

R-Annihilator – Hollow Modules

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Abstract: Let R be an associative ring with identity and let M be a unitary left R -module. We call a non-zero module M , R -annihilator-hollow module if every proper submodule of M is R -annihilator-small submodule of M . The sum A_M of all such submodules of M contains the Jacobson radical $J(M)$ and the singular submodule $Z(M)$. When M is finitely generated and faithful, we study A_M and K_M in this paper. Conditions when A_M is R -annihilator-small and $K_M = A_M$, $J(M) \subseteq A_M$ and $Z(M) \subseteq A_M$ are given.

Keywords: hollow modules, annihilators, R -annihilator-hollow modules

1. Introduction

Throughout this paper all rings are associative ring with identity and modules are unitary left modules. In [1], Nicholson and Zhou defined annihilator-small right(left) ideals as follows: a left ideal A of a ring R is called annihilator-small if $A+T=R$, where T is a left ideal, implies that $r(T)=0$, where $r(T)$ indicates the right annihilator.

Kalati and Keskin consider this problem for modules in [2] as follows:- let M be an R -module and $S=\text{End}(M)$. A submodule K of M is called annihilator-small if $K+T=M$, T a submodule of M , implies that $r_S(T)=0$, where r_S indicates the right annihilator of T over $S=\text{End}(M)$, where $r_S(T)=\{f \in S; f(T)=0, \forall t \in T\}$.

These observations lead us to introduce the following concept. A non-zero module M , is called

R -annihilator-hollow module if every proper submodule of M is R -annihilator-small submodule of M .

In fact, the set K_M of all elements k such that Rk is semisubmodule and annihilator-small. And contains both the Jacobson radical and the singular submodule when M is finitely generated and faithful.

The submodule A_M generated by K_M is a submodule of M analogue of the Jacobson radical that contains every R -annihilator-small submodules. In this work we give some basic properties of

R -annihilator-hollow modules and various.

Characterizations

We abbreviate the Jacobson radical as $\text{Rad}(M)$ and the singular submodule as $Z(M)$ for any R -module M . The notations $N \leq^e M$ mean that a submodule N of M is essential in the module M .

See [1] / [2].

2. R-annihilator Small submodules

In this section, we introduce the concept of the R -annihilator-small submodule and we illustrate it by examples. We also give some basic properties of this class of submodules.

We start this section by definition:

Definition (2.1):

We say that a submodule N of an R -module M is a R -annihilator-small submodule (R -a-small) if whenever $N+T=M$, T is a submodule of M , implies that $\text{Ann}_0(T)=0$, where $\text{Ann}_0(T)=\{r \in R; r.T=0\}$. Clearly $\text{Ann}_0(T)$ is a left ideal of R . We write $N \ll^a M$, see [3].

Let I be an ideal of a ring R . We say that I is R -a-small ideal of R if I is R -a-small submodule of R as an R -module.

Examples (2.2):

- 1) For an R -module M , M is not R -a-small submodule of M , where $M=M+0$ and $\text{ann } 0 = \{r \in R; r.0=0\} = R \neq 0$.
- 2) Let R be a commutative ring and I be an ideal of R . Then one can easily show that I is a-small ideal of R if and only if I is R -a-small ideal of R as

R -module, where $r(I)=\text{ann}(I)$ when R is a commutative ring.

- 1) Let N be a submodule of an R -module M and let $S=\text{End}M$. If N is a-small submodule of M then need not be N is R -a-small submodule of M as the following example shows:

Consider the module Z_6 as Z -module. $\{0\}$ is a small submodule of Z_6 and hence $\{0\}$ is a-small submodule of Z_6 , by remark(1.3.4). But $Z_6 = \{0\} + Z_6$ and $\text{ann } Z_6 = \{n \in Z; n.Z_6=0\} = 6Z \neq 0$. Thus $\{0\}$ is not Z -a-small submodule of Z_6 . See [7].

- 2) It is known that a non-zero small submodule can not be a direct summand. But this is not true for R -a-small submodules. For example, consider the module $M = Z_2 \oplus Z_2$ as Z_2 -module and

let $A = Z_2 \oplus 0$.

Clearly $M = A \oplus Z_2 = A \oplus ((\bar{1}, \bar{1}))$ and $\text{ann } 0 \oplus Z_2 = \text{ann}((\bar{1}, \bar{1})) = 0$. Thus A is Z -a-small submodule of M .

The following three Corollary give more properties of R -a-small submodules.

Corollary(2.3):

Let K and N be a submodules of an R -module M such that $K \leq N$. If

$\frac{N}{K}$ is R -a-small submodule of $\frac{M}{K}$, then N is R -a-small submodule of M .

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Proof:

Let N, K be submodules of an R -module M such that $K \leq N$ and

$\frac{N}{K}$ R -a-small submodule of $\frac{M}{K}$. Let $\pi: M \rightarrow \frac{M}{K}$ be the natural epimorphism. Therefore $\pi^{-1}(\frac{N}{K})$

is R -a-small submodule of M , by prop (2.1.5). But $\pi^{-1}(\frac{N}{K}) = N$. Thus N is R -a-small of M . See [7].

Corollary (2.4):

Let M be an R -module and let $K \leq N \leq L \leq M$ such that $\frac{L}{N}$ is R -a-small submodule of $\frac{M}{N}$ then $\frac{L}{K}$ is R -a-small submodule of $\frac{M}{K}$.

Proof:

Let $f: \frac{M}{K} \rightarrow \frac{M}{N}$ be the map defined by $f(x+K) = x+N, \forall x \in M$. One can easily to show f is an epimorphism. Since $\frac{L}{N}$ is R -a-small submodule of $\frac{M}{N}$, therefore $\frac{L}{K} = f^{-1}(\frac{L}{N})$ is R -a-small submodule of $\frac{M}{K}$, by prop(2.1.5) Thus $\frac{L}{K}$ is R -a-small submodule of $\frac{M}{K}$. See [7].

Corollary (2.5):

Let M be an R -module and let $K \leq N \leq M, K' \leq N' \leq M$,

if $\frac{N+N'}{K+K'}$ is R -a-small submodule of $\frac{M}{K+K'}$. Then:-

- 1- $\frac{N+K'}{K}$ is R -a-small submodule of $\frac{M}{K}$.
- 2- $\frac{K+N'}{K'}$ is R -a-small submodule of $\frac{M}{K'}$.
- 3- $\frac{N}{K} \oplus \frac{N'}{K'}$ is R -a-small submodule of $\frac{M}{K} \oplus \frac{M}{K'}$.

Proof:

1- Let $f_1: \frac{M}{K} \rightarrow \frac{M}{K+K'}$ be a map defined by $f_1(x+K) = x+K+K', \forall x \in M$ and let $f_2: \frac{M}{K'} \rightarrow \frac{M}{K+K'}$ be a map defined by $f_2(m+K') = m+K+K', m \in M$. One can easily show that each of f_1, f_2 is an epimorphism. Since $\frac{N+K'}{K+K'} \leq \frac{N+N'}{K+K'}$ and $\frac{N+N'}{K+K'}$ is R -a-small submodule of $\frac{M}{K+K'}$, then $\frac{N+K'}{K+K'}$ is R -a-small submodule of $\frac{M}{K+K'}$, by prop(2.1.4).

Thus $\frac{N+K'}{K} = f_1^{-1}(\frac{N+K'}{K+K'})$ is R -a-small submodule of $\frac{M}{K}$,
 By prop (2.1.5) ... (1)

2- Also $\frac{K+N'}{K+K'} \leq \frac{N+N'}{K+K'}$ and $\frac{N+N'}{K+K'}$ is R -a-small submodule of $\frac{M}{K+K'}$ then $\frac{K+N'}{K+K'}$ is R -a-small submodule of $\frac{M}{K+K'}$, by prop (2.1.4),

therefore $\frac{K+N'}{K'} = f_2^{-1}(\frac{K+N'}{K+K'})$ is R -a-small submodule of $\frac{M}{K'}$, by prop (2.1.5) ... (2)

3- From (1) and (2), we have

$\frac{N}{K} \oplus \frac{N'}{K'}$ R -a-small submodule of $\frac{M}{K} \oplus \frac{M}{K'}$, by prop (2-1-8).

Recall that an R -module M is called faithful if $\text{ann}(M)=0$, see [7].

Let R be an integral domain. Recall that an R -module M is called a torsion free R -module if $\text{ann}(x) = 0$, for every non-zero element x in M , see [4].

Proposition (2.6):

Let M be a faithful R -module. Then every small submodule of M is R -a-small.

Proof:

Let M be faithful R -module and let N be a small submodule of M . To show N is R -a-small submodule of M . Let $M = N+U$. Since N is small in M , then $M=U$ and hence $\text{ann}M = \text{ann}U$. So $\text{ann}U=0$. Thus N is R -a-small submodule of M . The following corollary follows immediately of proposition (2.6).

Corollary (2.7):

Let R be an integral domain and let M be a projective R -module. Then every proper submodule of M is R -a-small submodule of M .

In particular, every proper submodule of a free module over an integral domain is R -a-small.

Remark (2.8):

Let M be an R -module if there exists a submodule N of M such that N is R -a-small submodule of M . Then M is faithful.

Proof:

Since $M=N+M$ and N is R -a-small submodule of M , then $\text{ann}M=0$. Thus M is faithful.

Proposition (2.9):

Let I be an ideal of a commutative ring R and let M be an R -module if

$I M$ is R -a-small in M , then I is a-small ideal of R .

Proof:

Let $R=I+J$, where J is an ideal of R . Then $M=RM=(I+J)M=I M+J M$. Since $I M$ is R -a-small in M , then $\text{ann} J M=0$. But $\text{ann} J \leq \text{ann} J M$. Therefore $\text{ann} J=0$. Thus I is R -a-small in R .

Recall that an R -module M is called a multiplication module if for every submodule N of M , there exists an ideal I of R such that $N=I M$. Equivalently, an R -module M is a multiplication module if and only if $N=(N:M)M$, for every submodule N of M . where $(N:M)=\{r \in R ; rM \leq N\}$, see [5].

We end the section by the following corollary and proposition we give various characterizations of R -annihilator-small submodules.

Corollary (2.10):

Let M be a multiplication module over a commutative ring R and let N be a submodule of M . If N is R -a-small submodule of M , then $(N:M)$ is a-small ideal of R .

Proposition (2.11):

Let M be a module and K an R -a-small submodule of M . If $\text{Rad}(M)$ is a small submodule of M and $Z(M)$ is finitely

generated, then $K + \text{Rad}(M) + Z(M)$ is R-a-small submodule of M.

Proof:

Let $Z(M) = R z_1 + R z_2 + \dots + R z_n$, where $z_i \in Z(M) \forall i=1,2,\dots,n$.

To show $K + \text{Rad}(M) + Z(M)$ is R-a-small submodule of M, let $K + \text{Rad}(M) + Z(M) + X = M$, where X is a submodule of M. Since $\text{Rad}(M)$ is a small submodule of M, then $K + Z(M) + X = M$. But K is R-a-small submodule of M, therefore $\text{ann}(Z(M) + X) = \text{ann}(R z_1 + R z_2 + \dots + R z_n + X) = 0$. So $(\bigcap_{i=1}^n \text{ann} R z_i) \cap \text{ann} X = 0$. Since $z_i \in Z(M)$, then $\text{ann} Z_i \leq^e R, \forall i=1,2,\dots,n$. And hence $\bigcap_{i=1}^n \text{ann} R z_i \leq^e R$, by [12]. So $\text{ann} X = 0$. Thus $K + \text{Rad}(M) + Z(M)$ is R-a-small submodule of M.

3. R- annihilator hollow modules

In this section, we introduce the concept of the R-annihilator-hollow module and we study the basic properties of this type of module

We start this section by definition:

Definition (3.1):

Anon-zero module M is called R-annihilator-hollow module (R-a-hollow) if every proper submodule of M is R-annihilator-small submodule of M.

Examples (3.2):

1- An R-a-small submodule of an R-module M need not be small submodule.

For example, consider the module Z as Z- module. For every $n > 1$, claim that nZ is Z-a-small submodule of Z. To show that, let $Z = nZ + mZ$, where mZ is a submodule of Z. Since Z has no-zero divisors, then $\text{ann } mZ = \{r \in Z; r.mZ = 0\} = 0$. Thus nZ is Z-a-small submodule of Z. But it is known that $\{0\}$ is the only small submodule of Z, therefore Z as Z- module is R-annihilator-hollow module.

2- A small submodule of an R-module M need not be R-a-small submodule. For example. Consider Z_4 as Z-module. One can easily show that $\{0\}$ and $\{0, 2\}$ are small submodule of Z_4 . But $Z_4 = \{0\} + Z_4$ and $Z_4 = \{0, 2\} + Z_4$ and $\text{ann } Z_4 = \{n \in Z; n.Z_4 = 0\} = 4Z \neq 0$. Thus each of $\{0\}$ and $\{0, 2\}$ is not Z-a-small submodule of Z_4 , therefore Z_4 is not R-annihilator-hollow module.

3- Let $Z_{p^\infty} = \{x \in \frac{Q}{Z}; x = \frac{r}{p^n} + Z \text{ for some } r \in Z, n \in \mathbb{N}, p \text{ prime}\} \leq \frac{Q}{Z}$.

$$Z \not\cong \langle \frac{1}{p} + Z \rangle \not\cong \langle \frac{1}{p^2} + Z \rangle \not\cong \dots$$

$Z_{p^\infty} = \{0\} + Z_{p^\infty}$ if $\{0\}$ is R-annihilator-small submodule of Z_{p^∞} then $\text{ann } Z_{p^\infty} = \{r \in Z; n.Z_{p^\infty} = 0\} = 0$, therefore $\{0\}$ is R-annihilator-small submodule of Z_{p^∞} . $\langle \frac{1}{p} + Z \rangle + Z_{p^\infty} = Z_{p^\infty}$, $\text{ann } Z_{p^\infty} = 0$.

$\langle \frac{1}{p} + Z \rangle$ is R-annihilator-small submodule of Z_{p^∞} , therefore Z_{p^∞} as Z - module R-annihilator-hollow module, see [6].

4- Let $N \leq Q$ then N is R- annihilator- small submodule. Let $L \leq Q$ such that $Q = N + L$ then $\text{ann} L = 0$. Q is a torsion free then $\text{ann} A = 0$ for all $A \leq Q$, therefore N is R- annihilator-small submodule of Q then Q as Z-module is R-annihilator-hollow module but Q as Z-module is not hollow module.

The following two propositions give more properties of R-a-hollow module.

Proposition (3.3): Let $f: M \rightarrow M'$ be a homomorphism and let M' is R-a-hollow module such that for all $N \leq M$ such that $\text{Ker} f$ is small of M then M is R-a-hollow module.

Proof:

Let $N \not\cong M$ with $M = N + K$, where K is a submodule of M. To show $\text{ann } K = 0$.

$f(N) + f(K) = f(M) = M'$ (f is epimorphism), if $f(N) = M' = f(M)$ then $f^{-1}(f(N)) = M$. $N + \text{Ker} f = M$, since $\text{Ker} f$ is small of M then $N = M$ (which is a contradiction). Therefore $N \neq M$. Since $f(N)$ is R-a-small submodule of $f(M)$, then $f(N) \neq M' = f(M)$. Thus M' is R-a-hollow module. But $\text{ann} f(K) = 0$ and $\text{ann} K \leq \text{ann} f(K) = 0$, therefore $\text{ann} K = 0$, then N is R-a-small submodule of M. Thus M is R-a-hollow module.

Proposition (3.4): If $\frac{M}{K}$ is R-a-hollow module then M is R-

a-hollow module for all K submodule of M **Proof:** Suppos $\frac{M}{K}$ is R-a-hollow module and let $N \not\cong M$ such that $M = N + L$, L is submodule of M then $\text{ann} L = 0$.

$\frac{M}{K} = \frac{N+L}{K} = \frac{N+K}{K} + \frac{L+K}{K}$. Since $\frac{N+K}{K} \not\cong \frac{M}{K}$, then $\frac{N+K}{K}$ is R-a-small submodule of $\frac{M}{K}$, then $\text{ann} \frac{L+K}{K} = 0$.

Thus $\text{ann} L$ is submodule of $\text{ann} \frac{L+K}{K} = 0$, therefore $\text{ann} L = 0$ and M is R-a-hollow module. See [6].

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