Convergence Weakly to Asymptotic Common Fixed Point Theorems for Different Types of Proximal Point Schemes

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Abstract: ***In this paper, we introduce a proximal point schemes of szl –widering mapping which is independent of non-expansive mappings. Also, we discuss the weak convergence for these proximal point schemes in Hilbert space .***

Keywords: maximal monotone, converge strongly, nonexpansive mapping

1. Introduction and Preliminaries

Let X be a real Hilbert space and A be maximal monotone mapping. The Monotone operators have proven to a key class of objects in modern optimization and analysis see ([1]-[5]). The zero point problem for monotone operator A on a real Hilbert space X ,that is ,finding a point $z \in X$ such that $0 \in A(z)$.in order to solve this problem, many types of iterative algorithms have been studied such as [6]-[17]. Consider a single valued non-expansive mapping as: $J_{r_n} = (I + r_n A^{-1})(x)$,which is called resolvet mapping where $< r_n >$ be a sequence of positive real numbers. In[6,7] Xu , studied the convergence of the proximal point scheme

$$x \in C,\ x_{n+1} = \alpha_n x + (1-\alpha_n)\mathrm{T}_{x_n}, n = 1,2,3, \dots \quad (1)$$

where T is non_ expansive mapping and $< \alpha_n >$ be a sequence in (0,1) .In[8] Moudafi , studied the convergence of the proximal point schemes

$$x_t = tf(x_t) + (1\text{-}t)T_{x_t}\ as\ t \to \infty \quad (2)$$

$$x_{n+1} = \alpha_n f(x_n) + (1\text{-}\alpha_n)T_{x_n}\ as\ n \to \infty$$

where T is non_ expansive mapping , f b a contraction mapping and $< \alpha_n >$ be a sequence in (0,1).In [9] Xu, who extend Moudafi results . On other hand, Kamimura and Takahashi[10] ,studied the convergence strongly of the proximal point scheme

$$u \in C,\ x_{n+1} = \alpha_n u + (1-\alpha_n)\mathrm{J}_{r_{n}x_n}, n \geq 1 \quad (3)$$

in 2016[11, 12], Abed and Maibed studied the strong convergence of the many proximal point schemes. Now, consider X be a real Hilbert space, $\emptyset \neq$ C be a convex closed in X. We recall some definitions and lemmas which will used in the proofs.

Definition (1.1): [1]

A mapping $T: C \to X$ is calledLipschitz if there exists a real number $L > 0$ such that

$$\|T(x) - T(y)\| \leq L\|x - y\| \text{for each } x, y \in C. \quad (4)$$

When $L \in (0,1)$ then $T$is calledcontraction mapping and if $L = 1$then$T$ is called nonexpansive mapping

Lemma (1.2) [16]

Let $< a_n >$ and $< b_n >$ are sequences of nonnegative number such that

$a_{n+1} \leq a_n + b_n$, for each $n \geq 1$. If$\sum_{n=o}^{\infty} a_n$ converge then $\lim_{n\to\infty} a_n$exists.

Lemma (1.3): [17]

Let C be a nonempty convex closed subset of real Hilbert space $X$and T is non-expansive multivalued mapping such that $Fix(T) \neq \emptyset$.Then T is demi-closed, i.e.,$x_n \rightharpoonup p$ and $\lim_{n\to\infty} d(x_n, T(x_n)) = 0$.Then $p \in T(p)$.

Lemma (1.4) : [7]

If $\langle a_n \rangle$ be a sequence of non-negative real number such that:

$$a_{n+1} \leq (1 - \gamma_n)a_n + S_n, n \geq 0$$

Where $\langle \gamma_n \rangle$ is a sequence in (0,1) and $\langle S_n \rangle$ be a sequence in $\mathbb{R}$ such that:

$\sum_{n=0}^{\infty} \gamma_n = \infty$ and $\lim_{n\to\infty} sup \frac{S_n}{\gamma_n} \leq 0$ or $\sum_{n=1}^{\infty} |S_n| < \infty$.

Then$a_n \to 0$
as $n \to \infty$.

Lemma (1.5) : [18]

If $\langle a_n \rangle$ be a sequence nonnegative real numbers such that:

$$a_{n+1} \leq (1 - \gamma)a_n + \gamma_n S_n + \beta_n, n \geq 0 \quad (5)$$

Where $\langle \gamma_n \rangle, \langle \beta_n \rangle$ and $\langle S_n \rangle$ are satisfies the following:

1) $\gamma_n \subset [0,1]$; $\sum_{n=1}^{\infty} \gamma_n = \infty$
2) $\lim_{n\to\infty} sup\, S_n \leq 0$ or $\sum_{n=1}^{\infty} |\gamma_n S_n| < \infty$
3) $\beta_n \geq 0$ for each $n \geq 0$ such that $\sum_{n=0}^{\infty} \beta_n < \infty$.Then $\lim_{n\to\infty} a_n = 0$.

Lemma (1. 6): [19]

Let X be a Hilbert space and C be a nonempty convex closed subset of X if $\langle x_n \rangle$ be a sequence in X and $\|x_{n+1} - x\| \leq \|x_n - x\|$ for all $n \in N, x \in C$. Then $\langle P_c(x_n) \rangle$ converges strongly to a point in C.

Now, we introduce the concept ofszl –widering mapping

Definition (1.7): [20]

Let X be a normed space and C be a nonempty subset of X, then A mapping $T: C \to C$ is called szl –widering if for each

$s, l \in (0,1)$then there exists $z > 0$ such that the following equation holds:

$$\|Tx - Ty\|^2 \le (1-s)\|x-y\|^2 + l\|y - Ty\|.\|x - Tx - (y - Ty)\| + z|\langle x - Tx, y - Ty\rangle| \text{ , for each } x, y \in C \quad (6)$$

The concept of szl −widering mapping is independent of concepts of non-expansive mappings. As shown by the following examples:

Example (1.8):

(a) Let $T: R \to R$ such that T be identity mapping then T is non-expansive, but it is not $sz - widering$

(b) Let $T: R \to R$ such that $T(x) = 2x$Then T is szl −widering but not non-expansive mapping.

Lemma (1.9) :[20]

Let$\emptyset \neq C$ be a closed convex subset of Hilbert space $X$and$T$is szl −widering. Then it is demi-closed ,i.e., if there exists $\langle x_n \rangle$sequence in Csuch that $x_n \rightharpoonup p$ and $\|Tx_n - xn0$ then $p \in Fix(p)$.

Now, we introduce a proximal point schemes of non-expansive and szl −widering mappings and we discuss the weak convergence for these proximal point schemes under different conditions to asymptotic common fixed point of szl −widering mappings.

2. Main Results

Theorem (2.1)

If $\langle T_n \rangle$ be a sequence of bounded szl −widering mappings has .Define the proximal point scheme $\langle x_n \rangle$ as follows

$$x_{n+1} = a_n x_n + b_n T_n x_n + c_n y_n$$
$$y_n = J_{r_n}\big((1-a_n)x_n + a_n T_n x_n - r_n h x_n\big) \quad (7)$$

where $\langle a_n \rangle, \langle b_n \rangle$ *and* $\langle c_n \rangle$ are sequences in (0,1) such that:

1) $a_n + b_n + c_n = 1$ *and* $r_n < 2(1 - a_n)\alpha$
2) $\bigcap_{n=1}^{\infty} Fix(T_n) \bigcap (A+h)^{-1}(0) \neq \emptyset$

Then the proximal point scheme converges weakly to the asymptotic common fixed point of $T_n, \forall\, n \in N$.

Proof:

Let $p \in \bigcap_{n=1}^{\infty} Fix(T_n) \bigcap (A+h)^{-1}(0)$

Since, $y_n = J_{r_n}\big((1-a_n)x_n + a_n T_n x_n - r_n h x_n\big)$

So, $p \in J_{r_n}\big((1-a_n)p + a_n T_n p - r_n hp\big)$

$$\begin{aligned}\|y_n - p\|^2 &= \big\|J_{r_n}\big((1-a_n)x_n + a_n T_n x_n - r_n h x_n\big) - J_{r_n}\big((1-a_n)p + a_n T_n p - r_n hp\big)\big\| \\ &\le \|(1-a_n)(x_n - p) + a_n(T_n x_n - p) - r_n(hx_n - hp)\|^2 \\ &\le \left\| a_n(T_n x_n - p) + (1-a_n)[(x_n - p) - \frac{r_n}{(1-a_n)}(hx_n - hp)] \right\|^2\end{aligned}$$

Now, by definition of szl − widering then we get, for each $s_n, l_n \in (0,1)$ then there exist $z_n > 0$ (shortly we write them s, z and l respectively) such that

$$\begin{aligned}\|y_n - p\|^2 &\le a_n\|T_n x_n - p\|^2 + (1-a_n)\left\|(x_n - p) - \frac{r_n}{(1-a_n)}(hx_n - hp)\right\|^2 \\ &\le a_n(1-s)\|x_n - p\|^2 + l\|p - T_n p\|.\|x_n - T_n x_n - (p - T_n p)\| + z|\langle x_n - T_n x_n, p - T_n p\rangle| \\ &\quad + (1-a_n)\|x_n - p\|^2 - 2r_n\langle x_n - p, hx_n - hp\rangle + \frac{{r_n}^2}{(1-a_n)}\|hx_n - hp\|^2\end{aligned}$$

$$\begin{aligned}\|y_n - p\|^2 &\le a_n(1-s)\|x_n - p\|^2 + (1-a_n)\|x_n - p\|^2 - 2\alpha r_n\|hx_n - hp\|^2 + \frac{{r_n}^2}{(1-a_n)}\|hx_n - hp\|^2 \\ &\le \|x_n - p\|^2 - 2\left(\alpha - \frac{r_n}{(1-a_n)}\right) r_n\|hx_n - hp\|^2 \\ &\le \|x_n - p\|^2 - \left(\frac{2\alpha(1-a_n) - r_n}{(1-a_n)}\right) r_n\|hx_n - hp\|^2 \\ &\le \|x_n - p\|^2\end{aligned}$$

$$\begin{aligned}\|x_{n+1} - p\|^2 &\le a_n\|x_n - p\|^2 + b_n\|T_n x_n - p\|^2 + c_n\|y_n - p\|^2 \\ &\le a_n\|x_n - p\|^2 + b_n(1-s)\|x_n - p\|^2 + l\|p - T_n p\|.\|x_n - T_n x_n - (p - T_n p)\| \\ &\quad + z|\langle x_n - T_n x_n, p - T_n p\rangle| + c_n\|x_n - p\|^2 \\ &\le a_n\|x_n - p\|^2 + b_n\|x_n - p\|^2 + c_n\|x_n - p\|^2\end{aligned}$$

By lemma (1.2),

We get $\lim_{n\to\infty}\|x_n - p\|$exists (8)

Hence, $\langle x_n \rangle$ is bounded sequence, and hence ,$\langle f_n \rangle$and$\langle w_n \rangle$ also bounded.

Since$\|x_{n+1} - p\| \le \|x_n - p\| + \|T_n x_n - x_n\|$, therefore, by (8)

we get

$-\|T_n x_n - x_n\| \le \|x_n - p\| - \|x_{n+1} - p\| \to 0$ *as* $n \to \infty$

$\|T_n x_n - x_n\| \to 0$ *as* $n \to \infty$ (9)

Since $\langle x_n \rangle$ is bounded sequence.Then there exists subsequence $\langle x_{nk} \rangle$ of $\langle x_n \rangle$ such that $x_{nk} \rightharpoonup \tilde{x}$.

By (9), we get$\tilde{x}$ is an asymptotic common fixed point of $T_n, \forall\, n \in N$. ■

Theorem (2.2):

Let $\langle T_n \rangle$ be a sequence of szl −widering mappings on C and $\langle f_n \rangle$ be a sequence of non-expansive mappings on C.

Let$(\bigcap_{n=1}^{\infty} Fix(T_n)) \cap (\bigcap_{n=1}^{\infty} Fix(f_n)) \cap (A+h)^{-1}(0) \neq \emptyset$.If the proximal point scheme generated by:

$$x_{n+1} = a_n x_n + b_n T_n x_n + c_n f_n(x_n) + d_n\, y_n$$
$$y_n = J_{r_n}(a_n x_n + (1-a_n)T_n x_n - r_n h x_n)$$

where$\langle b_n \rangle, \langle c_n \rangle$ *and* $\langle d_n \rangle$are sequences in [0,1],$\langle a_n \rangle$ is sequence in (0,1]such that $2\alpha a_n > r_n$ and $a_n + b_n + c_n + d_n = 1$.Then the proximal point scheme $\langle x_n \rangle$ converges weakly to the common fixed point of T_n .Also, $\langle P_E(x_n) \rangle \to \tilde{x}$.

Where $E = \cap_{n\in N} Fix(f_n) \cap (A+h)^{-1}(0)$.

Proof:

Let $p \in (\bigcap_{n=1}^{\infty} Fix(T_n)) \cap (A+h)^{-1}(0) \cap (\bigcap_{n=1}^{\infty} Fix(f_n))$

$$p \in (A+h)^{-1}(0) \Rightarrow p = J_{r_n}(I - r_n h)p.$$
$$p = J_{r_n}(a_n p + (1-a_n)p - r_n hp)$$

$$\|y_n - p\|^2 = \|J_{r_n}(a_n x_n + (1-a_n)T_n x_n - r_n h x_n) - J_{r_n}(a_n p + (1-a_n)p - r_n hp)\|$$

$$\le \left\|(1-a_n)(T_n x_n - p) + a_n[(x_n - p) - \frac{r_n}{a_n}(hx_n - hp)]\right\|^2$$

$$\le (1-a_n)\|T_n x_n - p\|^2 + a_n\left\|(x_n - p) - \frac{r_n}{a_n}(hx_n - hp)\right\|^2$$

$$\le (1-a_n)(1-s)\|x_n - p\|^2 + l\|p - T_n p\|.\|(x_n - T_n x_n) - (p - T_n p)\| + z|\langle x_n - T_n x_n, p - T_n p\rangle| + a_n\|x_n - p\|^2 - 2r_n\langle x_n - p, hx_n - hp\rangle + \frac{r_n^{\,2}}{a_n}\|hx_n - hp\|^2$$

$$\|y_n - p\|^2 \le (1-a_n)\|x_n - p\|^2 + a_n\|x_n - p\|^2 - 2\alpha r_n\|hx_n - hp\|^2 + \frac{r_n^{\,2}}{a_n}\|hx_n - hp\|^2$$

$$\le \|x_n - p\|^2 - \left(\frac{2\alpha a_n - r_n}{a_n}\right) r_n\|hx_n - hp\|^2$$

$$\|y_n - p\|^2 \le \|x_n - p\|^2$$

$$\|x_{n+1} - p\|^2 \le a_n\|x_n - p\|^2 + b_n\|T_n x_n - p\|^2 + c_n\|f_n(x_n) - p\|^2 + d_n\|y_n - p\|^2$$

$$\le (a_n + b_n + c_n)\|x_n - p\|^2 + d_n\|x_n - p\|^2 = \|x_n - p\|^2$$

By lemma (1.2),
we get $\lim_{n\to\infty}\|x_n - p\|$exists (10)
Then $\langle x_n\rangle$ is bounded sequence, and hence$\langle J_{r_n}\rangle, \langle f_n\rangle$and $\langle w_n\rangle$ also bounded sequence.
Since $\|x_{n+1} - p\| \le \|x_n - p\| + \|T_n x_n - x_n\|$
By (10) we have ,

$$-\|T_n x_n - x_n\| \le \|x_n - p\| - \|x_{n+1} - p\| \to 0 \ as\ n \to \infty$$

$$\|T_n x_n - x_n\| \to 0 \ as\ n \to \infty \ (11)$$

Since $\langle x_n\rangle$ is bounded sequence
Then there exists subsequence $\langle x_{nk}\rangle$ of $\langle x_n\rangle$ such that $x_{nk} \rightharpoonup \tilde{x}$.
By (11)we get $\tilde{x}$ is an asymptotic common fixed point of $T_n, \forall\, n \in N$. ■

Theorem (2.3) :
If $\langle T_n\rangle$ be a bounded sequence of szl –widering mappings on C and $\langle f_n\rangle$ be a sequence of non-expansive mapping on C .If proximal point scheme $\langle x_n\rangle$ is defined as

$$x_{n+1} = a_n x_n + b_n f_n(x_n) + d_n[c_n x_n + (1-c_n)J_{r_n} y_n]$$

$$y_n = J_{r_n}((1-a_n)x_n + a_n T_n x_n - r_n h x_n)$$

Where $\langle a_n\rangle, \langle b_n\rangle$ and $\langle c_n\rangle$ are sequence in (0,1] such that
1) $a_n + b_n + c_n = 1$ and $2\alpha(1-a_n) > r_n$
2) $(\bigcap_{n=1}^{\infty} Fix(T_n)) \cap (\bigcap_{n=1}^{\infty} Fix(f_n)) \cap (A+h)^{-1}(0) \neq \emptyset$.

Then the proximal point scheme $\langle x_n\rangle$ has converges weakly to an asymptotic common fixed point.

Proof:
Let $p \in (\bigcap_{n=1}^{\infty} Fix(T_n)) \cap (\bigcap_{n=1}^{\infty} Fix(f_n)) \cap (A+h)^{-1}(0)$
Since $y_n = J_{r_n}((1-a_n)x_n + a_n T_n x_n - r_n h x_n)$

$$\|y_n - p\|^2 = \left\|J_{r_n}((1-a_n)x_n + a_n T_n(x_n) - r_n h(x_n)) - J_{r_n}((1-a_n)p + a_n T_n p - r_n hp)\right\|^2$$

Now, by definition of szl – widering then we get, for each $s_n, l_n \in (0,1)$ then there exist $z_n > 0$ (shortly we write them s, z and l respectively) such that

$$\|y_n - p\|^2 \le \|(1-a_n)x_n + a_n T_n(x_n) - r_n h x_n - (1-a_n)p - a_n T_n p + r_n hp\|^2$$

$$\le a_n\|T_n x_n - T_n p\|^2 + (1-a_n)\left\|(x_n - p) - \frac{r_n}{(1-a_n)}(hx_n - hp)\right\|^2$$

$$\le a_n(1-s)\|x_n - p\|^2 + l\|p - T_n p\|.\|(x_n - T_n x_n) - (p - T_n p)\| + z|\langle x_n - T_n x_n, p - T_n p\rangle| + (1-a_n)\|x_n - p\|^2 - 2r_n\langle x_n - p, h_n x_n - hp\rangle + \frac{r_n^{\,2}}{(1-a_n)}\|hx_n - hp\|^2$$

$$\|y_n - p\|^2 \le a_n\|x_n - p\|^2 + (1-a_n)\|x_n - p\|^2 - 2r_n\alpha\|hx_n - hp\|^2 + \frac{r_n^{\,2}}{(1-a_n)}\|hx_n - hp\|^2 \,;\ \alpha > 0$$

$$\le \|x_n - p\|^2 - \left(\frac{2\alpha(1-a_n) - r_n}{(1-a_n)}\right) r_n\|hx_n - hp\|^2$$

But, $2\alpha(1-a_n) > r_n$

$$\|y_n - p\|^2 \le \|x_n - p\|^2$$

$$\|x_{n+1} - p\|^2 \le a_n\|x_n - p\|^2 + b_n\|f_n(x_n) - p\|^2 + d_n\left\|c_n x_n + (1-c_n)J_{r_n} y_n - p\right\|^2$$

$$\le a_n\|x_n - p\|^2 + b_n\|x_n - p\|^2 + d_n\left[c_n\|x_n - p\|^2 + (1-c_n)\left\|J_{r_n} y_n - p\right\|^2\right]$$

$$\|x_{n+1} - p\|^2 \le (a_n + b_n)\|x_n - p\|^2 + d_n[c_n\|x_n - p\|^2 + (1-c_n)\|y_n - p\|^2]$$

$$\le (a_n + b_n)\|x_n - p\|^2 + d_n\|x_n - p\|^2 = \|x_n - p\|^2$$

By lemma (1.2),we get
$\lim_{n\to\infty}\|x_n - p\|$exists (12)
Then $\langle x_n\rangle$ is bounded sequence, and hence$\langle J_{r_n}\rangle, \langle f_n\rangle$ *and* $\langle w_n\rangle$ also bounded sequence .
Since $\|x_{n+1} - p\| \le \|x_n - p\| + \|T_n x_n - x_n\|$ therefore, by (12)

$$-\|T_n x_n - x_n\| \le \|x_n - p\| - \|x_{n+1} - p\| \to 0 \ as\ n \to \infty$$

$$\|T_n x_n - x_n\| \to 0 \ as\ n \to \infty \ (13)$$

Since $\langle x_n\rangle$ is bounded sequence. Then there exists subsequence $\langle x_{nk}\rangle$ of $\langle x_n\rangle$ such that $x_{nk} \rightharpoonup \tilde{x}$.By (13), we get$\tilde{x}$ is an asymptotic common fixed point of $T_n, \forall\, n \in N$. ■

Theorem (2.4):
If $\langle T_n\rangle$ be a bounded sequence of szl – widering mappings on $C$and $\langle f_n\rangle$ be sequence of non-expansive mapping on C such that $(\bigcap_{n=1}^{\infty} Fix(f_n)) \cap (\bigcap_{n=1}^{\infty} Fix(T_n)) \cap (A+h)^{-1}(0) \neq \emptyset$. If the proximal point scheme $\langle x_n\rangle$ is defined as:

$$x_{n+1} = a_n x_n + b_n T_n x_n + c_n f_n(x_n) + d_n T_n y_n$$

$$y_n = J_{r_n}(a_n x_n + (1-a_n)T_n x_n - r_n h x_n)$$

Where $\langle a_n\rangle, \langle b_n\rangle, \langle c_n\rangle$ *and* $\langle d_n\rangle$ are sequences in (0,1) and $\langle r_n\rangle$ be sequence in $\mathbb{R}^+$ such that $a_n + b_n + c_n + d_n = 1$ & $2a_n\alpha > r_n$.Then the proximal point scheme $\langle x_n\rangle$ has converges weakly to an asymptotic common fixed point

Proof:

Let $p \in (\cap_{n=1}^{\infty} Fix(f_n)) \cap (\cap_{n=1}^{\infty} Fix(T_n)) \cap (A + h-10 \neq \emptyset$

Since $p \in (A+h)^{-1}(0) \Rightarrow p = J_{r_n}(I - r_n h)p$.

$p = J_{r_n}(a_n p + (1-a_n)p - r_n h)$.

But $y_n = J_{r_n}(a_n x_n + (1-a_n)T_n x_n - r_n h x_n)$

$$\|y_n - p\|^2 = \left\|J_{r_n}(a_n x_n + (1-a_n)T_n x_n - r_n h x_n) - p\right\|^2$$
$$= \|J_{r_n}(a_n x_n + (1-a_n)T_n x_n - r_n h x_n) - J_{r_n}(a_n p + (1-a_n)p + r_n hp)\|^2$$

$$\|y_n - p\|^2 = \|a_n x_n + (1-a_n)T_n x_n - r_n h x_n - a_n p - (1-a_n)p + r_n hp\|^2$$
$$\le (1-a_n)\|T_n x_n - p\|^2 + a_n \left\|(x_n - p) - \frac{r_n}{a_n}(hx_n - hp)\right\|^2$$
$$\le (1-a_n)\|T_n x_n - p\|^2 + a_n\|x_n - p\|^2 - 2r_n\langle x_n - p, hx_n - hp\rangle + \frac{r_n{}^2}{a_n}\|hx_n - hp\|^2$$

Now, by definition of $szl-$ widering then we get, for each $s_n, l_n \in (0,1)$ then there exist $z_n > 0$ (shortly we write them s, z and l respectively) such that

$$\|y_n - p\|^2 \le (1-a_n)\{(1-s)\|x_n - p\|^2 + l\|p - T_n p\|.\|x_n - T_n x_n - (p - T_n p)\| + z|\langle x_n - T_n x_n, p - T_n p\rangle|\} + a_n\|x_n - p\|^2 - 2r_n\alpha\|h\,x_n - hp\|^2 + \frac{r_n{}^2}{a_n}\|h\,x_n - hp\|^2$$

$$\|y_n - p\|^2 \le (1-a_n)(1-s)\|x_n - p\|^2 + a_n\|x_n - p\|^2 - \frac{(2a_n\alpha - r_n)r_n}{a_n}\|h\,x_n - hp\|^2$$
$$\le (1-a_n)\|x_n - p\|^2 + a_n\|x_n - p\|^2 - \frac{(2a_n\alpha - r_n)r_n}{a_n}\|h\,x_n - hp\|^2$$
$$\le \|x_n - p\|^2$$

$$\|x_{n+1} - p\|^2 \le a_n\|x_n - p\|^2 + b_n\|T_n x_n - p\|^2 + c_n\|f_n(x_n) - p\|^2 + d_n\|T_n y_n - p\|^2$$

$$\|x_{n+1} - p\|^2 \le a_n\|x_n - p\|^2 + b_n(1-s)\|x_n - p\|^2 + b_n l\|p - T_n p\|.\|x_n - T_n x_n - (p - T_n p)\| + b_n z|\langle x_n - T_n x_n, p - T_n p\rangle| + c_n\|x_n - p\|^2 + d_n(1-s)\|y_n - p\|^2 + d_n l\|p - T_n p\|.\|y_n - T_n y_n - (p - T_n p)\| + d_n z|\langle y_n - T_n y_n, p - T_n p\rangle|$$

$$\|x_{n+1} - p\|^2 \le a_n\|x_n - p\|^2 + b_n(1-s_n)\|x_n - p\|^2 + \mathcal{K}_n\|x_n - p\|^2 + d_n(1-s)\|y_n - p\|^2$$
$$\le (1-d_n)\|x_n - p\|^2 + d_n\|x_n - p\|^2$$
$$\le \|x_n - p\|^2$$

By lemma (1.2),we get

$\lim_{n\to\infty}\|x_n - p\|$exists (14)

So, $\langle x_n\rangle$ is bounded sequence, and hence$\langle J_{r_n}\rangle, \langle f_n\rangle$and$\langle w_n\rangle$ also bounded.

Since $\|x_{n+1} - p\| \le \|x_n - p\| + \|T_n x_n - x_n\|$ therefore, by (14)

$$-\|T_n x_n - x_n\| \le \|x_n - p\| - \|x_{n+1} - p\| \to 0 \;as\; n \to \infty$$

$\|T_n x_n - x_n\| \to 0 \;as\; n \to \infty$ (15)

Since $\langle x_n\rangle$ is bounded sequence. Then there exists subsequence $\langle x_{nk}\rangle$ of $\langle x_n\rangle$ such that $x_{nk} \rightharpoonup \tilde{x}$. By (15)we get $\tilde{x}$ is an asymptotic common fixed poin of T_n. ■

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