A Matrix Trace Inequality for Products of Quaternion Hermitian Matrices

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Abstract: In this paper, the following matrix trace inequality for * _ products of quaternion hermitian matrices _ A _ and _ B, _ tr (A * B)² ≤ tr(A² * B²) _ is established, where _ k _ is positive integer.

Keywords: Triple representation of quaternion matrix, Hermitian matrix, * _ product, matrix trace inequality.

1. Introduction

The triple representation of complex matrices form a quaternion matrix, some new concept to quaternion division algebra where presented. [3]

The following matrix inequality for products of quaternion hermitian matrices _ A _ and _ B, _ tr (A * B)² ≤ tr(A² * B²) _ is established, where _ k _ is positive integer. [1]

Recently, there has been substantial interest in matrix trace inequalities for triple representation of complex and also hermitian matrices of the same order. [1, 2]

2. Lemmas and Theorem

Lemma 2.1: Suppose that _ P _ is a quaternion square matrix; then

\[ \text{tr}[P^{2n-r} * (P^H)^{2n-r}] \leq \text{tr}\left(\left[\sum_{i=0}^{n-r} (P^i)^2 (P^i)^H \right]P^2 \right) \]

where _ n _ and _ r _ are integers.

Proof:

\[ \text{tr}[P^{2n-r} * (P^H)^{2n-r}] = \text{tr}\left(\left[\sum_{i=0}^{n-r} (P^i)^2 (P^i)^H \right]P^2 \right) \]

\[ = \text{tr}\left(\left[\sum_{i=0}^{n-r} (P^i)^2 (P^i)^H \right]P^2 \right) \]

\[ = \text{tr}\left(\left[\sum_{i=0}^{n-r} (P^i)^2 (P^i)^H \right]P^2 \right) \]

\[ \leq \text{tr}\left(\left[\sum_{i=0}^{n-r} (P^i)^2 (P^i)^H \right]P^2 \right) \]

\[ \Rightarrow \text{tr}[P^{2n-r} * (P^H)^{2n-r}] \leq \text{tr}\left(\left[\sum_{i=0}^{n-r} (P^i)^2 (P^i)^H \right]P^2 \right) \]

Lemma 2.2: Suppose that _ P _ is a quaternion square matrix; then

\[ \text{tr}[P^{2k} * (P^H)^{2k}] \leq \text{tr}(P * P^H)^{2k}, \text{ where } k \text{ is a positive integer} \]

Proof:

Let \( S_k = (P * P^H)^{2k-1} - (P^H * P)^{2k-1} \)

When _ k _ = 1, from equation (3) we know that the matrix _ S_1 is quaternion hermitian. On the other hand, through direct calculation, we obtain

\[ 0 \leq \text{tr}S_1^2 = 2\text{tr}(P * P^H)^2 - 2\text{tr}(P^2 * (P^H)^2) \]

Hence Lemma 2.2 holds while _ k _ = 1

Suppose that Lemma 2.2 holds when _ k _ ≤ _ n _; in the following, we will prove that Lemma 2.2 is valid when _ k _ ≤ _ n+1 _.

From equation (3) it is easy to know that _ S_{n+1} _ is quaternion hermitian and

\[ 0 \leq \text{tr}S_{n+1}^2 = 2\text{tr}(P * P^H)^{2n+1} - 2\text{tr}(P^2 * (P^H)^{2n+1}) \]

That is,

\[ \text{tr}(P * P^H)^{2n+1} \leq \text{tr}(P^2 * (P^H)^{2n+1}) \]

Noticing that

\[ \text{tr}(P^2 * (P^H)^{2n+1}) = \text{tr}\left(\left[\sum_{i=0}^{2n+1} (P^i)^2 (P^i)^H \right]P^2 \right) \]

\[ = \text{tr}\left(P^2 * (P^H)^{2n+1} \right) \]

And combining the above equalities and inequality (4), we have

\[ \text{tr}(P * P^H)^{2n+1} \geq \text{tr}\left(\left[\sum_{i=0}^{2n+1} (P^i)^2 (P^i)^H \right]P^2 \right) \]

Let

\[ R_{n-1} = (P * P^H)^{2n-1} - (P^H * P)^{2n-1} \]

Then it is easy to verify that the matrix _ R_{n-1} _ in (6) is quaternion hermitian, and by inequality (5) and (6), we have

\[ \text{tr}(P * P^H)^{2n+1} \geq \text{tr}(R_{n-1} * R_{n-1}) = \text{tr}(R_{n-1}^2) \]

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Making use of Lemma 2.1 and inequality (7), we have
\[ tr(P * p^H)^{2n+1} \geq tr((R_{n-2}^2 * R_{n-2}^H)^2) \]
(8)

Making use of the induction assumption and inequality (8), we have
\[ tr(P * p^H)^{2n+1} \geq tr((R_{n-2}^2 * R_{n-2}^H)^2) \]
Furthermore,
\[ tr(P * p^H)^{2n+1} \geq tr((R_{n-2}^2 * R_{n-2}^H)^2) \]
Since \( R_{n-2} \) is quaternion hermitian. Repeating the above procedure, we have
\[ tr(P * p^H)^{2n+1} \geq tr((R_{n-2}^2 * R_{n-2}^H)^2) \]
Furthermore,
\[ tr(P * p^H)^{2n+1} \geq tr((P^2 * (P^H)^2)^2) \]
(9)

Making use of the induction assumption, we have
\[ tr(P^2 * (P^H)^2)^2 \geq tr((P^2)^2 * (P^H)^2)^2) \]
\[ = tr((P^2)^2 * (P^H)^2)^2) \]
(10)

Combining the inequalities (9) and (10), we have proved that Lemma 2.2 holds
when \( k = n+1 \).

The Proof is complete.

**Theorem 2.3**: Suppose that A and B are quaternion Hermitian matrices of the same order; then
\[ tr(A * B)^{2k} \leq tr(A^2k * B^2k) \]
(11)
where \( k \) is a positive integer.

**Proof**: When \( k = 1 \), it is easy to verify that the matrix
\( A * B - B * A \) is
Skew-hermitian, and
\[ tr(A * B - B * A)^2 \leq 0 \]
(12)
On the other hand, through direct calculation, we have
\[ (A * B - B * A)^2 = (A * B)^2 + (B * A)^2 - (A * B)^2 * (A) \]
\[ -(B * A)^2 * (B) \]
and
\[ tr(A * B - B * A)^2 = 2tr(A * B)^2 - 2tr(A^2 * B^2) \]
(13)
Combining inequality (11) and (12),
we have
\[ tr(A * B)^2 \leq tr(A^2 * B^2) \]
Hence Theorem 2.3 holds while \( k = 1 \).

Suppose that Theorem 2.3 holds when \( K \leq 1 \); in the following, we will prove that Theorem 2.3 is valid when \( k = n+1 \).

It is easy to verify that the matrix
\( (A * B)^n - [(A * B)^n]^H \) is quaternion skew hermitian, on
the other hand, through direct calculation, we have,
\[ 0 \geq tr((A * B)^n - [(A * B)^n]^H)^2 \]
\[ = 2tr((A * B)^{2n+1} - 2tr((A * B)^{2n} * [(A * B)^n]^H)^2) \]
(14)

Thus
\[ tr((A * B)^{2n+1} \leq tr((A * B)^{2n} * [(A * B)^n]^2) \]
(15)
Making use of Lemma 2.2, we have
\[ tr((A * B)^{2n} * [(A * B)^n]^2) \leq tr((A * B)^n * (A * B)^n)^2 = tr((A^2 * B^2)^n) \]
(16)

Making use of the induction assumption, we have
\[ tr(A^2 * B^2)^{2n} \leq tr(A^2)^{2n} * (B^2)^{2n} \]
(17)
Combining the inequalities (14) – (16), We have
\[ tr(A * B)^{2n+1} \leq tr((A^2)^{2n+1} * (B^2)^{2n+1}) \]

Thus we have proved that theorem 2.3 holds when \( k = n+1 \).

The Proof is complete.

**References**


