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R-Annihilator – Hollow Modules

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Abstract: ***Let R be an associative ring with identity and let M be a unitary left R- module .We call a non-zero module M, R-annihilator –hollow module if every proper submodule of M is R-annihilator–small submodule of M..The sum A_M of all such submodules of M contains the Jacobson radical J(M) and the singular submodule Z(M).When M is finitly generated and faithful, we study A_M and K_M in this paper .Conditions when A_M is R-annihilator-small and $K_M = A_M$, J(M) $\subseteq A_M$ and $Z(M) \subseteq A_M$ are given .***

Keywords: hollow modules, annihilators, R-annihilator- hollow modules

1. Introduction

Throughout this paper all rings are associative ring with identity and modules are unitary left modules. In [1] , Nicholson and Zhou defined annihilator-small right(left) ideals as follows : a left ideal A of a ring R is called annihilator-small if A+T=R, where T is a left ideal , implies that r(T)=0 , where r(T) indicates the right annihilator.

Kalati and.Keskin consider this problem for modules in [2] as follows:- let M be an R- module and S=End(M). A submodule K of M is called annihilator-small if K+T =M, T a submodule of M ,implies that $r_S(T)$ =0 , where r_S indicates the right annihilator of T over S= End(M), where $r_S(T)=\{f \in S ; f(T) = 0 , \forall t \in T\}$.

These observations lead us to introduce the following concept. A non-zero module M ,is called

R-annihilator –hollow module if every proper submodule of M is R-annihilator –small submodule of M.

In fact, the set K_M of all elements k such that Rk is semisubmodule and annihilator-small. And contains both the Jacobson radical and the singular submodule when M is finitely generated and faithful.

The submodule A_M generated by K_M is a submodule of M analogue of the Jacobson radical that contains every R-annihilator-small submodules. In this work we give some basic properties of

R-annihilator-hollow modules and various.

Characterizations

We abbreviate the Jacobson radical as Rad(M) and the singular submodule as Z(M) for any R- module M. The notations $N \leq^e M$ mean that a submodule N of M is essential in the module M.
See [1] / [2].

2. R- annihilator Small submodules

In this section, we introduce the concept of the R-annihilator -small submodule and we illustrate it by examples. We also give some basic properties of this class of submodules.
We start this section by definition:

Definition (2.1):
We say that a submodule N of an R-module M is a R-annihilator-small submodule (R-a-small) if whenever N+T=M, T is a submodule of M, implies that $Ann_\ell(T)=0$, where $Ann_\ell(T)=\{r \in R ; r.T = 0\}$. Clearly $Ann_\ell(T)$ is a left ideal of R. We write $N \ll^a M$, see [3].

Let I be an ideal of a ring R. We say that I is R-a-small ideal of R if I is R-a-small submodule of R as an R-module.

Examples (2.2):

1) For an R- module M, M is not R-a-small submodule of M, where M=M+0 and ann 0 ={r∈R ; r. 0 =0}=R≠0.
2) Let R be a commutative ring and I be an ideal of R. Then one can easily show that I is a-small ideal of R if and only if I is R-a-small ideal of R as

R-module, where r(I)=ann(I) when R is a commutative ring.

1) Let N be a submodule of an R-module M and let S= EndM. If N is a-small submodule of M then need not be N is R-a-small submodule of M as the following example shows:

Consider the module Z_6 as Z-module. $\{\bar{0}\}$ is a small submodule of Z_6 and hence $\{\bar{0}\}$ is a-small submodule of Z_6, by remark(1.3.4). But $Z_6=\{\bar{0}\}+Z_6$ and ann $Z_6=\{ n \in Z ; n. Z_6=0\} = 6Z \neq 0$. Thus $\{\bar{0}\}$ is not Z-a-small submodule of Z_6.See[7].

2) It is known that a non-zero small submodule can not be a direct summand. But this is not true for R-a-small submodules. For example, consider the module $M=Z_2 \oplus Z_2$ as Z_2-module and

let $A=Z_2 \oplus 0$.
Clearly $M=A \oplus Z_2= A \oplus((\bar{1},\bar{1}))$ and ann $0 \oplus Z_2$ = ann$((\bar{1},\bar{1}))$=0. Thus A is
Z-a-small submodule of M.

The following three Corollary give more properties of R-a-small submodules.

Corollary(2.3):
Let K and N be a submodules of an R- module M such that K≤ N. If
$\frac{N}{K}$ is R-a-small submodule of $\frac{M}{K}$, then N is R-a-small submodule of M.

Proof:

Let N, K be submodules of an R- module M such that K≤ N and $\frac{N}{K}$ R-a-small submodule of $\frac{M}{K}$. Let π: M $\to \frac{M}{K}$ be the natural epimorphism. Therefore $\pi^{-1}(\frac{N}{K})$ is R-a-small submodule of M , by prop (2.1.5). But $\pi^{-1}(\frac{N}{K})$ =N. Thus N is R-a-small of M.See[7].

Corollary (2.4):

Let M be an R-module and let $K \le N \le L \le M$ such that $\frac{L}{N}$ is R-a-small submodule of $\frac{M}{N}$ then $\frac{L}{K}$ is R-a-small submodule of $\frac{M}{K}$.

Proof:

Let f: $\frac{M}{K} \to \frac{M}{N}$ be the map defined by f(x+K)= x+N, ∀ x ∈ M. One can easily to show f is an epimorphism. Since $\frac{L}{N}$ is R-a-small submodule of $\frac{M}{N}$, therefore $\frac{L}{K} = f^{-1}\left(\frac{L}{N}\right)$ is R-a-small submodule of $\frac{M}{K}$, by prop(2.1.5) Thus $\frac{L}{K}$ is R-a-small submodule of $\frac{M}{K}$.See[7].

Corollary (2.5):

Let M be an R-module and let $K \le N \le M$, $K' \le N' \le M$, if $\frac{N+N'}{K+K'}$ is R-a-small submodule of $\frac{M}{K+K'}$. Then:-

1- $\frac{N+K'}{K}$ is R-a-small submodule of $\frac{M}{K}$.

2- $\frac{K+N'}{K'}$ is R-a-small submodule of $\frac{M}{K'}$.

3- $\frac{N}{K} \oplus \frac{N'}{K'}$ is R-a-small submodule of $\frac{M}{K} \oplus \frac{M}{K'}$.

Proof:

1- Let $f_1: \frac{M}{K} \to \frac{M}{K+K'}$ be a map defined by f(x+K)= x+K+ K′, ∀ x ∈ M and let $f_2: \frac{M}{K'} \to \frac{M}{K+K'}$ be a map defined by f(m+K′) = m+K+ K′, m ∈ M .One can easily show that each of f_1 , f_2 is an epimorphism. Since $\frac{N+K'}{K+K'} \le \frac{N+N'}{K+K'}$ and $\frac{N+N'}{K+K'}$ is R-a-small submodule of $\frac{M}{K+K'}$, then $\frac{N+K'}{K+K'}$ is R-a-small submodule of $\frac{M}{K+K'}$, by prop(2.1.4).

Thus $\frac{N+K'}{K} = f_1^{-1}\left(\frac{N+K'}{K+K'}\right)$ is R-a-small submodule of $\frac{M}{K}$,
By prop (2.1.5) ⋯ (1)

2- Also $\frac{K+N'}{K+K'} \le \frac{N+N'}{K+K'}$ and $\frac{N+N'}{K+K'}$ is R-a-small submodule of $\frac{M}{K+K'}$ then $\frac{K+N'}{K+K'}$ is R-a-small submodule of $\frac{M}{K+K'}$, by prop (2.1.4),
therefore $\frac{K+N'}{K'} = f_2^{-1}\left(\frac{K+N'}{K+K'}\right)$ is R-a-small submodule of $\frac{M}{K'}$, by prop (2.1.5) ⋯ (2)

3- From (1) and (2) , we have
$\frac{N}{K} \oplus \frac{N'}{K'}$ R-a-small submodule of $\frac{M}{K} \oplus \frac{M}{K'}$, by prop (2-1-8).
Recall that an R-module M is called faithful if ann(M)=0, see[7].

Let R be an integral domain. Recall that an R-module M is called a torsion free R-module if ann(x) =0,for every non-zero element x in M, see[4].

Proposition (2.6):

Let M be a faithful R-module. Then every small submodule of M is R-a- small.

Proof:

Let M be faithful R-module and let N be a small submodule of M. To show N is R-a-small submodule of M. Let M = N+U. Since N is small in M, then M=U and hence annM = annU. So annU=0. Thus N is R-a-small submodule of M. The following corollary follows immediately of proposition (2.6).

Corollary (2.7):

Let R be an integral domain and let M be a projective R-module. Then every proper submodule of M is R-a-small submodule of M.

In particular, every proper submodule of a free module over an integral domain is R-a-small.

Remark (2. 8):

Let M be an R-module if there exists a submodule N of M such that N is R-a-small submodule of M. Then M is faithful.

Proof:

Since M=N+M and N is R-a-small submodule of M, then ann M=0. Thus M is faithful.

Proposition (2.9):

Let I be an ideal of a commutative ring R and let M be an R-module if
I M is R-a-small in M, then I is a-small ideal of R.

Proof:

Let R=I+J, where J is an ideal of R. Then M=RM =(I+J)M=I M+J M. Since I M is R-a-small in M, then ann J M=0. But ann J ≤ ann J M. Therefore ann J=0. Thus I is R-a-small in R.

Recall that an R-module M is called a multiplication module if for every submodule N of M, there exists an ideal I of R such that N=I M. Equivalenty, an R-module M is a multiplication module if and only if N= (N:M)M, for every submodule N of M. where (N:M)= {r ∈ R ; rM ≤ N},see [5].

We end the section by the following corollary and proposition we give various characterizations of R-annihilator-small submodules.

Corollary (2.10):

Let M be a multiplication module over a commutative ring R and let N be a submodule of M. If N is R-a-small submodule of M, then (N:M) is a-small ideal of R.

Proposition (2.11):

Let M be a module and K an R-a-small submodule of M. If Rad(M) is a small submodule of M and Z(M) is finitely generated, then $K + Rad(M) + Z(M)$ is R-a-small submodule of M .

Proof:
Let $Z(M)=R z_1+ R z_2 + . . . + R z_n$, where $z_i \in Z(M)$ $\forall$ i=1,2,…,n.
To show K+Rad(M)+Z(M) is R-a-small submodule of M, let $K + Rad(M) + Z(M) + X = M$, where X is a submodule of M. Since Rad(M) is a small submodule of M, then $K + Z(M) + X = M$. But K is R-a-small submodule of M , therefore ann(Z(M)+X)= ann(R z_1+R z_2+. . . +R z_n + X)=0. So($\bigcap_{i=1}^{n}$ annR z_i) ∩ annX =0. Since $z_i \in Z(M)$,then ann $Z_i \leq^e$ R,∀i=1,2,…,n.And hence $\bigcap_{i=1}^{n}$ annR z_i $\leq^e$ R, by [12]. So ann X=0. Thus K+Rad(M)+Z(M)is R-a-small submodule of M.

3. R- annihilator hollow modules

In this section, we introduce the concept of the R-annihilator –hollow module and we study the basic properties of this type of module

We start this section by definition:

Definition (3.1):
Anon-zero module M is called R-annihilator-hollow module (R-a-hollow) if every proper submodule of M is R-annihilator-small submodule of M.

Examples (3.2):
1- An R-a-small submodule of an R-module M need not be small submodule.

For example, consider the module Z as Z- module. For every n>1, claim that nZ is Z-a-small submodule of Z. To show that, let Z=nZ +mZ, where mZ is a submodule of Z. Since Z has no-zero divisors, then ann mZ= {r∈Z ; r.mZ=0}=0. Thus nZ is Z-a-small submodule of Z. But it is known that {0}is the only small submodule of Z, therefore Z as Z- module is R-annihilator-hollow module.

2-A small submodule of an R-module M need not be R-a-small submodule. For example. Consider Z_4 as Z-module. One can easily show that $\{\bar{0}\}$ and $\{\bar{0},\bar{2}\}$ are small submodule of Z_4. But $Z_4=\{\bar{0}\}+ Z_4$ and $Z_4 =\{\bar{0},\bar{2}\} + Z_4$ and ann Z_4={n∈Z ; n. Z_4=0} =4Z≠0. Thus each of $\{\bar{0}\}$ and $\{\bar{0},\bar{2}\}$ is not Z-a-small submodule of Z_4, therefore Z_4 is not R-annihilator-hollow module.

3-Let $Z_{p^\infty} = \{x\in \frac{Q}{Z}$; $x =\frac{r}{p^n}$ +Z for some r∈Z, n∈N, p prime$\}\leq \frac{Q}{Z}$.
$Z \lneqq <\frac{1}{P}+Z> \lneqq <\frac{1}{p^2}+Z> \lneqq \ldots$
$Z_{p^\infty} = \{0\} + Z_{p^\infty}$ if {0} is R-annihilator-small submodule of Z_{p^∞} then ann Z_{p^∞} ={r∈Z; n. Z_{p^∞}=0} = 0 ,therefore{0} is R-annihilator-small submodule of Z_{p^∞}. $<\frac{1}{P}+Z> + Z_{p^\infty} = Z_{p^\infty}$, ann Z_{p^∞} = 0.
$<\frac{1}{P}+Z>$ is R-annihilator-small submodule of Z_{p^∞}, therefore Z_{p^∞} as Z - module R-annihilator-hollow module,see[6].

4-Let N≤Q then N is R- annihilator- small submodule. Let L≤Q such that Q = N + L then annL = 0. Q is a torsion free then annA=0 for all A ≤Q, therefore N is R- annihilator-small submodule of Q then Q as Z-module is R-annihilator-hollow module but Q as Z-module is not hollow module.

The following two propositions give more properties of R-a-hollow module.

Proposition (3.3): Let f: M→ M' be a homomorphism and let M' is R-a-hollow module such that for all N≤M such that Kerf is small of M then M is R-a-hollow module.

Proof:
Let N $\lneqq$ M whith M=N+K, where K is a submodule of M. To show ann K=0.

f(N)+f(K)=f(M)= M' (f is epimorphism),if f(N)= M'=f(M) then f^{-1}(f(N))=M. N+Kerf=M, since Kerf is small of M then N=M (which is a contradiction). Therefore N≠M. Since f(N) is R-a-small submodule of f(M), then f(N)≠ M' =f(M). Thus M' is R-a-hollow module. But annf(K)=0 and annK≤annf(K)=0, therefore annK=0, then N is R-a-small submodule of M. Thus M is R-a-hollow module.

Proposition (3.4): If $\frac{M}{K}$ is R-a-hollow module then M is R-a-hollow module for all K submodule of M **Proof:** Suppos $\frac{M}{K}$ is R-a-hollow module and let N $\lneqq$ M such that M=N+L, L is submodule of M then annL=0.
$\frac{M}{K} = \frac{N+L}{K} = \frac{N+K}{K} + \frac{L+K}{K}$. Since $\frac{N+K}{K} \lneqq \frac{M}{K}$, then $\frac{N+K}{K}$ is R-a-small submodule of $\frac{M}{K}$, then ann$\frac{L+K}{K}$=0 .
Thus annL is submodule of ann$\frac{L+K}{K}$ = 0, therefore annL=0 and M is R-a-hollow module.See[6].

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