R-Annilhinator – Hollow Modules

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Abstract: Let R be an associative ring with identity and let M be a unitary left R-module. We call a non-zero module M, R-annilhinator–hollow module if every proper submodule of M is R-annilhinator-small submodule of M. The sum $A_M$ of all such submodules of M contains the Jacobson radical J(M) and the singular submodule Z(M). When M is finitely generated and faithful, we study $A_M$ and $K_M$ in this paper. Conditions when $A_M$ is R-annilhinator-small and $K_M= A_M, J(M) \subseteq A_M$ and $Z(M) \subseteq A_M$ are given.

Keywords: hollow modules, annihilators, R-annilhinator–hollow modules

1. Introduction

Throughout this paper all rings are associative rings with identity and modules are unitary left modules. In [1], Nicholson and Zhou defined annihilator-small right(left) ideals as follows: a left ideal A of a ring R is called annihilator-small if $A+T=R$, where T is a left ideal, implies that $r(T)=0$, where $r(T)$ indicates the right annihilator.

Kalati and Keskin consider this problem for modules in [2] as follows:- let M be an R-module and S=End(M). A submodule K of M is called annihilator-small if $K+T=M$, T a submodule of M ,implies that $r_S(T)=0$, where $r_S$ indicates the right annihilator of T over $S=End(M)$, where $r_S(T)={f \in S; f(T)=0, \forall \in T}$.

These observations lead us to introducing the following concept. A non-zero module M, is called R-annilhinator–hollow module if every proper submodule of M is R-annilhinator–small submodule of M.

In fact, the set $K_M$ of all elements k such that Rk is semisubmodule and annihilator-small. And contains both the Jacobson radical and the singular submodule when M is finitely generated and faithful.

The submodule $A_M$ generated by $K_M$ is a submodule of M analogue of the Jacobson radical that contains every R-annilhinator-small submodules. In this work we give some basic properties of R-annilhinator-hollow modules and various.

Characterizations

We abbreviate the Jacobson radical as $Rad(M)$ and the singular submodule as $Z(M)$ for any R-module M. The notations $N \leq M$ mean that a submodule N of M is essential in the module M.

See [1]/[2].

2. R-annilhinator Small submodules

In this section, we introduce the concept of the R-annilhinator-small submodule and we illustrate it by examples. We also give some basic properties of this class of submodules.

We start this section by definition:

Definition (2.1): We say that a submodule N of an R-module M is a R-annilhinator-small submodule (R-a small) if whenever $N+T=M$, T is a submodule of M, implies that $Ann_T(N)=0$, where $Ann_T(N)={r \in R; r.T=0}$. Clearly $Ann_T(N)$ is a left ideal of R. We write $N \ll R$, see [3].

Let I be an ideal of a ring R. We say that I is R-a-small ideal of R if I is R-a small submodule of R as an R-module.

Examples (2.2): 1) For an R-module M, if N is a submodule of M, where $M=M+0$ and $ann 0 = r \in R; r.0 =0=R \neq 0$.

2) Let R be a commutative ring and I be an ideal of R. Then one can easily show that I is a small ideal of R if and only if I is R-a small ideal of R as R-module, where $r(I)=ann(I)$ when R is a commutative ring.

1) Let N be a submodule of an R-module M and let S=End(M). If N is a small submodule of M then need not be N is R-a small submodule of M as the following example shows:

Consider the module $Z_6$ as $Z$-module. $\{0\}$ is a small submodule of $Z_6$ and hence $\{0\}$ is a small submodule of $Z_6$, by remark(1.3.4). But $Z_6=\{0\}+Z_6$ and $ann Z_6=\{ n \in Z ; n.Z_6=0 \}$ $=6Z\neq0$. Thus $\{0\}$ is not $Z$-a small submodule of $Z_6$. See [7].

2) It is known that a non-zero small submodule can not be a direct summand. But this is not true for R-a small submodules. For example, consider the module $M=Z_2 \oplus Z_2$ as $Z_2$-module and let $A= Z_2 \oplus 0$.

Clearly $M=A \oplus Z_2= A \oplus (\bar{1}, \bar{1})$ and $ann 0 \oplus Z_2 = ann(\bar{1}, \bar{1})=0$. Thus A is $Z_2$-a small submodule of M.

The following three Corollary give more properties of R-a small submodules.

Corollary (2.3): Let K and N be a submodules of an R-module M such that $K \subseteq N$. If $\frac{N}{K}$ is R-a small submodule of $\frac{M}{K}$, then N is R-a small submodule of M.
Proof:
Let N, K be submodules of an R-module M such that K ≤ N and
\[
\frac{N}{K} \text{ R-a-small submodule of } \frac{M}{K}.
\]
Let \( \pi: M \to \frac{M}{K} \) be the natural epimorphism. Therefore
\[
\pi^{-1}\left(\frac{N}{K}\right)
\]
is R-a-small submodule of \( M \), by prop (2.1.5). But \( \pi^{-1}\left(\frac{N}{K}\right) = N \). Thus N is R-a-small of M. See[7].

Corollary (2.4):
Let M be an R-module and let \( K \leq L \leq M \) such that \( \frac{L}{N} \) is R-a-small submodule of \( \frac{M}{N} \) then \( \frac{L}{K} \) is R-a-small submodule of \( \frac{M}{K} \).

Proof:
Let \( f: \frac{M}{K} \to \frac{M}{N} \) be the map defined by \( f(x+K)= x+N, \ \forall \ x \in M \). One can easily to show f is an epimorphism. Since \( \frac{L}{N} \) is R-a-small submodule of \( \frac{M}{N} \), therefore \( \frac{L}{K} = f^{-1}\left(\frac{L}{N}\right) \) is R-a-small submodule of \( \frac{M}{K} \), by prop(2.1.5) Thus \( \frac{L}{K} \) is R-a-small submodule of \( \frac{M}{K} \). See[7].

Corollary (2.5):
Let M be an R-module and let \( K \leq N \leq M \), \( K' \leq N' \leq M \), if \( \frac{N+N'}{K+K'} \) is R-a-small submodule of \( \frac{M}{K+K'} \). Then:-
1- \( \frac{L}{K} \) is R-a-small submodule of \( \frac{M}{K+K'} \).
2- \( \frac{N+K}{K+K'} \) is R-a-small submodule of \( \frac{M}{K} \).
3- \( \frac{N+K}{K} \oplus \frac{M}{K} \) is R-a-small submodule of \( \frac{M}{K} \) See[7].

Proof:
1- Let \( f_1: \frac{M}{K} \to \frac{M}{K+K'} \) be a map defined by \( f(x+K)= x+K+K' \), \( \forall \ x \in M \) and let \( f_2: \frac{M}{K} \to \frac{M}{K+K'} \) be a map defined by \( f(m+K') = m+K+K' \), \( m \in M \). One can easily to show that each of \( f_1 \), \( f_2 \) is an epimorphism. Since \( \frac{N+K}{K+K'} \leq \frac{N+K}{K} \) and \( \frac{N+K}{K} \leq \frac{N+N'}{K+K'} \) is R-a-small submodule of \( \frac{M}{K+K'} \), then \( \frac{N+K}{K} \) is R-a-small submodule of \( \frac{M}{K} \), by prop(2.1.4). Thus \( \frac{N+K}{K} = f_1^{-1}\left(\frac{N+K}{K+K'}\right) \) is R-a-small submodule of \( \frac{M}{K} \). By prop (2.1.5) \( \cdots (1) \)

2- Also \( \frac{M}{K+K'} \leq \frac{N+N'}{K+K'} \) and \( \frac{N+N'}{K+K'} \) is R-a-small submodule of \( \frac{M}{K+K'} \), by prop (2.1.4), therefore \( \frac{N+N'}{K+K'} = f_2^{-1}\left(\frac{N+K}{K+K'}\right) \) is R-a-small submodule of \( \frac{M}{K} \), by prop (2.1.5) \( \cdots (2) \)

3- From (1) and (2), we have \( \frac{N}{K} \oplus \frac{N}{K} \oplus \frac{M}{K} \oplus \frac{M}{K} \), by prop (2-1.8).

Remark (2.8):
Let M be a R-module if there exists a submodule N of M such that N is R-a-small submodule of M. Then M is faithful.

Proof:
Since M=\( N+U \) and N is R-a-small submodule of M, then \( \text{ann} M = 0 \). Thus M is faithful.

Proposition (2.9):
Let I be an ideal of a commutative ring R and let M be an R-module if
I \( M \) is R-a-small in M, then I is a-small ideal of R.

Proof:
Let R=I+J, where J is an ideal of R. Then M=RM=(I+J)M=I+JM. Since I M is R-a-small in M, then \( \text{ann} M = 0 \). But ann J \( \leq \text{ann} J = 0 \). Therefore ann J=0. Thus I is R-a-small in R.

Recall that an R-module M is called a multiplication module if for every submodule N of M, there exists an ideal I of R such that N=IM. Equivalently, an R-module M is a multiplication module if and only if N= (N;M)M, for every submodule N of M, where \( (N;M)=\{ r \in R ; \ rM \leq N \} \), see [5].

We end the section by the following corollary and proposition we give various characterizations of R-annihilator-small submodules.

Corollary (2.10):
Let M be a multiplication module over a commutative ring R and let N be a submodule of M. If N is R-a-small submodule of M, then \( (N;M) \) is a-small ideal of R.

Proposition (2.11):
Let M be a module and \( K \) an R-a-small submodule of M. If \( \text{Rad}(M) \) is a submodule of M and \( Z(M) \) is finitely

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generated, then $K + \text{Rad}(M) + Z(M)$ is $R$-a-small submodule of $M$.

**Proof:**

Let $Z(M)=Rz_1 + Rz_2 + \ldots + Rz_n$, where $z_i \in Z(M) \forall i=1,2,\ldots,n$.

To show $K + \text{Rad}(M) + Z(M)$ is $R$-a-small submodule of $M$, let $K + \text{Rad}(M) + Z(M) = X$, where $X$ is a submodule of $M$. Since $\text{Rad}(M)$ is a small submodule of $M$, then $K + Z(M) + X = M$. But $K$ is R-small submodule of $M$, therefore $\text{ann}(Z(M)+X)=\text{ann}(Rz_1+Rz_2+\ldots + Rz_n+X)=0$.

So $(\cap_{i=1}^{n}\text{ann}Rz_i) \cap \text{ann} X=0$. Since $z_i \in Z(M)$, then $\text{ann}Z\subseteq R\forall i=1,2,\ldots,n$. And hence $(\cap_{i=1}^{n}\text{ann}Rz_i) \subseteq R$. By [12]. So $\text{ann} X=0$. Thus $K+\text{Rad}(M)+Z(M)$ is $R$-a-small submodule of $M$.

3. **R- annihilator hollow modules**

In this section, we introduce the concept of the R-annihilator –hollow module and we study the basic properties of this type of module

We start this section by definition:

**Definition (3.1):**

An non-zero module $M$ is called R-annihilator-hollow module (R-a-hollow) if every proper submodule of $M$ is R-annihilator-small submodule of $M$.

**Examples (3.2):**

1. An R-a-small submodule of an R-module $M$ need not be small submodule.

For example, consider the module $Z$ as $Z$-module. For every $n>1$, claim that $nZ$ is $Z$-a-small submodule of $Z$. To show that, let $Z=nZ + nZ$, where $nZ$ is a submodule of $Z$. Since $Z$ has no zero divisors, then $nZ = \{\alpha \in Z; n\alpha = 0\}$. Thus $nZ$ is $Z$-a-small submodule of $Z$. But it is known that $\{0\}$ is the only small submodule of $Z$, therefore $Z$ as $Z$-module is R-annihilator-hollow module.

2. A small submodule of an R-module $M$ need not be R-a-small submodule. For example. Consider $Z_4$ as $Z$-module. One can easily show that $\{0\}$ and $\{0,2\}$ are small submodule of $Z_4$. But $Z_4=[0]+Z_4$ and $Z_4 = \{0,2\} + Z_4$ and $\text{ann} Z_4 = \{n \in \mathbb{Z}; nZ_4=0\} \neq Z_4 \neq 0$. Thus each of $\{0\}$ and $\{0,2\}$ is not $Z$-a-small submodule of $Z_4$, therefore $Z_4$ is not R-annihilator-hollow module.

3. Let $Z_p^{\infty} = \{x \in \mathbb{Q} / \mathbb{Z}; x = \frac{r}{p^n} + Z$ for some $r \in \mathbb{Z}$, $n \in \mathbb{N}$, $p$ prime $\} \subseteq \mathbb{Q} / \mathbb{Z}$.

$Z_{p^{\infty}} \subseteq \frac{1}{p} + Z > \subseteq \frac{1}{p^2} + Z > \subseteq \ldots$

$Z_p^{\infty} = \{0\} + Z_p^{\infty}$ if $\{0\}$ is R-annihilator-small submodule of $Z_p^{\infty}$ then $\text{ann} Z_p^{\infty} = \{r \in \mathbb{Z}; n \in \mathbb{N}, Z_p^{\infty} = 0\} \neq 0$ , therefore $\{0\}$ is R-annihilator-small submodule of $Z_p^{\infty}$, $\frac{1}{p} + Z > + Z_p^{\infty} = Z_p^{\infty}$, then $\text{ann} Z_p^{\infty} = 0$.

$\frac{1}{p} + Z >$ is R-annihilator-small submodule of $Z_p^{\infty}$, therefore $Z_p^{\infty}$ as $Z$ - module R-annihilator-hollow module,see[6].

4. Let $N \subseteq Q$ then $N$ is R- annihilator- small submodule. Let $L \subseteq Q$ such that $Q = N + L$ then $\text{ann} N = 0$. $Q$ is a torsion free then $\text{ann} N = 0$ for all $A \subseteq \mathbb{Q}$, therefore $N$ is R- annihilator-small submodule of $Q$ and let $Q$ as $Z$-module is R-annihilator-hollow module but $Q$ as $Z$-module is not hollow module.

The following two propositions give more properties of R-a-hollow module.

**Proposition (3.3):** Let $f: M \rightarrow M'$ be a homomorphism and let $M'$ is R-a-hollow module such that for all $N \subseteq M$ such that Ker $f$ is small of $M$ then $M$ is R-a-hollow module.

**Proof:**

Let $N \subseteq M$ whith $M=N+K$, where $K$ is a submodule of $M$. To show $\text{ann} K=0$.

$f(N)+f(K)=f(M')$ (f is epimorphism) if $f(N)= M'=f(M)$ then $f^{-1}(f(N))=M$. $N + \text{Ker} f = M$, since Ker is small of $M$ then $N=M$ (which is a contradiction). Therefore $N=\text{M}$. Since $f(N)$ is a R-small submodule of $f(M)$, then $\text{ann} K= \text{ann} M'$. Thus $\text{M}$ is R-a-hollow module. But $\text{ann}(K)=0$ and $\text{ann} K \subseteq \text{ann}(K)=0$, therefore $\text{ann} K=0$, then $N$ is R-a-small submodule of $M$. Thus $\text{M}$ is R-a-hollow module.

**Proposition (3.4):** If $\frac{M}{K}$ is R-a-hollow module then $\text{M}$ is R-a-hollow module for all $K$ submodule of $M$

**Proof:**

Suppose $\frac{M}{K}$ is R-a-hollow module and let $N \subseteq M$ such that $M=N+L$, $L$ is submodule of $M$ then $\text{ann} L=0$.

$\frac{M}{K} = N + K = \frac{N+K}{K}$. Since $\frac{N+K}{K}$ is R-a-small submodule of $\frac{M}{K}$, then $\text{ann} \frac{N+K}{K}=0$.

Thus $\text{ann} N$ is submodule of $\text{ann} \frac{N+K}{K}=0$, therefore $\text{ann} N=0$ and $M$ is R-a-hollow module. See[6].

References


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