Spacetime Quantum Mechanics and Yukawa Potential

Vivek Chhetri

Department of Physics, Sikkim Government College

Abstract: ***The major understanding of Strong Force came in from the revolutionary work of Hideki Yukawa of Japan [1]. Yukawa postulated that strong forces are mediated by exchange of heavy particles called mesons. He correctly guessed the short range Yukawa potential and calculated the mass of the mesons. In this paper I will use Feynman's path integral approach to show that exchange of spin zero virtual particle produces an attractive force. Finally I will derive the Yukawa potential.***

Keywords: Path integral, Propagator, Spacetime, virtual particle

1. Introduction

In Classical Mechanics a particle has a definite path, but Richard Feynman found a powerful way of thinking about Quantum mechanics in which he challenged this basic assumption of Classical mechanics. In Feynman's path integral of the world also called the "sum over histories" approach to Quantum mechanics a particle travels along every possible path through spacetime with each trajectory Feynman associated two numbers the amplitude and phase.Theprobability of a particle going from A to B is found by adding up the waves associated with every possible path from A to B.This was the picture in the Quantum world for large objects the phase of all the paths cancel except one so we get a single classical path. When Feynman tried to put this picture in a Mathematical framework he was guided by a mysterious remark by Dirac [2] [3]

$$e^{i\int_{t1}^{t2} L(x,x')} \text{ corresponds to } <x_{2,}t_2|x_{1,}t_1>$$

Trying to make sense of Dirac's remark Feynman developed his path integral or "sum over histories "approach to Quantum mechanics. Feynman's model had the added advantage because spacetime formulation is easy to visualize and this Lagrangian approach is relativistically invariant. In my paper I will use the path integral approach for a scalar field and derive all the results guessed by Yukawa.

2. Path Integral for a Scalar Field

The path integral for a scalar field in d=(D+1) dimensional spacetime can be written as

$$Z=\int D\emptyset e^{i}\int d^{d}x\{1/2(\partial\emptyset)^2-v(\emptyset)\} \quad (1)$$

Let us work in the four dimensional spacetime after adding the interaction term our path integral becomes

$$Z=\int D\emptyset e^{i}\int dx^4\{1/2(\partial\emptyset)^2-v\emptyset)+J(x)\emptyset(x)\} \quad (2)$$

This equation (2) can be written as

$$Z=\int D\emptyset e^{i}\int dx^4\{ -1/2\emptyset(\partial^2+m^2)\emptyset +J\emptyset\} \quad (3)$$

Using the standard equation

$$\int dx_1 dx_2 .. dx_N\, e^{\frac{i}{2}x.A.x+iJ.x}=E\ e^{-\frac{i}{2}J.\bar{A}.J} \quad (4)$$

Where E is a constant and $\bar{A}=A^{-1}$

Using $A_{ij}A_{jk}^{-1}=\delta_{ij}$ it is easy to see from equation (3) and (4)that A corresponds to

$$-(\partial^2+m^2).$$

Hence we can write

$$-(\partial^2+m^2)\, D(x-y)=\delta^4(x-y) \quad (5)$$

Hence using equation (3) and (4) we can write the path integral for a scalar field Z(J) as

$$Z(J)=Ce^{i/2}\iint dx^4 dy^4 J(x)D(x-y)J(y) \quad (6)$$

$$Z(J)=Ce^{iW(J)} \quad (7)$$

$$W(J)=1/2\iint dx^4 dy^4 J(x)D(x-y)J(y) \quad (8)$$

D(x-y) is called the free propagator and it plays a very important role in Quantum field theory.

3. Free Propagator

The function D(x-y) is known as propagator it is closely related to the Green's function. From equation (7) it is easy to see that

$$D(x-y)=\int d^4x/(2\pi)^4 e^{ik(x-y)}/k^2-m^2+i\varepsilon \quad (9)$$

Using the Fourier transform

$$J(k)=\int d^4x e^{-ikx} J(x) \quad (10)$$

$$J^*(k)=\int d^4x e^{ikx} J(x)=J(-k) \quad (11)$$

If J(x) is real, we can write W(J) as

$$W(J) = -1/2 \int d^4x/(2\pi)^4 J^*(k)\ 1/k^2-m^2+i\varepsilon\ \ J(k) \quad (12)$$

The propagator plays a very important role in field theory it is amplitude a particle starts at some point in the past and ends up at another point in the future. It is an alternative to wavefunctions.The propagators have a neat mathematical property they are Green's functions.

If $J(x)=J_1(x)+J_2(x)$ where $J_1(x)$ and $J_2(x)$ are concentrated in two local region 1 and 2 of spacetime. W(J) will contain four terms of the form $J^*_1J_1$, $J^*_2J_2$, J^*_1J and $J^*_2J_1$. If we neglect the self interaction we have two terms.

$$W(J) = -1/2 \int d^4x/(2\pi)^4 J^*_2(k)\ 1/k^2-m^2+i\varepsilon\ \ J_1(k) \quad (13)$$
$$W(J) = -1/2 \int d^4x/(2\pi)^4 J^*_1(k)\ 1/k^2-m^2+i\varepsilon\ \ J_2(k) \quad (14)$$

We interpret equation (13) as follows. In region 1 of spacetime there exist a source that sends out disturbance in the field which is later absorbed by a sink in region 2 of spacetime. Experimentalists call this disturbance in the field a particle of mass m.It is easy to see that $i/k^2-m^2+i\varepsilon$ is playing the role of the propagator for a scalar field.

4. Virtual Particle and Force

Let $J(x)=J_1(x)+J_2(x)$ where $J_a(x)=\delta^3(\bar{x} - \bar{x}_a)$. In other words J(x) is a sum of sources that are time independent infinitely sharp spikes located at x_1 and x_2 in space. As before W (J) contain four terms we neglect self interaction now W (J) can be written as

$$-1/2\int d^0x d^0y \iint d^0k/2\pi e^{ik0(x0-y0)} \int d^3k/(2\pi)^3 e^{ik(x1-x2)}/k^2-m^2+i\varepsilon \quad (15)$$

If we can choose K^0 equal to zero these particles are not on the mass shell these particles are virtual particles our equation (15) becomes

$$W(J)=\int d^0x \int d^3k/(2\pi)^3 e^{ik.(x1-x2)}/\bar{k}^2+m^2 \quad (16)$$

Since $Z(J)=Ce^{iW(J)}$ for virtual particles this becomes

$$Z=<0|e^{-iHT}|0>=e^{-iET} \quad (17)$$

We can write

$$E = -\int d^3k/(2\pi)^3 e^{ik.(x1-x2)}/\bar{k}^2+m^2 \quad (18)$$

The integral works out to be

$$E= -1/4\pi r e^{-mr} \quad (19)$$

The Energy is negative. The exchange of spin zero particles has lowered the energy this means that it produces an attractive force whose potential varies as$\sim 1/re^{-mr}$ exactly as Yukawa has guessed.

5. Conclusion

The exchange of a particle leading to a force was a profound new conceptual advance in modern physics. A practical version of path integrals was the Feynman diagrams for elementary particles. Path integrals are also applied in statistical mechanics, superconductivity with great success. The path integral approach to Quantum mechanics is so rich that leading Physicists of today still suspect it has hidden insights and perspectives which is yet to be discovered. It is still used by Physicists today to understand the mysteries of Quantum Gravity [9] [10].

It was the genius and magical imagination of Yukawa to guess the mechanism of this new interaction and calculate the mass of mesons by guessing the short range potential in his own words Yukawa says…

"I decided to carry the theory (of Heisenberg) one step further…it seemed that it was a third force unrelated to gravity and electromagnetism. If one visualizes the force field as a game of catch between the protons and neutrons, the crux of the problem was the nature of the ball or the particle".

We have just proved Yukawa to be right by a new modern technique of path integrals.

References

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