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A Matrix Trace Inequality for Products of Quaternion Hermitian Matrices

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Abstract: *In this paper, the following matrix trace inequality for $*\_$ products of quaternion hermitian matrices A and B, $tr\,(A*B)^{2^k} \le tr(A^{2^k} * B^{2^k})$ is established, where k is positive integer.*

Keywords: Triple representation of quaternion matrix, Hermitian matrix, $*\_$ product, matrix trace inequality.

1. Introduction

The triple representation of complex matrices form a quaternion matrix, some new concept to quaternion division algebra where presented. [3]

The following matrix inequality for products of quaternion hermition matrices A and B, $tr\,(A*B)^{2^k} \le tr(A^{2^k} * B^{2^k})$ is established, where k is positive integer. [1]

Recently, there has been substantial interest in matrix trace inequalities for triple representation of complex and also hermition matrices of the same order.[1,2]

2. Lemmas and Theorem

Lemma 2.1:
Suppose that P is a quaternion square matrix; then

$$tr[p^{2^{n-r}} * (P^H)^{2^{n-r}}]^{2^r} \le tr\left\{\left(P^{2^{2^r(n-r-1)}} * P^{H^{2^{2^r(n-r-1)}}}\right)\right\} tr[P^2 * P^{H^2}]^{r+1} \quad (1)$$

where n and r are integers.

Proof:

$$\begin{aligned}
tr[p^{2^{n-r}} * (P^H)^{2^{n-r}}]^{2^r} &= tr\{[P_0^{2^{n-r}} + P_1^{2^{n-r}} j + P_2^{2^{n-r}} k] * [P_0^{H^{2^{n-r}}} + P_1^{H^{2^{n-r}}} j + P_2^{H^{2^{n-r}}} k]\}^{2^r} \\
&= tr\{[P_0^{2^{n-r}} P_0^{H^{2^{n-r}}} + P_1^{2^{n-r}} P_1^{H^{2^{n-r}}} j + P_2^{2^{n-r}} P_2^{H^{2^{n-r}}} k]\}^{2^r} \\
&= tr\{(P_0^2 P_0^{H^2})^{n-r} + (P_1^2 P_1^{H^2})^{n-r} j + (P_2^2 P_2^{H^2})^{n-r} k\}^{2^r} \\
&= tr\{(P_0^2 P_0^{H^2})^{n-r-1}(P_0^2 P_0^{H^2}) + (P_1^2 P_1^{H^2})^{n-r-1}(P_1^2 P_1^{H^2}) j + (P_2^2 P_2^{H^2})^{n-r-1}(P_2^2 P_2^{H^2})\}^{2^r} \\
&= tr\{P_0^{2^{n-r-1}} P_0^{H^{2^{n-r-1}}} P_0^2 P_0^{H^2} + P_1^{2^{n-r-1}} P_1^{H^{2^{n-r-1}}} P_1^2 P_1^{H^2} j + P_2^{2^{n-r-1}} P_2^{H^{2^{n-r-1}}} P_2^2 P_2^{H^2} k\}^{2^r} \\
&= tr\{(P^{2^{n-r-1}} * P^{H^{2^{n-r-1}}})(P^2 * P^{H^2})\}^{2^r} \\
&= tr\{\left(P^{2^{(n-r-1)2^r}} * P^{H^{2^{(n-r-1)2^r}}}\right)\left(P^{2\times 2^r} * P^{H^{2\times 2^r}}\right)\} \\
&= tr\{\left(P^{2^{2^r(n-r-1)}} * P^{H^{2^{2^r(n-r-1)}}}\right)\left(P^{2^{r+1}} * P^{H^{2^{r+1}}}\right)\} \\
&\le tr\left\{\left(P^{2^{2^r(n-r-1)}} * P^{H^{2^{2^r(n-r-1)}}}\right)\right\} tr[p^2 * P^{H^2}]^{r+1}
\end{aligned}$$

The proof is completed.

Lemma 2.2: Suppose that P is quaternion square matrix; then
$tr[P^{2^k} * (P^H)^{2^k}] \le tr(P * P^H)^{2^k}$, where k is a positive integer (2)

Proof:

$$\text{Let } S_k = (P * P^H)^{2^{k-1}} - (P^H * P)^{2^{k-1}} \quad (3)$$

When k=1, from equation (3) we know that the matrix S_1 is quaternion hermitian. On the other hand, through direct calculation, we obtain

$$0 \le trS_1^2 = 2tr(P * P^H)^2 - 2tr[P^2 * (P^H)^2]$$

Hence Lemma 2.2 holds while k = 1

Suppose that Lemma 2.2 holds when $k \le n$; in the following, we will prove that
Lemma 2.2 is valid when k = n+1.

From equation (3) it is easy to know that S_{n+1} is quaternion hermitian and

$$0 \le trS_{n+1}^2 = 2tr(P * P^H)^{2^{n+1}} - 2tr[(P * P^H)^{2^n} * (P^H * P)^{2^n}]$$

That is,

$$tr[P * P^H]^{2^{n+1}} \le tr[(P * P^H)^{2^n} * (P^H * P)^{2^n}] \quad (4)$$

Noticing that

$$\begin{aligned}
tr\left[(P * P^H)^{2^n} * (P^H * P)^{2^n}\right] &= tr\{[(P * P^H)^{2^{n-1}} * (P * P^H)^{2^{n-1}})] * [(P^H * P)^{2^{n-1}} * (P^H * P)^{2^{n-1}}]\} \\
&= tr\{[(P * P^H)^{2^{n-1}} * (P^H * P)^{2^{n-1}}][(P^H * P)^{2^{n-1}} * (P * P^H)^{2^{n-1}}]\}
\end{aligned}$$

And combining the above equalities and inequality (4), we have

$$tr[P * P^H]^{2^{n+1}} \ge tr\{[(P * P^H)^{2^{n-1}} * (P^H * P)^{2^{n-1}}] * [(P^H * P)^{2^{n-1}} * (P * P^H)^{2^{n-1}}]\} \quad (5)$$

Let

$$R_{n-1} = (P * P^H)^{2^{n-1}} * (P^H * P)^{2^{n-1}} \quad (6)$$

Then it is easy to verify that the matrix R_{n-1} in (6) is quaternion hermitian, and by inequality (5) and (6), we have

$$tr(P * P^H)^{2^{n+1}} \ge (tr(R_{n-1} * R_{n-1}^H)) = tr(R_{n-1}^2) \quad (7)$$

Making use of Lemma 2.1 and inequality (7), we have

$$tr(P*P^H)^{2^{n+1}} \geq tr(R_{n-1}^2) = tr(R_{n-2}*R_{n-2}^H)^2 \quad (8)$$

Making use of the induction assumption and inequality (8), we have

$$tr(P*P^H)^{2^{n+1}} \geq tr[(R_{n-2})^2 * (R_{n-2}^H)^2]$$

Furthermore,

$$tr(P*P^H)^{2^{n+1}} \geq tr(R_{n-2})^{2^2}$$

Since R_{n-2} is quaternian hermitian. Repeating the above procedure, we have

$$tr(P*P^H)^{2^{n+1}} \geq tr(R_0)^{2^n} = tr[(P*P^H)*(P^H*P)]^{2^n}$$

Furthermore,

$$tr(P*P^H)^{2^{n+1}} \geq tr[P^2*(P^H)^2]^{2^n} \quad (9)$$

Making use of the induction assumption, we have

$$tr[P^2*(P^H)^2]^{2^n} \geq tr\{(P^2)^{2^n}*[(P^H)^2]^{2^n}\} = tr[P^{2^{n+1}}*(P^H)^{2^{n+1}}] \quad (10)$$

Combining the inequalities (9) and (10), we have proved that Lemma 2.2 holds
when k = n+1.
The Proof is complete.

Theorem 2.3 : Suppose that A and B are quaternion Hermitian matrices of the same order; then

$$tr(A*B)^{2^k} \leq tr(A^{2^k}*B^{2^k}), \quad (11)$$

where k is a positive integer.

Proof : When k = 1, it is easy to verify that the matrix $A*B-B*A$ is Skew- hermitian, and $tr(A*B-B*A)^2 \leq 0$ (12)

On the other hand, through direct calculation, we have

$$(A*B-B*A)^2 = (A*B)^2 + (B*A)^2 - (A*B^2*A) - (B*A^2*B)$$

and

$$tr(A*B-B*A)^2 = 2tr(A*B)^2 - 2tr(A^2*B^2) \quad (13)$$

Combining inequality (11) and (12),
we have

$$tr(A*B)^2 \leq tr(A^2*B^2)$$

Hence Theorem 2.3 holds while k = 1

Suppose that Theorem 2.3 holds when $K \leq 1$; in the following, we will prove that Theorem 2.3 is valid when k = n+1.

It is easy to verify that the matrix $(A*B)^{2^n} - [(A*B)^{2^n}]^H$ is quaternion skew hermitian, on the other hand, through direct calculation, we have,

$$0 \geq tr\{(A*B)^{2^n} - [(A*B)^{2^n}]^H\}^2$$
$$= 2tr(A*B)^{2^{n+1}} - 2tr\{(A*B)^{2^n}*[(A*B)^H]^{2^n}\}$$

Thus

$$tr(A*B)^{2^{n+1}} \leq tr\{(A*B)^{2^n}*[(A*B)^H]^{2^n}\} \quad (14)$$

Making use of Lemma 2.2, we have

$$tr\{(A*B)^{2^n}*[(A*B)^H]^{2^n}\} \leq tr[(A*B)*(A*B)^H]^{2^n} = tr(A^2*B^2)^{2^n} \quad (15)$$

Making use of induction assumption, we have

$$tr(A^2*B^2)^{2^n} \leq tr\left[(A^2)^{2^n}*(B^2)^{2^n}\right] = tr[A^{2^{n+1}}*B^{2^{n+1}}] \quad (16)$$

Combining the inequalities (14) – (16), We have

$$tr(A*B)^{2^{n+1}} \leq tr(A^{2^{n+1}}*B^{2^{n+1}}) \quad (17)$$

Thus we have proved that theorem 2.3 holds when k = n+1.
The proof is complete.

References

[1] Bellman. R: some inequalities for positive definite matrices, in "General Inequalities 2, Proceedings 2nd International conference on General Inequalities"; (E.F. Beckenback, Ed.), PP.89-90, Birkhauser, Basel, (1980)
[2] Coope. I.D : "On matrix trace inequalities and related topics for products of Hermition matrices"; J. Math. Appl.188, 999-1001 (1994)
[3] Gunasekaran. K and Rahamathunisha. M: "on tricomplex representation methods and application of matrices over quaternion division algebra"; Applied Science periodical volume XVIII, No.1, Feb (2016)
[4] Gunasekaran. K, Kannan. K and Rahamathunisha. M: "On Tri Complex representation methods of Matrix trace Inequality"; Journal of untra scientist pf Physical Science. http://dx.doi.org/10.22147/jusps-A/290105,JUSPS-A Vol.29(1), 30-32(2017)
[5] Neudecker. H: "A Matrix trace inequality"; J. Math. Anal. Appl. 166, 302-303 (1992)
[6] Yang.Y: "A. Matrix trace inequality"; J. Math. Anal. Appl.133, 573-574 (1988)