New Implementation of Paired Triple Connected Domination Number of a Graph

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Abstract: A set \( S \subseteq V \) is a paired triple connected dominating set if \( S \) is a triple connected dominating set of \( G \) and the induced subgraph \(<S>\) has a perfect matching. The paired triple connected domination number \( \gamma_{ptc}(G) \) is the minimum cardinality taken over all paired triple connected dominating sets in \( G \). The minimum number of colours required to colour all the vertices so that adjacent vertices do not receive the same colour and is denoted by \( \chi(G) \). In [5], Mahadevan G et. al., characterized the classes of the graphs whose sum of paired triple connected domination number and chromatic number equals \( 2n – 1 \). In this paper we characterize the classes of all graphs whose sum of paired triple connected domination number and chromatic number equals \( 2n – 2, 2n – 3, 2n – 4 \), for any \( n \geq 5 \).

Keywords: Paired triple connected domination number, Chromatic number

AMS (2010): 05C69

1. Introduction

Throughout this paper, by a graph we mean a finite, simple, connected and undirected graph \( G(V, E) \). For notations and terminology, we follow [2]. The number of vertices in \( G \) is denoted by \( n \). Degree of a vertex \( v \) is denoted by \( deg(v) \). We denote a cycle on \( n \) vertices by \( C_n \), a path of \( n \) vertices by \( P_n \), complete graph on \( n \) vertices by \( K_n \). The friendship graph, denoted by \( F_n \), can be constructed by identifying \( n \) copies of the cycle \( C_3 \) at a common vertex. A spider is a tree which has atmost one vertex of degree \( \geq 3 \). A wounded spider \( S^n(K_{1,n-1}) \) is the graph formed by subdividing (exactly once) atmost \( n-1 \) of the edges of a star \( K_{1,n-1} \). If \( S \) is a subset of \( V \), then \(<S>\) denotes the vertex induced subgraph of \( G \) induced by \( S \). A subset \( S \) of \( V \) is called a dominating set of \( G \) if every vertex in \( V-S \) is adjacent to at least one vertex in \( S \). The domination number \( \gamma(G) \) of \( G \) is the minimum cardinality of all such dominating sets in \( G \). One can get a comprehensive survey of results on various types of domination number of a graph in [16]. The chromatic number \( \chi(G) \) is defined as the minimum number of colors required to color all the vertices such that adjacent vertices do not receive the same color. Many authors have introduced different types of domination parameters by imposing conditions on the dominating set [14, 15]. Recently, the concept of triple connected graphs has been introduced by Paulraj Joseph J. et. al.[13] by considering the existence of a path containing any three vertices of \( G \). They have studied the properties of triple connected graphs and established many results on them. A graph \( G \) is said to be triple connected if any three vertices lie on a path in \( G \). All paths, cycles, complete graphs and wheels are some standard examples of triple connected graphs. In [4] Mahadevan G. et. al., introduced triple connected domination number of a graph and found many results on them. A subset \( S \) of \( V \) of a nontrivial connected graph \( G \) is said to be triple connected dominating set, if \( S \) is a dominating set and the induced subgraph \(<S>\) is triple connected. The minimum cardinality taken over all triple connected dominating sets is called the triple connected domination number of \( G \) and is denoted by \( \gamma_{ptc}(G) \). In [5] Mahadevan G. et. al., introduced paired triple connected domination number of a graph and found many results on them. A subset \( S \) of \( V \) of a nontrivial connected graph \( G \) is said to be paired triple connected dominating set, if \( S \) is a triple connected dominating set and the induced subgraph \(<S>\) has a perfect matching. The minimum cardinality taken over all paired triple connected dominating sets is called the paired triple connected domination number of \( G \) and is denoted by \( \gamma_{ptc}(G) \).

Several authors have studied the problem of obtaining an upper bound for the sum of domination parameter and a graph theoretic parameter and characterized the corresponding extremal graphs. In [12], Paulraj Joseph J and Arumugam S proved that \( \gamma + \kappa \leq p \), where \( \kappa \) denotes the vertex connectivity of the graph. They also proved that \( \gamma + 1 \leq p + 1 \) and characterized the corresponding extremal graphs. They also proved similar results for \( \gamma \) and \( \chi \). In [10], Mahadevan G Selvam A, IravithulBasira A characterized the extremal of graphs for which the sum of the complementary connected domination number and chromaticnumber. In [5], Mahadevan G proved that \( \gamma_{ptc} + \chi \leq 2n-1 \), and characterized the corresponding extremal graph.

Motivated by the above results, in this paper, we characterize all graphs for which the sum of paired triple connected domination number and chromatic number equals \( 2n–2, 2n – 3, 2n – 4 \) for any \( n \geq 5 \).

2. Previous Results

Theorem 2.1 [5] For any connected graph \( G \) with \( n \geq 5 \), we have \( 4 \leq \gamma_{ptc}(G) \leq n – 1 \).

Notation 2.2 Let \( G \) be a connected graph with \( m \) vertices \( v_1, v_2, \ldots, v_m \). The graph obtained from \( G \) by attaching \( n_1 \) times a pendant vertex of \( P_1 \) on the vertex \( v_1 \), \( n_2 \) times a pendant vertex of \( P_2 \) on the vertex \( v_2 \) and so on,is denoted by
Theorem 3.1 For any connected graph $G$ with $n \geq 5$ vertices, $\gamma_{pcs}(G) + \chi(G) = 2n - 2$ if and only if $G$ is isomorphic to $K_d(P_2), K_{6}$ or any one of the graphs shown in Figure 3.1.

**Proof:** Let $G$ be a connected graph with $n \geq 5$ vertices. Suppose $G$ is isomorphic to $K_d(P_2), K_{6}$ or any of the graphs given in Figure 2.1, then clearly $\gamma_{pcs}(G) + \chi(G) = 2n - 2$.

Conversely, Let $\gamma_{pcs}(G) + \chi(G) = 2n - 2$. This is possible if $\gamma_{pcs}(G) = n - 1$ and $\chi(G) = n - 1$ or if $\gamma_{pcs}(G) = n - 2$ and $\chi(G) = n$.

Case (i) $\gamma_{pcs}(G) = n - 1$ and $\chi(G) = n - 1$.

Since $\chi(G) = n - 1$, $G$ contains a clique $K_{n-1}$ on $n - 1$ vertices. Let $x$ be the vertex other than the $n - 1$ vertices in $K_{n-1}$. Since $G$ is connected $x$ is adjacent to a vertex $v$ in $K_{n-1}$.

Now $S = \{x, v, v_1, v_2\}$ (for $i \neq j \neq k$) is a paired triple connected dominating set of $G$. Since $\chi_{ctc}(G) = n - 1$, so that $p = 3$. Hence $K_{n-1} \cong K_3 \leq <v_1, v_2, v_3, v_4>$. Let $x$ be adjacent to $v_1$ in $K_3$. If $d(x) = 1$, then $G \cong G_6$. If $x$ is adjacent to $v_3, v_2, v_1$ in $K_3$. If $d(x) = 3$, then $G \cong G_7$. In all the other cases, no graph exists.

Case (ii) $\gamma_{pcs}(G) = n - 2$ and $\chi(G) = n$.

But $\chi(G) = n$, we have $G$ is isomorphic to $K_n$. For $K_n$, $\gamma_{pcs}(K_n) = 4$ so that $n = 6$. Hence $G \cong K_6$.

**Theorem 3.2** For any connected graph $G$ with $n \geq 5$ vertices, $\gamma_{pcs}(G) + \chi(G) = 2n - 3$ if and only if $G$ is isomorphic to $W_5, F_2, K_3, K_d(P_3), K_d(2P_2), K_d(P_3), K_d(P_2, P_2, 0)$ or any one of the graphs shown in Figure 3.2.

**Proof:** Let $G$ be a connected graph with $n \geq 5$ vertices. Suppose $G$ is isomorphic to $W_5, F_2, K_3, K_d(P_3), K_d(2P_2), K_d(P_3), K_d(P_2, P_2, 0)$ or any one of the graphs given in Figure 2.2, then clearly $\gamma_{pcs}(G) + \chi(G) = 2n - 3$.

Conversely, Let $\gamma_{pcs}(G) + \chi(G) = 2n - 3$. This is possible if $\gamma_{pcs}(G) = n - 1$ and $\chi(G) = n - 2$ or if $\gamma_{pcs}(G) = n - 2$ and $\chi(G) = n - 1$ or if $\gamma_{pcs}(G) = n - 3$ and $\chi(G) = n$.

Case (i) $\gamma_{pcs}(G) = n - 1$ and $\chi(G) = n - 2$. 
Since $\chi(G) = n - 2$, $G$ contains a clique $K_{n-2}$ on $n - 2$ vertices. Let $T = \{x, y\}$ be the vertices other than the $n - 2$ vertices in $K_{n-2}$. Then the induced subgraph of $T$ may be $K_2$ or $K_{1,1}$.

Subcase (i) $<T> = K_2$

Since $G$ is connected, there exists a vertex $x$ in $T$ that is adjacent to a vertex $v_i$ in $K_{n-2}$. Now $S = \{x, v_i, v_j\}$ (for $i \neq j$) forms a paired triple connected dominating set of $G$. Since $\gamma_{ptc}(G) = n - 1$, so that $n = 5$. Hence $K_{n-2} = K_{1,1}$.

Let $x$ be adjacent to $u$ in $K_{n-2}$. If $d(x) = 2$, $d(y) = 1$, then $G \cong K_2(P_2)$. Let $x$ be adjacent to $u$ and $v$ in $K_{n-2}$. If $d(x) = 3$, $d(y) = 1$, then $G \cong G_2$. Let $x$ be adjacent to $u$ and $y$ be adjacent to $w$ in $K_{n-2}$. If $d(x) = d(y) = 2$, then $G \cong G_2$. Let $x$ be adjacent to $u$ in $K_{n-2}$ and $y$ be adjacent to $w$ in $K_{n-2}$. If $d(x) = d(y) = 2$, then $G \cong G_2$. Let $x$ be adjacent to $u$ in $K_{n-2}$ and $y$ be adjacent to $w$ in $K_{n-2}$. If $d(x) = d(y) = 2$, then $G \cong G_2$. Let $x$ be adjacent to $u$ and adjacent to $v$ and $w$ in $K_{n-2}$. If $d(x) = d(y) = 2$, then $G \cong G_2$. In all the other cases, no new graph exists.

Subcase (ii) $<T> = K_{1,1}$

Since $G$ is connected, any vertex in $T$ is adjacent to a vertex $v_i$ in $K_{n-2}$. Then $S = \{x, v_i, v_j\}$ (for $i \neq j$) forms a paired triple connected dominating set of $G$. Since $\gamma_{ptc}(G) = n - 1$, so that $n = 5$. Hence $K_{n-2} = K_1$.

Let $x$ and $y$ be adjacent to $u$ in $K_{n-2}$. If $d(x) = d(y) = 2$, then $G \cong K_2(P_2)$. Let $x$ be adjacent to $u$ and $v$ in $K_{n-2}$ and $y$ be adjacent to $w$ in $K_{n-2}$. If $d(x) = d(y) = 2$, then $G \cong G_2$. Let $x$ be adjacent to $u$ and $v$ in $K_{n-2}$ and $y$ be adjacent to $w$ in $K_{n-2}$. If $d(x) = d(y) = 2$, then $G \cong G_2$. Let $x$ be adjacent to $u$ and $v$ in $K_{n-2}$ and $y$ be adjacent to $w$ in $K_{n-2}$. If $d(x) = d(y) = 2$, then $G \cong G_2$. In all the other cases, no new graph exists.

Since $G$ is connected, $x$ in $T$ is adjacent to $v_i$ in $K_{n-2}$ and $y$ in $T$ is adjacent to $v_j$ (for $i \neq j$). Then $S = \{x, v_i, v_j\}$ (for $i \neq j$) forms a paired triple connected dominating set of $G$. Since $\gamma_{ptc}(G) = n - 1$, so that $n = 5$. Hence $K_{n-2} = K_1$.

Let $x$ be adjacent to $u$ in $K_{n-2}$ and $y$ be adjacent to $v$ in $K_{n-2}$. If $d(x) = d(y) = 1$, then $G \cong K_2(P_2, P_2, 0, 0)$. In all the other cases, no new graph exists.

Case (iii) $\gamma_{ptc}(G) = n - 3$ and $\chi(G) = n$.

Since $\chi(G) = n$, we have $G$ is isomorphic to $K_n$. But for $K_n$, $\gamma_{ptc}(K_n) = 4$ so that $n = 7$. Hence $G \cong K_7$.

**Theorem 3.3** For any connected graph $G$ with $n \geq 5$ vertices, $\gamma_{ptc}(G)+\chi(G) = 2n-4$. This is possible if $G \cong S^*(K_{1,3}), P_5, C_6(P_2), K_9(P_3), K_9(2P_2), K_9(P_3, P_3, 0, 0), K_9(P_2, P_2, 2, 0), K_9(2, 2), K_9(3), K_9(4), K_9(5), K_9$ or any one of the following graphs given in Figure 3.3.

**Proof**: Let $G$ be a connected graph with $n \geq 5$ vertices. Suppose $G$ is isomorphic to $P_5, S^*(K_{1,3}), C_6(P_2), K_9(P_3), K_9(2P_2), K_9(P_3, P_3, 0, 0), K_9(P_2, P_2, 2, 0), K_9(2, 2), K_9(3), K_9(4), K_9(5), K_9$ or any one of the graphs given in Figure 2.3, then clearly $\gamma_{ptc}(G)+\chi(G) = 2n-4$.

Conversely, let $\gamma_{ptc}(G)+\chi(G) = 2n-4$. This is possible if $\gamma_{ptc}(G) = n-1$ and $\chi(G) = n-3$ or if $\gamma_{ptc}(G) = n-2$ and $\chi(G) = n-2$ or if $\gamma_{ptc}(G) = n-3$ and $\chi(G) = n-1$ or if $\gamma_{ptc}(G) = n-4$ and $\chi(G) = n$.

**Case (i)** $\gamma_{ptc}(G) = n-1$ and $\chi(G) = n-3$.

Let $x$ be adjacent to $v_i, v_2$ and $v_j$ in $K_5$. If $d(x) = 4$, then $G \cong G_6$. In all the other cases, no graph exists.
Since $\gamma(G) = n-3$, $G$ contains a clique $K$ on $n-3$ vertices or does not contain a clique $K$ on $n-3$ vertices.

Let $G$ contains a clique $K$ on $n - 3$ vertices.

Let $S = V(G) - V(K) = \{v_1, v_2, v_3\}$. Then the induced subgraph $<S>$ has the following possible cases. $<S> = K_1, K_2, P_3, K_3 \cup K_1$.

Subcase i. $<S> = K_1$.
Let $v_1, v_3 \in V(K)$ be the vertices of $K_3$. Since $G$ is connected, there exists a vertex $u_1 \in K_3$ which is adjacent to any one of $v_1, v_2, v_3$. Let $u_j$ be adjacent to $v_i$ then $\{v_j, u_1, v_3\} \cup \{v_2, u_1, v_3\}$ is a $\gamma_{ptc}$ set of $G$, so that $\gamma_{ptc} = 4$ and $n = 5$, which is a contradiction. Hence no graph exists.

Subcase ii. $<S> = K_2$.
Let $v_1, v_2, v_3$ be the vertices of $K_3$. Since $G$ is connected, there exists a vertex $u_1 \in K_3$ which is adjacent to any one of $v_1, v_2, v_3$. Let $u_j$ be adjacent to $v_i$ then $\{v_j, u_1, u_2\}$ is a $\gamma_{ptc}$ set of $G$, so that $\gamma_{ptc} = 4$ and $n = 5$, which is a contradiction. Hence no graph exists.

Subcase iii. $<S> = K_3$.
Let $v_1, v_2, v_3$ be the vertices of $K_3$. Since $G$ is connected, there exists a vertex $u_1 \in K_3$ which is adjacent to any one of $v_1, v_2, v_3$. Let $u_j$ be adjacent to $v_i$ then $\{v_j, u_1, u_2\}$ is a $\gamma_{ptc}$ set of $G$, so that $\gamma_{ptc} = 4$ and $n = 5$, which is a contradiction. Hence no graph exists.

Case (ii) $\gamma_{ptc} = n - 2$ and $\chi = n - 2$.

Since $\chi = n - 2, G$ contains a clique $K$ on $n - 2$ vertices or does not contain a clique $K$ on $n - 2$ vertices.

Let $S = V(G) - V(K) = \{v_1, v_2\}$. Then $<S> = K_2, \overline{K}_2$.

Subcase i. $<S> = K_2$.
Let $v_1, v_2$ be the vertices of $K_2$. Since $G$ is connected, there exists a vertex $u_1 \in K_{n-3}$ which is adjacent to $v_1$ and $u_j$ for $i \neq j$ then $\{v_1, u_1, u_2\}$ is a $\gamma_{ptc}$ set of $G$, so that $\gamma_{ptc} = 4$ and $n = 6$ and hence $K = K_4 = \{u_1, u_2, v_1, v_2\}$.

Let $u_1$ be adjacent to $v_1$ and $u_2$ be adjacent to $v_1$ then $\{v_1, u_1, u_2\}$ is a $\gamma_{ptc}$ set of $G$, so that $\gamma_{ptc} = 4$ and $n = 5$ and hence $K = K_2 = \{u_1, v_2, v_3\}$.

Let $u_1$ be adjacent to $v_1$ and $u_2$ be adjacent to $v_1$ then $\{v_1, u_1, u_2\}$ is a $\gamma_{ptc}$ set of $G$, so that $\gamma_{ptc} = 4$ and $n = 5$ and hence $K = K_2 = \{u_1, v_2, v_3\}$.

Since $G$ is connected, there exists a vertex $u_1 \in K_{n-3}$ which is adjacent to $v_1$ and $v_3$. In this case $\{v_2, v_1, u_1, u_2\}$ is a $\gamma_{ptc}$ set of $G$, so that $\gamma_{ptc} = 4$ and $n = 5$ and hence $K = K_2 = \{u_1, v_2, v_3\}$.

Since $G$ is connected, there exists a vertex $u_1 \in K_{n-3}$ which is adjacent to $v_1$ and $v_3$. In this case $\{v_2, v_1, u_1, u_2\}$ is a $\gamma_{ptc}$ set of $G$, so that $\gamma_{ptc} = 4$ and $n = 5$ and hence $K = K_2 = \{u_1, v_2, v_3\}$.

Since $G$ is connected, there exists a vertex $u_1 \in K_{n-3}$ which is adjacent to $v_1$ and $v_3$. In this case $\{v_2, v_1, u_1, u_2\}$ is a $\gamma_{ptc}$ set of $G$, so that $\gamma_{ptc} = 4$ and $n = 5$ and hence $K = K_2 = \{u_1, v_2, v_3\}$.
Let $u_1$ be adjacent to $v_1$ and $u_2$ be adjacent to $v_2$. If $\deg(v_1) = 1$, then $G \cong K_{n}(P_3, P_3, 0, 0)$. Let $u_1$ be adjacent to $v_1$ and $u_2$ be adjacent to $v_2$, and $u_4$ be adjacent to $v_4$. If $\deg(v_1) = 2$, then $G \cong G_{10}$. Let $u_1$ be adjacent to $v_1$ and $u_2$ be adjacent to $v_2$. If $\deg(v_1) = 3$, then $G \cong G_{11}$. Let $u_1$ be adjacent to $v_1$ and $u_2$ be adjacent to $v_2$. If $\deg(v_1) = 2$, then $G \cong G_{12}$. Let $u_1$ be adjacent to $v_1$ and $u_2$ be adjacent to $v_2$. If $\deg(v_1) = 3$, then $G \cong G_{13}$. Let $u_1$ be adjacent to $v_1$ and $u_2$ be adjacent to $v_2$. If $\deg(v_1) = 4$, then $G \cong G_{14}$. Let $u_1$ be adjacent to $v_1$ and $u_2$ be adjacent to $v_2$. If $\deg(v_1) = 5$, then $G \cong G_{15}$.

Since $\chi = n$, we have $G$ is isomorphic to $K_n$. Let $v$ be adjacent to $v_1$ and $v_2$ be adjacent to $v_1$. If $\deg(v) = 2$, then $G \cong G_{16}$. Let $u_1$ be adjacent to $v_1$ and $u_2$ be adjacent to $v_2$. If $\deg(v) = 3$, then $G \cong G_{17}$. Let $u_1$ be adjacent to $v_1$ and $u_2$ be adjacent to $v_2$. If $\deg(v) = 4$, then $G \cong G_{18}$. Let $u_1$ be adjacent to $v_1$ and $u_2$ be adjacent to $v_2$. If $\deg(v) = 5$, then $G \cong G_{19}$.

Case iii. $\gamma_{pct} = n - 3$ and $\chi = n - 1$.

Since $\chi = n - 1$, $G$ contains a clique $K$ on $n - 1$ vertices.

Let $v_1$ be the vertex not on $K_{n-1}$. Since $G$ is connected, there exists a vertex $v$ is adjacent to each vertex of $K_{n-1}$. Let $u_i$ and $u_j$ be adjacent to $v$. In this case, $\{v_1, u_1, u_2, u_3\}$ is a $\gamma_{pct}$ set of $G$, so that $\gamma_{pct} = 4$ and $n = 7$ and hence $K = K_6 = <u_1, u_2, u_3, u_4, u_5, u_6>$.

Let $u_1$ be adjacent to $v_1$. If $\deg(v_1) = 3$, then $G \cong K_6(P_3)$.

Case iv. $\gamma_{pct} = n - 4$ and $\chi = n$.

Since $\chi = n$, we have $G$ is isomorphic to $K_n$. But for $K_n$, $\gamma_{pct} = 4$, so that $n = 8$. Hence $G \cong K_8$.

4. Conclusion and Future Scope

The authors obtained a large classes of graphs whose sum of paired triple connected number and chromatic number equals to $2n - 5$ for $n \geq 5$, which will be reported later.

References