (b) by Definition (2.1). The proof of (c), on can use (a) and (b).

Corollary 3.9. Let $\boldsymbol{D} = \{D_i;\ i = 1, 2, 3, \dots, n\}$ be a finite family of arbitrary non-empty d. g. 's and $H \subseteq D$, then $\eta_u(V(H)) \geq \max\{\eta_{mi}(V(H)): i = 1, 2, \dots, n\}$.

Proof. By using Proposition (3.8) (c), we have $Bd_n(V(H)) \subseteq Bd_{mi}(V(H)); i = 1, 2, \dots, n$

$$\Rightarrow |Bd_n(V(H))| \leq |Bd_{mi}(V(H))|$$

$$\Rightarrow \frac{|Bdn(V(H))|}{|V(D)|} \leq \frac{|Bdmi(V(H))|}{|V(D)|}$$

$$\Rightarrow 1 - \frac{|Bdn(V(H))|}{|V(D)|} \geq 1 - \frac{|Bdmi(V(H))|}{|V(D)|}$$

$$\Rightarrow \eta_n(V(H)) \geq \max\{\eta_{mi}(V(H)): i = 1, 2, \dots, n\}.$$

Definition 3.10. Let $\boldsymbol{D} = \{D_i;\ i = 1, 2, 3, \dots, n\}$ be a finite family of non-empty arbitrary d. g., then the subd. g. 's $H, K \subseteq D$ are called:

(a) R-bottom equal ($H \eqsim_R K$) if $L_n(V(H)) = L_n(V(K))$,

(b) R-top equal ($H \simeq_R K$) if $U_n(V(H)) = U_n(V(K))$,

(c) R-equal ($H \approx_R K$) if ($H \eqsim_R K$) and ($H \simeq_R K$).

Example 3.11. In Example (2.3), we have,

(a) R-bottom equal ($H \eqsim_R K$) if $L_n(V(H)) = L_n(V(K))$
Let $H = (V(H), E(H))$: $V(H) = \{\varpi_1\}$, $E(H) = \{(\varpi_1, \varpi_1)\}$ and $K = (V(K), E(K))$: $V(K) = \{\varpi_3\}$, $E(K) = \{(\varpi_3, \varpi_3)\}$ then $L_n(V(H)) = \{\varpi_4, \varpi_5\}$, $L_n(V(K)) = \{\varpi_4, \varpi_5\}$. Therefore, $L_n(V(H)) = L_n(V(K))$.

(b) R-top equal ($H \simeq_R K$) if $U_n(V(H)) = U_n(V(K))$
Let $H = (V(H), E(H))$: $V(H) = \{\varpi_3\}$, $E(H) = \{(\varpi_3, \varpi_3)\}$ and $K = (V(K), E(K))$: $V(K) = \{\varpi_1, \varpi_3\}$, $E(K) = \{(\varpi_1, \varpi_1), (\varpi_3, \varpi_1), (\varpi_3, \varpi_3)\}$ then $U_n(V(H)) = \{\varpi_3\}$, $U_n(V(K)) = \{\varpi_3\}$. Therefore, $U_n(V(H)) = U_n(V(K))$.

(c) R-equal ($H \approx_R K$) if ($H \eqsim_R K$) and ($H \simeq_R K$)
Let $H = (V(H), E(H))$: $V(H) = \{\varpi_1, \varpi_5\}$, $E(H) = \{(\varpi_1, \varpi_1), (\varpi_1, \varpi_5), (\varpi_5, \varpi_5)\}$ and $K = (V(K), E(K))$: $V(K) = \{\varpi_2, \varpi_4\}$, $E(K) = \{(\varpi_2, \varpi_2), (\varpi_2, \varpi_4), (\varpi_4, \varpi_4)\}$ then $L_n(V(H)) = \{\varpi_1, \varpi_2, \varpi_4, \varpi_5\}$, $L_n(V(K)) = \{\varpi_1, \varpi_2, \varpi_4, \varpi_5\}$, $U_n(V(H)) = \phi$, $U_n(V(H)) = \phi$. Therefore, $L_n(V(H)) = L_n(V(K))$ and $U_n(V(H)) = U_n(V(K))$.

Definition 3.12. Let $\boldsymbol{D} = \{D_i;\ i = 1, 2, 3, \dots, n\}$ be a finite family of non-empty arbitrary d. g. and subd. g. 's $H, K \subseteq D$. Then

(a) H is called R-bottom included in K (denoted by $H \subseteq_R K$) if $L_n(V(H)) \subseteq L_n(V(K))$,

(b) H is called R-top included in K (denoted by $H \sqsubseteq_R K$) if $U_n(V(H)) \subseteq U_n(V(K))$,

(c) H is called R-roughly included in K (denoted by $H \,\overline{\subseteq}_R\, K$) if ($H \subseteq_R K$) and ($H \sqsubseteq_R K$).

Example 3.13. In Example (2.3), we have,

(a) H is called R-bottom included in K (denoted by $H \subseteq_R K$) if $L_n(V(H)) \subseteq L_n(V(K))$
Let $H = (V(H), E(H))$: $V(H) = \{\varpi_3\}$, $E(H) = \{(\varpi_3, \varpi_3)\}$ and $K = (V(K), E(K))$: $V(K) = \{\varpi_4\}$, $E(K) = \{(\varpi_4, \varpi_4)\}$

then $L_n(V(H)) = \{\varpi_4, \varpi_5\}$, $L_n(V(K)) = \{\varpi_1, \varpi_4, \varpi_5\}$. Therefore, $L_n(V(H)) \subseteq L_n(V(K))$.

(b) H is called R-top included in K (denoted by $H \overline{\sqsubset}_R K$) if $U_n(V(H)) \subseteq U_n(V(K))$

Let $H = (V(H), E(H))$: $V(H) = \{\varpi_2, \varpi_5\}$, $E(H) = \{(\varpi_2, \varpi_2), (\varpi_2, \varpi_5), (\varpi_5, \varpi_5)\}$ and $K = (V(K), E(K))$: $V(K) = \{\varpi_1, \varpi_2, \varpi_3\}$, $E(K) = \{(\varpi_1, \varpi_1), (\varpi_1, \varpi_2), (\varpi_2, \varpi_2), (\varpi_3, \varpi_1), (\varpi_3, \varpi_3)\}$ then $U_n(V(H)) = \phi$, $U_n(V(K)) = \{\varpi_3\}$. Therefore, $U_n(V(H)) \subseteq U_n(V(K))$.

(c) H is called R-roughly included in K (denoted by $H \overline{\subseteq}_R K$) if $(H \subseteq_R K)$ and $(H \overline{\sqsubset}_R K)$

Let $H = (V(H), E(H))$: $V(H) = \{\varpi_1, \varpi_2, \varpi_5\}$, $E(H) = \{(\varpi_1, \varpi_1), (\varpi_1, \varpi_2), (\varpi_1, \varpi_5), (\varpi_2, \varpi_2), (\varpi_5, \varpi_2), (\varpi_5, \varpi_5)\}$ and $K = (V(K), E(K))$: $V(K) = \{\varpi_1, \varpi_3, \varpi_4\}$, $E(K)$ = then $L_n(V(H)) = \{\varpi_1, \varpi_2, \varpi_4, \varpi_5\}$, $L_n(V(K)) = V(D)$, $U_n(V(H)) = \{\varpi_3\}$, $U_n(V(H)) = \{\varpi_1, \varpi_3\}$. Therefore, $L_n(V(H)) \subseteq L_n(V(K))$ and $U_n(V(H)) \subseteq U_n(V(K))$.

Definition 3.14. Let $\boldsymbol{D} = \{D_i;\ i = 1, 2, 3, \dots, n\}$ be a finite family of non-empty arbitrary d. g. and subd. g. 's $H, K \subseteq D$. Then

(a) $\varpi \underline{\in}_R V(H)$ if $\varpi \in L_n(V(H))$,

(b) $\varpi \overline{\in}_R V(H)$ if $\varpi \in U_n(V(H))$,

(c) $\varpi \overline{\underline{\in}}_R V(H)$ if $\varpi \in L_n(V(H))$ and $\varpi \in U_n(V(H))$.

Example 3.15. In Example (2.3), we have,

(a) $\varpi \underline{\in}_R V(H)$ if $\varpi \in L_n(V(H))$

Let $H = (V(H), E(H))$: $V(H) = \{\varpi_4\}$, $E(H) = \{(\varpi_4, \varpi_4)\}$, then $L_n(V(H)) = \{\varpi_1, \varpi_4, \varpi_5\}$. Therefore, $\varpi \underline{\in}_R V(H)$.

(b) $\varpi \overline{\in}_R V(H)$ if $\varpi \in U_n(V(H))$,

Let $H = (V(H), E(H))$: $V(H) = \{\varpi_1, \varpi_3\}$, $E(H) = \{(\varpi_1, \varpi_1), (\varpi_3, \varpi_1), (\varpi_3, \varpi_3)\}$, then $U_n(V(H)) = \{\varpi_3\}$. Therefore, $\varpi \overline{\in}_R V(H)$.

(c) $\varpi \overline{\underline{\in}}_R V(H)$ if $\varpi \in L_n(V(H))$ and $\varpi \in U_n(V(H))$.

Let $H = (V(H), E(H))$: $V(H) = \{\varpi_2, \varpi_3, \varpi_5\}$, $E(H) = \{(\varpi_2, \varpi_2), (\varpi_2, \varpi_3), (\varpi_3, \varpi_3), (\varpi_5, \varpi_2), (\varpi_5, \varpi_5)\}$, then $L_n(V(H)) = \{\varpi_2, \varpi_3, \varpi_4, \varpi_5\}$, $U_n(V(H)) = \{\varpi_2, \varpi_3\}$. Therefore, $\varpi \overline{\underline{\in}}_R V(H)$.

4. Conclusions

A generalization of approximation operators in rough set theory is introduced using a finite number of a finite d. g. 's and based on mixed degree systems. Proposition (3.8) and Corollary (3.9) show the effectiveness of this new approach in increasing the accuracy of the approximation of d. g. 's since $\eta_n(V(H)) \geq \max\{\eta_{mi}(V(H)): i = 1, 2, \dots, n\}$. It is clear from Proposition (3.8) that by using the lower and upper approximation defined in Definition (2.1); we decrease the boundary region of this d. g. by using the lower and upper approximation defined in Definition (2.2.1) in [21].

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