

# Convergence Weakly to Asymptotic Common Fixed Point Theorems for Different Types of Proximal Point Schemes

Salwa Salman Abed<sup>1</sup>, Zena Hussein Maibed<sup>2</sup>

Department of Mathematics, College of Education for Pure Science, Ibn Al-Haithem, University of Baghdad

**Abstract:** In this paper, we introduce a proximal point schemes of  $szl$ -widering mapping which is independent of non-expansive mappings. Also, we discuss the weak convergence for these proximal point schemes in Hilbert space.

**Keywords:** maximal monotone, converge strongly, nonexpansive mapping

## 1. Introduction and Preliminaries

Let  $X$  be a real Hilbert space and  $A$  be maximal monotone mapping. The Monotone operators have proven to a key class of objects in modern optimization and analysis see ([1]-[5]). The zero point problem for monotone operator  $A$  on a real Hilbert space  $X$ , that is, finding a point  $z \in X$  such that  $0 \in A(z)$ . In order to solve this problem, many types of iterative algorithms have been studied such as [6]-[17]. Consider a single valued non-expansive mapping as:  $J_{r_n} = (I + r_n A^{-1})(x)$ , which is called resolvent mapping where  $\langle r_n \rangle$  be a sequence of positive real numbers. In [6,7] Xu, studied the convergence of the proximal point scheme

$$x \in C, x_{n+1} = \alpha_n x + (1 - \alpha_n)T_{x_n}, n = 1, 2, 3, \dots (1)$$

where  $T$  is non-expansive mapping and  $\langle \alpha_n \rangle$  be a sequence in  $(0, 1)$ . In [8] Moudafi, studied the convergence of the proximal point schemes

$$x_t = tf(x_t) + (1-t)T_{x_t} \text{ as } t \rightarrow \infty (2)$$

$$x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n)T_{x_n} \text{ as } n \rightarrow \infty$$

where  $T$  is non-expansive mapping,  $f$  be a contraction mapping and  $\langle \alpha_n \rangle$  be a sequence in  $(0, 1)$ . In [9] Xu, who extend Moudafi results. On other hand, Kamimura and Takahashi [10], studied the convergence strongly of the proximal point scheme

$$u \in C, x_{n+1} = \alpha_n u + (1 - \alpha_n)J_{r_n} x_n, n \geq 1 (3)$$

in 2016 [11, 12], Abed and Maibed studied the strong convergence of the many proximal point schemes. Now, consider  $X$  be a real Hilbert space,  $\emptyset \neq C$  be a convex closed in  $X$ . We recall some definitions and lemmas which will used in the proofs.

**Definition (1.1):** [1]

A mapping  $T: C \rightarrow X$  is called Lipschitz if there exists a real number  $L > 0$  such that

$$\|T(x) - T(y)\| \leq L\|x - y\| \text{ for each } x, y \in C. (4)$$

When  $L \in (0, 1)$  then  $T$  is called contraction mapping and if  $L = 1$  then  $T$  is called nonexpansive mapping

**Lemma (1.2)** [16]

Let  $\langle a_n \rangle$  and  $\langle b_n \rangle$  are sequences of nonnegative number such that  $a_{n+1} \leq a_n + b_n$ , for each  $n \geq 1$ . If  $\sum_{n=0}^{\infty} a_n$  converge then  $\lim_{n \rightarrow \infty} a_n$  exists.

**Lemma (1.3):** [17]

Let  $C$  be a nonempty convex closed subset of real Hilbert space  $X$  and  $T$  is non-expansive multivalued mapping such that  $Fix(T) \neq \emptyset$ . Then  $T$  is demi-closed, i.e.,  $x_n \rightarrow p$  and  $\lim_{n \rightarrow \infty} d(x_n, T(x_n)) = 0$ . Then  $p \in T(p)$ .

**Lemma (1.4):** [7]

If  $\langle a_n \rangle$  be a sequence of non-negative real number such that:

$$a_{n+1} \leq (1 - \gamma_n)a_n + S_n, n \geq 0$$

Where  $\langle \gamma_n \rangle$  is a sequence in  $(0, 1)$  and  $\langle S_n \rangle$  be a sequence in  $\mathbb{R}$  such that:

$$\sum_{n=0}^{\infty} \gamma_n = \infty \text{ and } \lim_{n \rightarrow \infty} \sup \frac{S_n}{\gamma_n} \leq 0 \text{ or } \sum_{n=1}^{\infty} |S_n| < \infty.$$

Then  $a_n \rightarrow 0$  as  $n \rightarrow \infty$ .

**Lemma (1.5):** [18]

If  $\langle a_n \rangle$  be a sequence nonnegative real numbers such that:

$$a_{n+1} \leq (1 - \gamma_n)a_n + \gamma_n S_n + \beta_n, n \geq 0 (5)$$

Where  $\langle \gamma_n \rangle, \langle \beta_n \rangle$  and  $\langle S_n \rangle$  are satisfies the following:

- 1)  $\gamma_n \in [0, 1]$ ;  $\sum_{n=1}^{\infty} \gamma_n = \infty$
- 2)  $\lim_{n \rightarrow \infty} \sup S_n \leq 0$  or  $\sum_{n=1}^{\infty} |\gamma_n S_n| < \infty$
- 3)  $\beta_n \geq 0$  for each  $n \geq 0$  such that  $\sum_{n=0}^{\infty} \beta_n < \infty$ . Then  $\lim_{n \rightarrow \infty} a_n = 0$ .

**Lemma (1.6):** [19]

Let  $X$  be a Hilbert space and  $C$  be a nonempty convex closed subset of  $X$  if  $\langle x_n \rangle$  be a sequence in  $X$  and  $\|x_{n+1} - x\| \leq \|x_n - x\|$  for all  $n \in N, x \in C$ . Then  $\langle P_C(x_n) \rangle$  converges strongly to a point in  $C$ .

Now, we introduce the concept of  $szl$ -widering mapping

**Definition (1.7):** [20]

Let  $X$  be a normed space and  $C$  be a nonempty subset of  $X$ , then a mapping  $T: C \rightarrow C$  is called  $szl$ -widering if for each

$s, l \in (0,1)$  then there exists  $z > 0$  such that the following equation holds:

$$\begin{aligned} & \|Tx - Ty\|^2 \leq (1-s)\|x - y\|^2 \\ & + l\|y - Ty\| \cdot \|x - Tx - (y - Ty)\| \\ & + z|\langle x - Tx, y - Ty \rangle|, \text{ for each } x, y \\ & \in C \quad (6) \end{aligned}$$

The concept of  $szl$ -widering mapping is independent of concepts of non-expansive mappings. As shown by the following examples:

**Example (1.8):**

- (a) Let  $T: R \rightarrow R$  such that  $T$  be identity mapping then  $T$  is non-expansive, but it is not  $sz$ -widering
- (b) Let  $T: R \rightarrow R$  such that  $T(x) = 2x$  Then  $T$  is  $szl$ -widering but not non-expansive mapping.

**Lemma (1.9) :[20]**

Let  $\emptyset \neq C$  be a closed convex subset of Hilbert space  $X$  and  $T$  is  $szl$ -widering. Then it is demi-closed i.e., if there exists  $\{x_n\}$  sequence in  $C$  such that  $x_n \rightarrow p$  and  $\|Tx_n - x_n\| \rightarrow 0$  then  $p \in Fix(p)$ .

Now, we introduce a proximal point schemes of non-expansive and  $szl$ -widering mappings and we discuss the weak convergence for these proximal point schemes under different conditions to asymptotic common fixed point of  $szl$ -widering mappings.

**2. Main Results**

**Theorem (2.1)**

If  $\{T_n\}$  be a sequence of bounded  $szl$ -widering mappings has. Define the proximal point scheme  $\{x_n\}$  as follows

$$\begin{aligned} x_{n+1} &= a_n x_n + b_n T_n x_n + c_n y_n \\ y_n &= J_{r_n}((1 - a_n)x_n + a_n T_n x_n - r_n h x_n) \quad (7) \end{aligned}$$

where  $\{a_n\}, \{b_n\}$  and  $\{c_n\}$  are sequences in  $(0,1)$  such that:

- 1)  $a_n + b_n + c_n = 1$  and  $r_n < 2(1 - a_n)\alpha$
- 2)  $\bigcap_{n=1}^{\infty} Fix(T_n) \cap (A + h)^{-1}(0) \neq \emptyset$

Then the proximal point scheme converges weakly to the asymptotic common fixed point of  $T_n, \forall n \in N$ .

**Proof:**

Let  $p \in \bigcap_{n=1}^{\infty} Fix(T_n) \cap (A + h)^{-1}(0)$

Since,  $y_n = J_{r_n}((1 - a_n)x_n + a_n T_n x_n - r_n h x_n)$

So,  $p \in J_{r_n}((1 - a_n)p + a_n T_n p - r_n h p)$

$$\begin{aligned} \|y_n - p\|^2 &= \left\| J_{r_n}((1 - a_n)x_n + a_n T_n x_n - r_n h x_n) \right. \\ &\quad \left. - J_{r_n}((1 - a_n)p + a_n T_n p - r_n h p) \right\|^2 \\ &\leq \left\| (1 - a_n)(x_n - p) + a_n(T_n x_n - p) \right. \\ &\quad \left. - r_n(hx_n - hp) \right\|^2 \\ &\leq \left\| a_n(T_n x_n - p) + (1 - a_n)[(x_n - p) \right. \\ &\quad \left. - \frac{r_n}{(1 - a_n)}(hx_n - hp)] \right\|^2 \end{aligned}$$

Now, by definition of  $szl$ -widering then we get, for each  $s_n, l_n \in (0,1)$  then there exist  $z_n > 0$  (shortly we write them  $s, z$  and  $l$  respectively) such that

$$\begin{aligned} \|y_n - p\|^2 &\leq a_n \|T_n x_n - p\|^2 + (1 - a_n) \left\| (x_n - p) - \frac{r_n}{(1 - a_n)}(hx_n - hp) \right\|^2 \\ &\leq a_n(1 - s)\|x_n - p\|^2 + l\|p - T_n p\| \cdot \|x_n - T_n x_n - (p - T_n p)\| \\ &\quad + z|\langle x_n - T_n x_n, p - T_n p \rangle| + (1 - a_n)\|x_n - p\|^2 \\ &\quad - 2r_n \langle x_n - p, hx_n - hp \rangle + \frac{r_n^2}{(1 - a_n)} \|hx_n - hp\|^2 \\ \|y_n - p\|^2 &\leq a_n(1 - s)\|x_n - p\|^2 + (1 - a_n)\|x_n - p\|^2 \\ &\quad - 2\alpha r_n \|hx_n - hp\|^2 + \frac{r_n^2}{(1 - a_n)} \|hx_n - hp\|^2 \\ &\leq \|x_n - p\|^2 - 2\left(\alpha - \frac{r_n}{(1 - a_n)}\right) r_n \|hx_n - hp\|^2 \\ &\leq \|x_n - p\|^2 - \left(\frac{2\alpha(1 - a_n) - r_n}{(1 - a_n)}\right) r_n \|hx_n - hp\|^2 \\ \|x_{n+1} - p\|^2 &\leq a_n \|x_n - p\|^2 + b_n \|T_n x_n - p\|^2 \\ &\quad + c_n \|y_n - p\|^2 \\ &\leq a_n \|x_n - p\|^2 + b_n(1 - s)\|x_n - p\|^2 \\ &\quad + l\|p - T_n p\| \cdot \|x_n - T_n x_n - (p - T_n p)\| \\ &\quad + z|\langle x_n - T_n x_n, p - T_n p \rangle| + c_n \|x_n - p\|^2 \\ &\leq a_n \|x_n - p\|^2 + b_n \|x_n - p\|^2 + c_n \|x_n - p\|^2 \end{aligned}$$

By lemma (1.2),

We get  $\lim_{n \rightarrow \infty} \|x_n - p\|$  exists (8)

Hence,  $\{x_n\}$  is bounded sequence, and hence,  $\{f_n\}$  and  $\{w_n\}$  also bounded.

Since  $\|x_{n+1} - p\| \leq \|x_n - p\| + \|T_n x_n - x_n\|$ , therefore, by (8)

we get

$$\|T_n x_n - x_n\| \leq \|x_n - p\| - \|x_{n+1} - p\| \rightarrow 0 \text{ as } n \rightarrow \infty$$

$$\|T_n x_n - x_n\| \rightarrow 0 \text{ as } n \rightarrow \infty \quad (9)$$

Since  $\{x_n\}$  is bounded sequence. Then there exists subsequence  $\{x_{n_k}\}$  of  $\{x_n\}$  such that  $x_{n_k} \rightarrow \tilde{x}$ .

By (9), we get  $\tilde{x}$  is an asymptotic common fixed point of  $T_n, \forall n \in N$ . ■

**Theorem (2.2):**

Let  $\{T_n\}$  be a sequence of  $szl$ -widering mappings on  $C$  and  $\{f_n\}$  be a sequence of non-expansive mappings on  $C$ .

Let  $(\bigcap_{n=1}^{\infty} Fix(T_n)) \cap (\bigcap_{n=1}^{\infty} Fix(f_n)) \cap (A + h)^{-1}(0) \neq \emptyset$ . If the proximal point scheme generated by:

$$\begin{aligned} x_{n+1} &= a_n x_n + b_n T_n x_n + c_n f_n(x_n) + d_n y_n \\ y_n &= J_{r_n}(a_n x_n + (1 - a_n)T_n x_n - r_n h x_n) \end{aligned}$$

where  $\{b_n\}, \{c_n\}$  and  $\{d_n\}$  are sequences in  $[0,1], \{a_n\}$  is sequence in  $(0,1]$  such that  $2\alpha a_n > r_n$  and  $a_n + b_n + c_n + d_n = 1$ . Then the proximal point scheme  $\{x_n\}$  converges weakly to the common fixed point of  $T_n$ . Also,  $\langle P_E(x_n) \rangle \rightarrow \tilde{x}$ .

Where  $E = \bigcap_{n \in N} Fix(f_n) \cap (A + h)^{-1}(0)$ .

**Proof:**

Let  $p \in (\bigcap_{n=1}^{\infty} Fix(T_n)) \cap (A + h)^{-1}(0) \cap (\bigcap_{n=1}^{\infty} Fix(f_n))$

$$\begin{aligned} p \in (A + h)^{-1}(0) &\Rightarrow p = J_{r_n}(I - r_n h)p. \\ p &= J_{r_n}(a_n p + (1 - a_n)p - r_n h p) \end{aligned}$$

$$\begin{aligned} \|y_n - p\|^2 &= \|J_{r_n}(a_n x_n + (1 - a_n)T_n x_n - r_n h x_n) \\ &\quad - J_{r_n}(a_n p + (1 - a_n)p - r_n h p)\|^2 \\ &\leq \left\| (1 - a_n)(T_n x_n - p) + a_n \left[ (x_n - p) - \frac{r_n}{a_n}(h x_n - h p) \right] \right\|^2 \\ &\leq (1 - a_n) \|T_n x_n - p\|^2 + a_n \left\| (x_n - p) - \frac{r_n}{a_n}(h x_n - h p) \right\|^2 \\ &\leq (1 - a_n)(1 - s) \|x_n - p\|^2 \\ &\quad + l \|p - T_n p\| \cdot \|(x_n - T_n x_n) - (p - T_n p)\| \\ &\quad + z | \langle x_n - T_n x_n, p - T_n p \rangle | + a_n \|x_n - p\|^2 \\ &\quad - 2r_n \langle x_n - p, h x_n - h p \rangle \\ &\quad + \frac{r_n^2}{a_n} \|h x_n - h p\|^2 \\ \|y_n - p\|^2 &\leq (1 - a_n) \|x_n - p\|^2 + a_n \|x_n - p\|^2 \\ &\quad - 2\alpha r_n \|h x_n - h p\|^2 + \frac{r_n^2}{a_n} \|h x_n - h p\|^2 \\ &\leq \|x_n - p\|^2 \\ &\quad - \left( \frac{2\alpha a_n - r_n}{a_n} \right) r_n \|h x_n - h p\|^2 \\ \|y_n - p\|^2 &\leq \|x_n - p\|^2 \\ \|x_{n+1} - p\|^2 &\leq a_n \|x_n - p\|^2 + b_n \|T_n x_n - p\|^2 \\ &\quad + c_n \|f_n(x_n) - p\|^2 + d_n \|y_n - p\|^2 \\ &\leq (a_n + b_n + c_n) \|x_n - p\|^2 \\ &\quad + d_n \|x_n - p\|^2 \\ &= \|x_n - p\|^2 \end{aligned}$$

By lemma (1.2), we get  $\lim_{n \rightarrow \infty} \|x_n - p\|$  exists (10)  
 Then  $\langle x_n \rangle$  is bounded sequence, and hence  $\langle J_{r_n} \rangle$ ,  $\langle f_n \rangle$  and  $\langle w_n \rangle$  also bounded sequence.  
 Since  $\|x_{n+1} - p\| \leq \|x_n - p\| + \|T_n x_n - x_n\|$   
 By (10) we have,  
 $-\|T_n x_n - x_n\| \leq \|x_n - p\| - \|x_{n+1} - p\| \rightarrow 0$  as  $n \rightarrow \infty$   
 $\|T_n x_n - x_n\| \rightarrow 0$  as  $n \rightarrow \infty$  (11)  
 Since  $\langle x_n \rangle$  is bounded sequence  
 Then there exists subsequence  $\langle x_{n_k} \rangle$  of  $\langle x_n \rangle$  such that  $x_{n_k} \rightarrow \tilde{x}$ .  
 By (11) we get  $\tilde{x}$  is an asymptotic common fixed point of  $T_n, \forall n \in N$ . ■

**Theorem (2.3) :**

If  $\langle T_n \rangle$  be a bounded sequence of *szl* -widering mappings on  $C$  and  $\langle f_n \rangle$  be a sequence of non-expansive mapping on  $C$ . If proximal point scheme  $\langle x_n \rangle$  is defined as

$$\begin{aligned} x_{n+1} &= a_n x_n + b_n f_n(x_n) + d_n [c_n x_n + (1 - c_n) J_{r_n} y_n] \\ y_n &= J_{r_n} ((1 - a_n) x_n + a_n T_n x_n - r_n h x_n) \end{aligned}$$

Where  $\langle a_n \rangle$ ,  $\langle b_n \rangle$  and  $\langle c_n \rangle$  are sequence in  $(0, 1]$  such that

- 1)  $a_n + b_n + c_n = 1$  and  $2\alpha(1 - a_n) > r_n$
- 2)  $(\bigcap_{n=1}^{\infty} \text{Fix}(T_n)) \cap (\bigcap_{n=1}^{\infty} \text{Fix}(f_n)) \cap (A + h)^{-1}(0) \neq \emptyset$ .

Then the proximal point scheme  $\langle x_n \rangle$  has converges weakly to an asymptotic common fixed point.

**Proof:**

Let  $p \in (\bigcap_{n=1}^{\infty} \text{Fix}(T_n)) \cap (\bigcap_{n=1}^{\infty} \text{Fix}(f_n)) \cap (A + h)^{-1}(0)$

$$\text{Since } y_n = J_{r_n} ((1 - a_n) x_n + a_n T_n x_n - r_n h x_n)$$

$$\begin{aligned} \|y_n - p\|^2 &= \|J_{r_n} ((1 - a_n) x_n + a_n T_n(x_n) - r_n h(x_n)) \\ &\quad - J_{r_n} ((1 - a_n) p + a_n T_n p - r_n h p)\|^2 \end{aligned}$$

Now, by definition of *szl* -widering then we get, for each  $s_n, l_n \in (0, 1)$  then there exist  $z_n > 0$  (shortly we write them  $s, z$  and  $l$  respectively) such that

$$\begin{aligned} \|y_n - p\|^2 &\leq \|(1 - a_n) x_n + a_n T_n(x_n) - r_n h x_n \\ &\quad - (1 - a_n) p - a_n T_n p + r_n h p\|^2 \\ &\leq a_n \|T_n x_n - T_n p\|^2 \\ &\quad + (1 - a_n) \left\| (x_n - p) - \frac{r_n}{(1 - a_n)} (h x_n - h p) \right\|^2 \\ &\leq a_n (1 - s) \|x_n - p\|^2 \\ &\quad + l \|p - T_n p\| \cdot \|(x_n - T_n x_n) - (p - T_n p)\| \\ &\quad + z | \langle x_n - T_n x_n, p - T_n p \rangle | \\ &\quad + (1 - a_n) \|x_n - p\|^2 \\ &\quad - 2r_n \langle x_n - p, h x_n - h p \rangle \\ &\quad + \frac{r_n^2}{(1 - a_n)} \|h x_n - h p\|^2 \\ \|y_n - p\|^2 &\leq a_n \|x_n - p\|^2 + (1 - a_n) \|x_n - p\|^2 \\ &\quad - 2r_n \alpha \|h x_n - h p\|^2 \\ &\quad + \frac{r_n^2}{(1 - a_n)} \|h x_n - h p\|^2; \alpha > 0 \\ &\leq \|x_n - p\|^2 \\ &\quad - \left( \frac{2\alpha(1 - a_n) - r_n}{(1 - a_n)} \right) r_n \|h x_n - h p\|^2 \end{aligned}$$

But,  $2\alpha(1 - a_n) > r_n$   
 $\|y_n - p\|^2 \leq \|x_n - p\|^2$   
 $\|x_{n+1} - p\|^2 \leq a_n \|x_n - p\|^2 + b_n \|f_n(x_n) - p\|^2$   
 $\quad + d_n \|c_n x_n + (1 - c_n) J_{r_n} y_n - p\|^2$   
 $\leq a_n \|x_n - p\|^2 + b_n \|x_n - p\|^2$   
 $\quad + d_n [c_n \|x_n - p\|^2$   
 $\quad + (1 - c_n) \|J_{r_n} y_n - p\|^2]$   
 $\|x_{n+1} - p\|^2 \leq (a_n + b_n) \|x_n - p\|^2$   
 $\quad + d_n [c_n \|x_n - p\|^2 + (1 - c_n) \|y_n - p\|^2]$   
 $\leq (a_n + b_n) \|x_n - p\|^2 + d_n \|x_n - p\|^2$   
 $= \|x_n - p\|^2$

By lemma (1.2), we get  $\lim_{n \rightarrow \infty} \|x_n - p\|$  exists (12)  
 Then  $\langle x_n \rangle$  is bounded sequence, and hence  $\langle J_{r_n} \rangle$ ,  $\langle f_n \rangle$  and  $\langle w_n \rangle$  also bounded sequence .

Since  $\|x_{n+1} - p\| \leq \|x_n - p\| + \|T_n x_n - x_n\|$  therefore, by (12)

$$-\|T_n x_n - x_n\| \leq \|x_n - p\| - \|x_{n+1} - p\| \rightarrow 0 \text{ as } n \rightarrow \infty$$

$$\|T_n x_n - x_n\| \rightarrow 0 \text{ as } n \rightarrow \infty \text{ (13)}$$

Since  $\langle x_n \rangle$  is bounded sequence. Then there exists subsequence  $\langle x_{n_k} \rangle$  of  $\langle x_n \rangle$  such that  $x_{n_k} \rightarrow \tilde{x}$ . By (13), we get  $\tilde{x}$  is an asymptotic common fixed point of  $T_n, \forall n \in N$ . ■

**Theorem (2.4):**

If  $\langle T_n \rangle$  be a bounded sequence of *szl* -widering mappings on  $C$  and  $\langle f_n \rangle$  be sequence of non-expansive mapping on  $C$  such that  $(\bigcap_{n=1}^{\infty} \text{Fix}(f_n)) \cap (\bigcap_{n=1}^{\infty} \text{Fix}(T_n)) \cap (A + h)^{-1}(0) \neq \emptyset$ . If the proximal point scheme  $\langle x_n \rangle$  is defined as:

$$\begin{aligned} x_{n+1} &= a_n x_n + b_n T_n x_n + c_n f_n(x_n) + d_n T_n y_n \\ y_n &= J_{r_n} (a_n x_n + (1 - a_n) T_n x_n - r_n h x_n) \end{aligned}$$

Where  $\langle a_n \rangle$ ,  $\langle b_n \rangle$ ,  $\langle c_n \rangle$  and  $\langle d_n \rangle$  are sequences in  $(0, 1)$  and  $\langle r_n \rangle$  be sequence in  $\mathbb{R}^+$  such that  $a_n + b_n + c_n + d_n = 1$  &  $2a_n \alpha > r_n$ . Then the proximal point scheme  $\langle x_n \rangle$  has converges weakly to an asymptotic common fixed point

**Proof:**

Let  $p \in (\bigcap_{n=1}^{\infty} \text{Fix}(f_n)) \cap (\bigcap_{n=1}^{\infty} \text{Fix}(T_n)) \cap (A + h - 10 \neq \emptyset)$

Since  $p \in (A + h)^{-1}(0) \Rightarrow p = J_{r_n}(I - r_n h)p$ .

$$p = J_{r_n}(a_n p + (1 - a_n)p - r_n h).$$

But  $y_n = J_{r_n}(a_n x_n + (1 - a_n)T_n x_n - r_n h x_n)$

$$\|y_n - p\|^2 = \|J_{r_n}(a_n x_n + (1 - a_n)T_n x_n - r_n h x_n) - p\|^2$$

$$= \|J_{r_n}(a_n x_n + (1 - a_n)T_n x_n - r_n h x_n) - J_{r_n}(a_n p + (1 - a_n)p + r_n h p)\|^2$$

$$\|y_n - p\|^2 = \|a_n x_n + (1 - a_n)T_n x_n - r_n h x_n - a_n p - (1 - a_n)p + r_n h p\|^2$$

$$\leq (1 - a_n)\|T_n x_n - p\|^2 + a_n \|x_n - p\|^2 - 2r_n \langle x_n - p, h x_n - h p \rangle + \frac{r_n^2}{a_n} \|h x_n - h p\|^2$$

Now, by definition of  $s, z, l$  - widening then we get, for each  $s, l_n \in (0,1)$  then there exist  $z_n > 0$  (shortly we write them  $s, z$  and  $l$  respectively) such that

$$\|y_n - p\|^2 \leq (1 - a_n)\{(1 - s)\|x_n - p\|^2 + l\|p - T_n p\| \cdot \|x_n - T_n x_n - (p - T_n p)\| + z\langle x_n - T_n x_n, p - T_n p \rangle\} + a_n \|x_n - p\|^2 - 2r_n \alpha \|h x_n - h p\|^2 + \frac{r_n^2}{a_n} \|h x_n - h p\|^2$$

$$\|y_n - p\|^2 \leq (1 - a_n)(1 - s)\|x_n - p\|^2 + a_n \|x_n - p\|^2 - \frac{(2a_n \alpha - r_n)r_n}{a_n} \|h x_n - h p\|^2$$

$$\leq (1 - a_n)\|x_n - p\|^2 + a_n \|x_n - p\|^2 - \frac{(2a_n \alpha - r_n)r_n}{a_n} \|h x_n - h p\|^2$$

$$\leq \|x_n - p\|^2$$

$$\|x_{n+1} - p\|^2 \leq a_n \|x_n - p\|^2 + b_n \|T_n x_n - p\|^2 + c_n \|f_n(x_n) - p\|^2 + d_n \|T_n y_n - p\|^2$$

$$\|x_{n+1} - p\|^2 \leq a_n \|x_n - p\|^2 + b_n (1 - s)\|x_n - p\|^2 + b_n l\|p - T_n p\| \cdot \|x_n - T_n x_n - (p - T_n p)\| + b_n z\langle x_n - T_n x_n, p - T_n p \rangle + c_n \|x_n - p\|^2 + d_n (1 - s)\|y_n - p\|^2 + d_n l\|p - T_n p\| \cdot \|y_n - T_n y_n - (p - T_n p)\| + d_n z\langle y_n - T_n y_n, p - T_n p \rangle$$

$$\|x_{n+1} - p\|^2 \leq a_n \|x_n - p\|^2 + b_n (1 - s_n)\|x_n - p\|^2 + \delta_n \|x_n - p\|^2 + d_n (1 - s)\|y_n - p\|^2$$

$$\leq (1 - d_n)\|x_n - p\|^2 + d_n \|x_n - p\|^2$$

$$\leq \|x_n - p\|^2$$

By lemma (1.2), we get

$$\lim_{n \rightarrow \infty} \|x_n - p\| \text{ exists (14)}$$

So,  $\langle x_n \rangle$  is bounded sequence, and hence  $\langle J_{r_n} \rangle, \langle f_n \rangle$  and  $\langle w_n \rangle$  also bounded.

Since  $\|x_{n+1} - p\| \leq \|x_n - p\| + \|T_n x_n - x_n\|$  therefore, by (14)

$$-\|T_n x_n - x_n\| \leq \|x_n - p\| - \|x_{n+1} - p\| \rightarrow 0 \text{ as } n \rightarrow \infty$$

$$\|T_n x_n - x_n\| \rightarrow 0 \text{ as } n \rightarrow \infty \text{ (15)}$$

Since  $\langle x_n \rangle$  is bounded sequence. Then there exists subsequence  $\langle x_{n_k} \rangle$  of  $\langle x_n \rangle$  such that  $x_{n_k} \rightarrow \tilde{x}$ . By (15) we get  $\tilde{x}$  is an asymptotic common fixed point of  $T_n$ . ■

**References**

- [1] H.H. Bauschke and P.L. Combettes, Convex Analysis and Monotone Operator Theory in Hilbert Spaces, Springer-Verlag, 2011.
- [2] J.M. Borwein and J.D. Vanderwerff, Convex Functions, Cambridge University Press, 2010.
- [3] R.S. Burachik and A.N. Iusem, Set-Valued Mappings and Enlargements of Monotone Operators, Springer-Verlag, 24, 2008.
- [4] S. Simons, From Hahn-Banach to Monotonicity, Springer-Verlag, 2008.
- [5] C. Zălinescu, Convex Analysis in General Vector Spaces, world Scientific Publishing, 2002.
- [6] H.K.Xu, "Another Control Condition In An Iterative Method for Nonexpansive Mappings, Bull. Austral. Math. Soc. 65 (2002), 109-113.
- [7] H.K.Xu, "Iterative Algorithm for Nonlinear Operators", J. London Math. Soc. 66(2002) 240-256
- [8] A.Moudafi, Viscosity Approximation Method for Fixed Point Problems, Journal of Mathematical Analysis and Applications, 241(2000) 46-55.
- [9] H.K.Xu, "Viscosity Approximation Methods for Nonexpansive Mapping", J. Math. Anal. Appl. 298(2004)279-291.
- [10] S. Kamimura, W. Takahashi, Approximating Solutions of Maximal Monotone Operators In Hilbert spaces, J. Approx. Theory 106 (2000) 226-240.
- [11] S.S. Abed, Z.H. Maibed, Convergence Theorems of Iterative Schemes For Nonexpansive Mappings, Journal of Advances in Mathematics, 12(2016)6845-6851.
- [12] S.S. Abed, Z.H. Maibed, Convergence Theorems for Maximal Montone Operators By Family of Nonspreading Mappings, (IJSR), 2015, (2017)2319\_7064.
- [13] F. Kohsaka and W. Takahashi, "Fixed Point Theorems For a Class Of Nonlinear Mappings Relate To Maximal Monotone Operators In Banach Spaces", Arch. Math (Basel), 91 (2008) 166 - 177.
- [14] H.Piri, Solution of Variational Inequalities on Fixed Points of Nonexpansive Mappings, Bulletin of The Iranian Mathematical Society vol.39 no 4(2013)743-764.
- [15] U. Singthong and S. Suontal, "Equilibrium Problems and Fixed Point Problems For Non-spreading Type Mapping In Hilbert Space", J. Nonlinear, Anal. Appl, 2(2011), 51-61.
- [16] H. Manaka and W. Takahashi, Weak Convergence Theorems for Maximall Monotone Operators With Nonspreading Mappings in Hilbert Space, CUBO, A. Math. journal, Vol 13, NO.01, 2011. (11-24).
- [17] D.A. Ruiz, G.L. Acedo and V.M. Marquez, "Firmly nonexpansive mappings", J. Nonlinear analysis, vol 15(2014)1.
- [18] M. Eslamian, "Rockafellars Proximal Point Algorithm For A Finite Family Of Monotone Operators", U. P. B. sci. Bull. vol 76 ISS 1 (2014).

- [19] W. Takahashi and M. Toyoda, "Weak Convergence Theorem For Non-expansive Mappings and Monotone Mappings", J. optim theory Appl. 118 (2003) 417-428.
- [20] S.S. Abed ,Z .H. Maibed, " Theorems for Proximal Point Schemes by Sequences of  $szl$  –Widering Mappings" ,Global Journal of Math Sciences :Theory and Practical(GJMS).

