Z-Nearly Prime Submodules

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Abstract: ***Let R be a commutative ring with identity and N be a proper submodule N of R-moduLe M is called prime if whenever $rx \in N$; $r \in R$, $x \in M$, implies either $x \in N$ or $r \in [N:M]$. In this paper we say that N is Z-nearly prime submodule of R-moduLe M, if whenever $f \in M^* = Hom(M,R)$, $x \in M$ such that $f(x) \cdot x \in N$, then either $x \in N + J(M)$ or $f(x) \in [N + J(M):M]$, $J(M)$ is the Jacobson radical of M. We prove some result of this type of submodules.***

Keywords: prime submodule,Z-prime submodule, nearly prime submodule, nearly regular module

1. Introduction

Throughout this paper$R$ is commutative ring with one. And M is a unitary R-module.A proper submodule N of an R-module M is called a prime submodule if for each $r \in R$, $x \in M$, such that $rx \in N$, then either $x \in N$ or $r \in [N:M]$, where $[N:M] = \{r : r \in R, rM \subseteq N\}$, [1]. There are several generalization of the notion of prime submodules as Z-prime. The definition of Z-prime is came in [2], as following: we say that a proper submodule N of an R-module M is called Z-prime if for each $x \in M$, $f \in M^* = Hom(M,R)$, such that $f(x).x \in N$ implies that either $x \in N$ or $f(x) \in [N:M]$. In [3] we study nearly prime as a generalization of prime submodules and they define a nearly prime submodules as follows: a proper submodule N of an R-module M is called nearly prime, if whenever $rx \in N$, $r \in R$, $x \in M$ implies either $x \in N + J(M)$ or $r \in [N + J(M):M]$.

In this article, we study Z-nearly prime submodules as generalization of Z-prime submodules, we give basic properties and we illustrate the relation between Z-nearly prime submodule and Z-prime submodule. It is clear that each Z-prime submodule is Z-nearly prime submdule, but the converse is not true in general. However, we gives a condition under which the two concepts one equivalent. And we study the Z-nearly prime submodules in other modules such as F-regular, injective, divisible, injective hull modules.

2. Z-Nearly Prime Submodules

In this section wedefine the concepts of Z-nearly prime submodule andinvestigate some properties. As generalization of Z-prime submodles,

Definition (2.1):

A proper submodule N of R-moduLe M is called Z-nearly prime submodule of M if whenever $f \in M^* = Hom(M,R)$, $x \in M$ such that $f(x).x \in N$, then either $x \in N + J(M)$ or $f(x) \in [N + J(M):M]$; $J(M)$ is the Jacobson radical of M . Specially, an ideal I of a ring R is Z-nearly prime ideal of R if and only if I is a Z-nearly prime submodule of R-module M.

Remarks and examples (2.2):

1) It clear that every$Z$-prime submodule N of an R-module M is Z-nearly prime submoduleof M,but the converse is not true in general.
2) Let $M = Z \oplus Z$ as Z-module and consider the submodule$N = 2Z \oplus 0$ of $Z \oplus Z$, then N is not Z-nearly prime, since if we take$f: Z \oplus Z \to Z$, define by $f(n \cdot$ *m=2n , f3,03,0=63,0=18,0∈N* but *6∉N+JZ:Z=0* and $(3,0) \notin N + J(Z)$, thus N is not Z-nearly prime submodule of Z as Z-module.
3) If $N =< 2 >$, then N is Z-nearly prime submodule of Z as Z-module, where $J(Z) = 0$, $[N:Z] =< 2 >$.
4) The direct sum of two Z-nearly prime submodules of an R-module M_1 and M_2 need not to be Z-nearly prime submoules of M_1 andM_2 for example:

Let $M = Z \oplus Z$. Let$N_1 = 2Z$ is a Z-prime submodule of Z as Z-module, hence is$Z$-nearly prime and $N_2 = (0)$ is Z-prime, hence is Z-nearly prime but $2Z \oplus 0$ is not Z-nearly prime in $Z \oplus Z$, by (2).

Proposition (2.3):

If N is Z-nearly prime submodule of an R-module M, K is any proper submodule of M such that $K \nsubseteq N$ and $J(M) \subseteq K$, then $N \cap K$ is Z-nearly prime submodule in M

Proof:

Since $K \nsubseteq N$, then $N \cap K$ is a proper submodule in M. Let $f \in M^* = Hom(M,R)$, $x \in M$ such that $f(x).x \in N \cap K$. We want to prove either $x \in (N \cap K) + J(M)$ or $f(x) \in [(N \cap K) + J(M):M]$, suppose that $x \notin (N \cap K) + J(M) = (N + J(M)) \cap K$, then $x \notin N + J(M)$, but N is Z-nearly prime of an R-module M, then $f(x) \in [N + J(M):M]$, then $f(x)M \subseteq N + J(M)$. Since $f(x).x \in N \cap K$, then $f(x)x \in K$, then $f(x)M \subseteq K$, thus $f(x)M \subseteq \left(N + J(M)\right) \cap K = (N \cap K) + J(M)$. Then $f(x) \in [(N \cap K) + J(M):M]$. Which implies that$N \cap K$ is Z-nearly prime in M.

Recall thata ring R is said to be a good ring if $J(R).M = J(M)$; where $M$is an R-module,[4]

Corollary (2.4):

Let R be a good ring and N be an R-module M, K be any proper submodule of M such that $K \nsubseteq N$, $J(K) = K$, then $N \cap K$ is Z-nearly prime in M

Proof:
Since $K \nsubseteq N$, then $N \cap K$ is a proper submodule in M. Since R be a good ring and $J(K) = K$, then $J(M) \subseteq K$, by above proposition $N \cap K$ is Z-nearly prime in M.

Proposition (2.5):
Let N and K be two Z-nearly prime submodules of M and either $J(M) \subseteq N$ or $J(M) \subseteq K$, then $N \cap K$ is Z-nearly prime of M

Proof:
Since $N \cap K \subseteq N$ and N is a Z-nearly prime submodule of M, then $N \cap K$ is a proper submodule in M. Let $f \in M^* = Hom(M,R)$, $x \in M$ such that $f(x).x \in N \cap K$. We want to show that either $x \in (N \cap K) + J(M)$ or $f(x) \in [(N \cap K) + J(M): M]$, suppose that $f(x) \notin [(N \cap K) + J M:M$, then $f x M \nsubseteq (N \cap K) + J(M)$, then $f x M \nsubseteq N + J(M)$ and $f(x)M \nsubseteq K + J(M)$. Since N and K are two Z-nearly prime submodules of M, then either $x \in N + J(M)$ or $f(x) \in [N + J(M): M]$, and either $x \in K + J(M)$ or $f(x) \in [K + J(M): M]$, then $x \in (N + J(M)) \cap (K + J(M))$. If $J(M) \subseteq N$, then $x \in N \cap (K + J(M))$, then $x \in (N \cap K) + J(M)$. If $J(M) \subseteq K$, then $x \in \big(N + J(M)\big) \cap K$, hence $x \in (N \cap K) + J(M)$, which implies that $N \cap K$ is Z-nearly prime of M.

Proposition(2.6):
Let N be a submodule of an R–module M. If $[N + J(M): M]$ is a maximal ideal of R, then N is a $Z-$nearly prime submodule of M .

Proof:
Let $f \in Hom(M,R)$, $x \in M$ such that $f(x).x \in N$,if $f(x) \notin [N + J(M): M]$ and since $[N + J(M): M]$ is a maximal ideal of R, then $R = \langle f(x)\rangle + [N + J(M): M]$, where $\langle f(x)\rangle$is an ideal of R generated by $f(x)$, hence there exists $s \in R$ and $k \in [N + J(M): M]$ and $1 = sf(x) + k$ .Thus, $x = sf(x)x + kx \in N + J(M)$, therefore $N$is a$Z-$nearly prime submodule of M.

The next example show that the converse of remark (2.2)(1)is nottrue.
Let $M = Z \oplus Z_{16}$ as Z-module, the submodule $N = Z \oplus (0)$of $M$is not Z-prime, see [2], but it is a Z-nearly prime submodule of M, because $[N + J(M): M] = 2Z$ is a maximal ideal of Z and hence by proposition (2.6), $N = Z \oplus (0)$ is a Z-nearly prime submodule of M ; $J(M) = \{(0,\bar{0}), (0,\bar{2}), (0,\bar{4}), (0,\bar{6}), (0,\bar{8})(0,\overline{10}), (0,\overline{12}),(0,\overline{14})\}$.

Proposition (2.7):
Let N be a proper submodule ofan R–module$M$ such that $[K : M] \not\subset [N + J(M) : M]$ for each submodule$K$ of $M$and $N + J(M) \subsetneq K$.If $[N + J(M) : M]$ is a prime ideal of R, then N is aZ-nearly prime submodule of M.

Proof:
Let $f \in Hom(M,R)$, $x \in M$ such that $f(x).x \in N$ and suppose $x \notin N + J(M)$. It is clear that the submodule$K = \langle x\rangle + [N + J(M)]$,and so $[K : M] \not\subset [N + J(M) : M]$.Then there exists $s \in [K : M]$ and $s \notin [N + J(M) : M]$ .Thus, $sM \subseteq K$ and $sM \not\subset N + J(M)$ .But $sM \subseteq K$ implies $f(x)sM \subseteq rK = f(x)[N + J(M) + \langle x\rangle] \subseteq N + J(M)$ and $f(x)s \in [N + J(M) : M]$ .Since $[N + J(M) : M]$ is a prime ideal and $s \notin [N + J(M) : M]$, $f(x) \in [N + J(M) : M]$ .Therefore N is a Z-nearlyprime submoduleof M.

Proposition(2. 8):
Let N be a submodule of an R-module $M$and $P = [N + J(M) :_R \langle e\rangle]$. If the ideal $[N + J(M): \langle e\rangle] = P$, for each $e \in M$; $e \notin N + J(M)$, then N is a Z-nearlyprime submodule of M.

Proof:
Let $f \in Hom(M,R)$, $x \in M$ such that $f(x).x \in N$ and suppose $x \notin N + J(M)$. Thus $f(x) \in [N + J(M) : \langle x\rangle]$.But $[N + J(M) : \langle x\rangle] = P$, so $f(x) \in P$. Therefore$N$ is aZ-nearlyprime submoduleof M.

Now, we give the following lemma:

Lemma(2.9):
Let N be a submodule of an R-module M .If the submodule $[N + J(M) :_M <r>] = N + J(M)$,for each $r \in R$, $r \notin P$, then the ideal $[N + J(M) :_R <e>] = P$, for each $e \in M$; $e \notin N + J(M)$.

Proof:
Let $e \in M$; $e \notin N + J(M)$ .It is clear that $P \subseteq [N + J(M):_R \langle e\rangle]$. Let $r \in [N + J(M):_R \langle e\rangle]$, then $re \in N + J(M)$ .Suppose $r \notin P = [N + J(M):_R M]$. Since $[N + J(M) :_M \langle r\rangle] = N + J(M)$ and $e \in [N + J(M) :_M \langle r\rangle]$, so $e \in N + J(M)$, which a contradicts our assumption.Thus, $r \in P$ for each $e \in M$ such that $e \notin N + J(M)$.Therefore $[N + J(M): \langle e\rangle] = P$.

Corollary (2.10):
Let N be a submodule of an R-module $M$and $P = [N + J(M): M]$.If $[N + J(M):_M <r>] = N + J(M)$, for each $r \in R$, then N is a Znearlyprime submodule of M .

Proposition (2.11):
Let M be an R-module and N be a submodule of M. If $[N + J(M): M] = [N + J(M): K]$ for each submodule K of M such that $K \supsetneq N + J(M)$, then N is Z-nearly prime submodule of M.

Proof:
To prove N is aZ-nearly primesubmodule of an $R-$ module$M$. Let $f \in Hom(M,R)$,$x \in M$ such that $f(x).x \in N$ and suppose $x \notin N + J(M)$. Let $K = [N + J(M)] + <x> \supsetneq N + J(M)$. Then $x \in K$ and so $(x) \in [N: K] \subseteq [N + J(M): K] = [N + J(M): M]$. It follows that $f(x) \in [N + J(M): M]$ and hence N is Z-nearly prime submodule of.

Now we state and prove the following lemma.

Lemma(2.12):
Let N be a submodule of an R-module M .If the submodule $[N + J(M) :_M <r>] = N + J(M)$, for each $r \in R$, $r \notin P$, then the ideal $[N + J(M) :_R <e>] = P$ for each $e \in M$; $e \notin N + J(M)$.

Proof:
Let $e \in M$; $e \notin N + J(M)$ .It is clear that $P \subseteq [N + J(M) :_R <e>]$ .Let $r \in [N + J(M) :_R <e>]$, then $re \in N + J(M)$ .Suppose $r \notin P = [N + J(M) :_R M]$. Since $[N + J(M) :_M <r>] = N + J(M)$ and $e \in [N + J(M) :_M <r>]$, so $e \in N + J(M)$ which a contradicts our assumption .Thus, $r \in P$ for each $e \in M$ such that $e \notin N + J(M)$ .Therefore $[N + J(M): <e>] = P$.

Corollary (2.13):
Let N be a submodule of an R-module M and $P = [N + J(M): M]$ .If $[N + J(M):_M <r>] = N + J(M)$, for each $r \in R$, then N is a nearlyprime submodule of M.

3. Z-nearly Prime Submodules

In this section we study the Z-nearly prime submodules in others modules such as F -regular, Z-regular, injective, divisible, injective hull modules. First we need the following proposition:

Proposition(3.1):
If N is a Z-nearly Prime submodule of an R–module M and $J(M) = 0$, then N is a Z-prime submodule of M .

Proof:
It is clear.
Recall that an R – module M is said to be F–regular if each submodule of M is pure,[5].

Corollary(3.2):
If N is aZ–nearlyprime submodule of an R-module M and M is a F-regular, then N is a Z-prime submoduleof M.

Proof:
Since M is an F–regular R-module, so $J(M) = 0$, [7].Hence the result follows by proposition (3.1).

Corollary (3.3):
If N is a Z –nearly Prime submodule of an R-module M and$R / ann(x)$ is anF-regular ring for every $0 \neq x \in M$, then N is a Z-prime submodule of M .

Proof:
Since $R/ann(x)$ is a regular ring for every $0 \neq x \in M$, then M is an$F$–regular R-module by [7].Hence the result follows by Corollary (3.2).

Recall that an R -module M is called Z -regular (for simplicity just regular) if $\forall m \in M, \exists L \in M* = Hom(M, R)$ such that $= L(m).m$, [8].

Corollary(3.4):
If N is a Z–Prime submodule of Z-regular R–module M, then N is a primesubmodule of M.

Proof:
Since M is a Z–regular R–module, then M is a F–regular R-module by [7].Hence the result follows by Corollary (3.2).

Corollary(3.5):
If $N$is a Z–nearly prime submodule of an R–module M and $J(N) = J(M) \cap N$ for each N submodule of M, then N is a Z-prime submodule of M

Proof :
Since $J(N) = J(M) \cap N$, so $J(M) = 0$ by [6, proposition(1-33),p.22]. Hence the result follows by proposition (3.1).

Now, because of the fact R is good ring if and only if $J(N) = J(M) \cap N$for each submodule$N$ of an R–module M , [4]. Then the following is a consequence of corollary(3.5).

Corollary(3.6):
If N is a Z-nearly prime submodule of an R–module M and$R$ is a good ring, then N is a Z-prime submodule of.

Proof:
Since R is a good ring, then J(N) = J(M) ∩ N for each submodule N of M by [9] .Therefore, $J(M) = 0$ by [6,proposition(1-33), p.22] and hence N is a Z-prime submodule of M.

Recall that an R–module M is called divisible if and only if $rM = M$, $\forall\, 0 \neq r \in R$,[4].

By using this concept,we have the following:

Proposition(3.7):
Let R is PID and M is a divisible R–module such that $J(M) \neq M$. If N is a Z–nearly prime submoduleof M, then N is a Z-prime submodule of M.

Proof :
Since M is a divisible R -module and $J(M) \neq M$, so $J(M) = 0$ by [6,proposition(1-4),p.12]. Hence the result follows immediately from proposition (3.1).

Corollary(3.8):
Let M be an injective define on an integral domain $R$and $J(M) \neq M$. If N is a Z-nearlyPrime submodule, then N is a Z-prime submoduleof M.

Proof:
Since M is an injective R–module, so M is a divisible R–module by, [4]. But $J(M) \neq M$, so $J(M) = 0$ by, [6,proposition(1-4),p.12] .Hence the result follows immediately from proposition(3.1).

Recall that a submodule$N$ of an R–module M is said to be essential, if has non- trivial intersection with every non- zero submodule of M,[4] .

Recall that an R-module M is called an injective hull or injective envelope of a module M if it is an essential extension of M and an injective module,[4].

By using this concept,we can give the following result:

Corollary(3.9)

Let M be an injective hull on an integral domain R and $(M) \neq M$. If N is a Z-nearlyPrime submodule, then N is a Z-prime submoduleof M.

Proof :

Since M is an injective hull R-module,so M is an injective and hence M is a divisible R-module by [4]. But $J(M) \neq M$, so $J(M) = 0$ by [6,proposition(1-4),p.12]. Hence the result follows immediately from proposition(3.1).

Corollary(3.10):

Let M has no proper essential extensions R- module and $J(M) \neq M$. If N is a Z–nearly Prime submodule, then N is a Z-prime submoduleof M.

Proof:

Since Mhave no proper essential extensions$R$–module, so M is an injective by [4] and hence M is a divisible R-module by [4]. But $J(M) \neq M$, so $J(M) = 0$ by [6,proposition(1-4), p.12] .Hence the result follows immediately from proposition(3.1).

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