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On the Existence of Concurrent Vector Fields in a Finsler Space-II

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1. Introduction

Let M^n be an n-dimensional differentiable manifold and F^n be an n-dimensional Finsler space equipped with metric function *L(x, y)*, metric tensor $g_{ij}(x,y)$, angular metric tensor h_{ij} and torsion tensor C_{ijk}. The h-and v-covariant derivatives of a vector field X_i are defined as Rund [9] :

a) $X_i\big|_j = \partial_j X_i - N_j^r \Delta_r X_i - X_r F_{ij}^r,$

b) $X_i\big|_j = \Delta_j X_i - X_r C_{ij}^r,$

where $N_j^r = F_{oj}^r$, ∂_j and Δ_j respectively denote partial differentiation with respect to x^j and y^j, such that an index *o* means contraction by *y*.

The Concurrent vector fields in a Finsler space were first of all defined and studied by Tachibana [11] followed by Matsumoto [4] and others in the following form :

Definition 1.1: A vector field $X^i(x)$ is said to be concurrent in a Finsler space F^n if it satisfies :

(i) $X^i C_{ijk} = 0,$

(ii) $X^i\big|_j = -\delta_j^i.$

In the paper [7] provide that Definition 1.1 of concurrent vector fields in its present form is untenable and we gave an alternative definition as follows:

Definition 1.2: A vector field $X^i(x)$ is said to be concurrent in a Finsler space F^n if it satisfies:

(i) $X^i A_{ijk} = \alpha h_{jk},$

(ii) $X^i\big|_j = -\delta_j^i.$

where □ is an arbitrary non-zero scalar function of *x* and *y*, $A_{ijk} = LC_{ijk}$ and $A_j = LC_j$.

Furthermore, we have also defined and studied a special type of vector field [8] called P-concurrent vector field as follows :-

Definition 1.3: A vector field $X^i(x)$ in a Finsler space F^n with a non-vanishing tensor P_{ijk} will be defined as a P-concurrent vector field if it satisfies

(i) $X^i P_{ijk} = 0,$

(ii) $X^i\big|_j = -\delta_j^i$ (1.3)

In (1978) Singh [10] studied concurrent vector fields using Berwald's covariant derivative instead of Carton and proved that "A vector field X^i on F^n is concurrent in the Berwald's sense if and only if it is concurrent in the Cartan's sense". It is interesting to note that apparently there seems to be no difficulty in taking Berwald's covariant derivative in Definition 1.1, but after some analysis it is found that this is not true. The purpose of the present paper is to study the existence of concurrent vector fields using Berwald's covariant derivative and to give an alternative definition of vector fields which we shall call B-concurrent. Some properties of concurrent, P-concurrent and B-concurrent vector fields in a Finsler space have also been studied in this paper.

2. Concurrent vector fields in the sense of Berwald

For a Berwald connection parameter [9] given by $G_{jk}^h = F_{jk}^h + C_{jk|0}^h$, the covariant derivative of a tensor field T_b^a is given by

(2.1) $T_{b(j)}^a = T_{b|j}^a + T_b^r C_{rj|0}^a - T_r^a C_{bj|0}^r$

Definition 2.1: A vector field $X^i(x)$ is said to be concurrent [10] in a Finsler space F^n if it satisfies :

(2.2) (i) $X^i C_{ijk} = 0,$ (ii) $X_{(j)}^i = -\delta_j^i$

Now we shall study the existence of such concurrent vector field in a Finsler space.

Two-dimensional Finsler space F^2: From equation (2.2)(i) we can obtain $X^j C_j = 0$. Differentiating $C_j = Cm_j$, covariantly, using Berwald's covariant derivative (2.1) and Definition 2.1, we get

(2.3) $X^j C_{j(k)} = -C_r X^j C_{jk|0}^r.$

Furthermore, since it is easy to get

(2.4) $X^h C_{j(k)} = C_k$,

therefore comparing (2.3) and (2.4), we get

(2.5) $C_k + X^j C_r C^r_{jk|0} = 0$.

Differentiating $X^j C^r_{jk} = 0$, with the help of equation (2.1), on simplification we get

(2.6) $X^j C^r_{jk|0} = 0,$

Which by virtue of (2.5) leads to $C_k = 0$. Hence we have :

Theorem 2.1: In a two-dimensional non-Riemannian Finsler space F^2, a concurrent vector field in the sense of Berwald given by Definition 2.1 does not exist.

Three-dimensional Finsler space F^3: In F^3, any vector X^j can be expressed as

(2.7) $X^j = \alpha l^j + \beta m^j + \lambda n^j$

which on multiplication by m_j gives by virtue of (2.2)(i) □ =0 Substituting □ =0 in (2.7) and differentiating it with respect to x^k, using (2.1), we can obtain

$$\text{(2.8)}\ \alpha_{(k)} = -l_k,\quad \gamma h_k = m_k + \gamma C^j_{rk|0} m_j n^r,\quad \gamma_{(k)} = -(n_k + \gamma C^j_{rk|0} n_j n^r)$$

which gives $\gamma h_0 = 0$ ie., either $\gamma = 0$ or $h_0 = 0$.

Here $\gamma = 0$, means $X^j = \alpha l^j$ and $\lambda_{(k)} = 0$, i.e., $n_k = 0$ which is absurd. In case $h_0 = 0$, we obtain from (2.7) on differentiation and multiplication by X^k, $X^k (h_k - C^j_{rk|0} m_j n^r) = 0$, which does not seem possible unless X^k is in the direction of l^k. Hence we have :

Theorem 2.2: In a three-dimensional non-Riemannian Finsler space F^3, a concurrent vector field in the sense of Berwald given by Definition 2.1 does not exist.

C-reducible Finsler space: In a C-reducible space F^n, C_{ijk} is expressed as Matsumoto [3] :

(2.9) $C_{ijk} = (C_k h_{ij} + C_i h_{jk} + C_j h_{ki})/(n+1)$

If X^j is a concurrent vector field equation (2.9), by virtue of (2.2) leads to $X_j = X^r l_r l_j$, which on application of Berwald's covariant differentiation gives $h_{jk} = 0$. Hence we have :

Theorem 2.3: A C-reducible Finsler space F^n, with a concurrent vector field in the sense of Berwald given by Definition 2.1, reduces to a Riemannian space.

In the next section we shall be giving the modified definition of concurrent vector field in a Finsler space.

3. Modified Definition of Concurrent Vector Field

Definition 3.1: A vector field $X^r(x)$ in a Finsler space F^n will be defined as a B-concurrent vector field if it satisfies

(3.1) (i) $X^r A_{rjk} = \alpha h_{jk}$ (ii) $X^r_{(j)} = -\delta^r_j$,

where □ is an arbitrary non-zero scalar function of *x* and *y*, $A_{rjk} = L\, C_{rjk}$ and $A_j = L C_j$.

Two-dimensional Finsler space F^2: From equation (3.1) for a two-dimensional Finsler space we can write

(3.2) $X^r A_{rjk} = \alpha m_j m_k$,

which easily gives

$$\text{(3.3)}\ X^r A_{rjk(h)} = A_{hjk} + \alpha_{(h)} m_j m_k - \alpha m_t (m_k P^t_{jh} + m_j P^t_{kh})$$

and

$$\text{(3.4)}\ \varsigma_{(h,k)} \{ X^r A_{rjk(h)} - (\alpha m_r P^r_{jh} + m_j \alpha_{(h)}) m_k \} = 0,$$

where $\varsigma_{(h,k)}$ means interchange of indices and subtraction. Hence we have :

Theorem 3.1: In a two-dimensional Finsler space F^2, a B-concurrent vector field X^r satisfies (3.4).

From equation (3.2) we can obtain

(3.5) $X^r A_{r(k)} = A_k + \alpha_{(k)}$,

which on differentiation implies

(3.6) $X^r_{(m)} A_{r(k)} + X^r A_{r(k)(m)} = A_{k(m)} + \alpha_{(k)(m)}$.

From equation (3.6), we can easily obtain

(3.7) $\varsigma_{(k,m)} \{ X^r A_{r(k)(m)} - \alpha_{(k)(m)} \} = 0,$

which by virtue of [9],

$\alpha_{(k)(m)} - \alpha_{(m)(k)} = -(\Delta_r \alpha) H^r_{km}$, leads to

(3.8) $X^r \{ A_{r(k)(m)} - A_{r(m)(k)} \} = -(\Delta_r \alpha) H^r_{km}$.

Hence we have:

Theorem 3.2: In a two-dimensional Finsler space F^2, admitting a B-concurrent vector field X^r, the necessary and sufficient condition for $X^r A_{r(k)(m)}$ to be symmetric in *k* and *m* is that the tensor H^r_{km} vanishes.

We know that in a two-dimensional Finsler space A_{rjk} is expressed as [9]

(3.9) $A_{rjk} = L C C_r C_j C_k ,$

$$(3.10)\ A_{rjk} = L[C_1 m_r m_j m_k - C_2(m_r m_j n_k + m_j m_k n_r + m_k m_r n_j + C_3(m_r n_j n_k + m_j n_k n_r + m_k njn_r + C_2 n_r n_j n_k]$$

Let X^r be a vector field in F^3, which is represented by

(3.11) $X^r = (l^r \cos\theta + m^r \cos\phi + n^r \cos\psi),$

where cos θ, cos ϕ and cos ψ are directions cosines of this vector. Applying Definition 3.1, together with equations (3.10) and (3.11), on comparing coefficients, we get

(3.12)

$$L(C_1 \cos\phi - C_2 \cos\psi) = \alpha, L(C_3 \cos\phi - C_2 \cos\psi) = \alpha, C_2 \cos\phi - C_3 \cos\psi,$$

which leads to :

Theorem 3.4: In a three-dimensional Finsler space F^3, admitting a B-concurrent vector field X^r, coefficients C_1, C_2 and C_3 satisfy $C_1 C_3 = 2C_2{}^2 + C_3{}^2$.

Differentiating equation (3.11) with respect to x^k, with the help of Berwald's covariant derivative and multiplying the resulting equation by y^k and contracting for *r* and *k*, we obtain

(3.13)

$$h_{(0)} \cos\psi + \phi_{(0)} \sin\phi = 0,\ h_{(0)} \cos\phi - \psi_{(0)} \sin\psi = 0$$

From equations (3.12) and (3.13) we can obtain:

Theorem 3.5: In a three-dimensional Finsler space F^3, admitting a B-concurrent vector field $X^r, \tan\psi = C_3 h_{(0)} / C_2 \psi_{(0)}$.

In a three-dimensional C-reducible Finsler space equation (3.10) together with [3]

(3.14) $A_{rjk} = L(C_k h_{rj} + C_r h_{jk} + C_j h_{kr}) / 4,$

leads on simplification to $C_1+C_3=C$ and $C_2=0$. Hence with the help of Theorem 3.4, we obtain :

Theorem 3.6: In a three-dimensional C-reducible Finsler space F^3, admitting a B-concurrent vector field $X^r, C_1 = C_3 = C/2$.

We know that a P-reducible Finsler space F^3, satisfies equation [6]

(3.15) $[(C_1 - 3C_3)^2 + (4C_2)^2]_{|0} = 0.$

therefore substituting $C_1=C_3$, $C_2=0$, equation (3.15) gives $C_3 C_{3|0}=0$. Hence we have:

Theorem 3.7: In a three-dimensional P-reducible Finsler space F^3, admitting a B-concurrent vector field $X^r, C_3 C_{3|0} = 0$

4. Curvature properties of B-concurrent vector fields

Differentiating equation (3.1) with respect to Berwald's covariant derivative, we get

(4.1) $X^r A_{rjk(m)} = A_{mjk} + \alpha_{(m)} h_{jk} + \alpha h_{jk(m)},$

which on further differentiation and subtraction leads to

$$(4.2)\; X^r(A_{rjk(m)(n)} - A_{rjk(n)(m)}) = -\Delta_r(\alpha h_{jk})H^r{}_{mn} - \alpha[H_{jkmn} - H_{kjmn} + l_r(l_k H^r{}_{jmn} + l_j H^r{}_{kmn})]$$

Substituting the value of left hand side with the help of required commutation formula [9], we obtain

$$(4.3)\; X^r A_{tjk} H^t{}_{rmn} = 2\alpha(l_r l_k H^r{}_{jmn} + l_r l_j H^r{}_{kmn} - H_{kjmn}),$$

which on interchange of *j* and *k* and subtraction gives on simplification $H_{jkmn} = H_{kjmn}$. Hence we have:

Theorem 4.1: In an n-dimensional Finsler space admitting a B-concurrent vector field, Berwald's curvature tensor H_{jkmn} is symmetric in first two indices.

If we use the symmetry of Berwald's curvature tensor in first two indices, equation (4.2) implies

$$(4.4)\; X^r(A_{rjk(m)(n)} - A_{rjk(n)(m)}) = -\Delta_r(\alpha h_{jk})H^r_{mn} - \alpha l_r(l_k H^r_{jmn} + l_j H^r_{kmn})$$

From equation (4.4), we can easily obtain :

Corollary 4.1: In an n-dimensional Finsler space admitting a B-concurrent vector field the sufficient condition for vanishing of $X^r(A_{rjk(m)(n)} - A_{rjk(n)(m)})$ is given by $H^r_{mn} = 0$.

Assuming left hand side of (4.4) vanishes, by multiplying right hand side (4.4) by $y^i y^k$, we get $y_r H^r_{mn} = 0$. Hence we have:

Corollary 4.2: In an n-dimensional Finsler space admitting a B-concurrent vector field, the necessary condition for vanishing of $X^r(A_{rjk(m)(n)} - A_{rjk(n)(m)})$ is $y_r H^r_{mn} = 0$.

Since we know that [9]

$$(4.5)\; H_{jkmn} - H_{kjmn} = 2R_{jkmn} + (1/2)g^{pq}\{g_{jp(m)}g_{kq(n)} - g_{jp(n)}g_{kp(m)}\}$$

therefore from Theorem 4.1, after substituting the values of $g_{jp(m)}$ etc., we obtain

$$(4.6)\; R_{jkmn} = P^r_{jn} P_{rkm} - P^r_{jm} P_{rkn}.$$

Hence we have :

Theorem 4.2: In an n-dimensional Finsler space admitting a B-concurrent vector field, the curvature tensor R_{jkmn} is expressed in terms of torsion tensor P_{rkm}.

Since we know that R_{jkmn} satisfies curvature identity [9]

$$(4.7)\; \sum_{(k,m,n)}\{R_{jkmn} + C_{jkr}K^r_{0mn}\} = 0.$$

where symbol $\sum_{(k,m,n)}$ means sum of three terms obtained by cyclic permutation of k,m,n. From equation (4.6) and (4.7), we obtain

$$(4.8)\; \sum_{(k,m,n)}\{C_{jkr}K^r_{0mn}\} = 0.$$

Hence we have :

Theorem 4.3: In an n-dimensional Finsler space admitting a B-concurrent vector field, the curvature tensor K^r_{0mn} satisfies (4.8).

Since we know that R_{jkmn} satisfies Bianchi identity[9]

$$(4.9)\; \sum_{(m,n,h)}\{R_{jkmn|h} + l^t R^l_{tnh} P_{jkml}\} = 0$$

therefore from equation (4.6) and (4.9), we get

$$(4.10)\; \sum_{(m,n,h)}[\varsigma_{(j,k)}\{P^r_{jm}(P_{rkh|n})\}] = 0.$$

Hence we have :

Theorem 4.4: An n-dimensional Finsler space admitting a B-concurrent vector field, satisfies Bianchi identity (4.10).

We know that [6], in a P-reducible Finsler space P_{rjk} is expressed as

(4.11)

$$P_{rjk} = (n+1)^{-1}(A_{r|0}h_{jk} + A_{j|0}h_{kr} + A_{k|0}h_{rj}).$$

therefore substituting from (4.11) in (4.6), we obtain on simplification

(4.12)

$$R_{jkmn} = (n+1)^{-2}\{A_{k|0}(A_{m|0}h_{jn} - A_{n|0}h_{jm}) - A_{j|0}(A_{m|0}h_{kn} - A_{n|0}h_{km}) + A^r{}_{|0}A_{r|0}(h_{jn}h_{km} - h_{jm}h_{kn})\},$$

which leads to

(4.13)
$$R_{jm} = (n+1)^{-2}\{(3-n)A_{m|0}A_{j|0} - (n-1)A^{r}{}_{|0}A_{r|0}h_{jm}\}$$

and

(4.14) $$R = (2-n)(n+1)^{-1}\, A^{r}{}_{|0}A_{r|0}\,.$$

Hence we have:

Theorem 4.5: In an n-dimensional Finsler space admitting a B-concurrent vector field, the curvature tensor R_{jkmn}, the Ricci-tensor R_{jm} and the curvature scalar R are respectively expressed by (4.12), (4.13) and (4.14).

Equation (4.10) by virtue of equation (4.11) will give rise to

(4.15)
$$\Sigma_{(m,n,h)}[\varsigma_{(j,k)}\{A_{m|0}(A_{j|0|n}h_{kh} + A_{k|0|h}h_{jn}) + (A_{h|0|n} - A_{n|0|h})A_{k|0}h_{jm} + 2A^{r}{}_{|0}A_{r|0|m}h_{kn}h_{jh}\}] = 0.$$

Hence we have:

Theorem 4.6: In an n-dimensional P-reducible Finsler space admitting a B-concurrent vector field, equation (4.15) represents the Bianchi-identity.

In a three-dimensional P-reducible Finsler space F^3 of R-K type [6], P_{jkh} is expressed as

(4.16) $$P_{jkh} = L[C_3h_0(h_{jk}n_h + h_{kh}n_j + h_{hj}n_k) + C_{3|0}(h_{jk}m_h + h_{kh}m_j + h_{hj}m_k)],$$

which when used in (4.6) leads to

(4.17)
$$R_{jkmn} = L^2\varsigma_{(j,k)}[\{(C_{3|0})^2 + (C_3h_0)^2\}h_{km}h_{nj} + h_{km}\{m_nm_j(C_{3|0})^2 + n_nn_j(C_3h_0)^2\} + h_{nj}\{m_mm_k(C_{3|0})^2 + n_mn_k(C_3h_0)^2\} + 2C_3C_{3|0}\,h_0\{h_{mn}m_jn_k + h_{nj}m_mn_k + h_{km}m_jn_j\}].$$

From equation (4.17) we can obtain

(4.18)
$$\Sigma_{(k,m,n)}\{R_{jkmn}\} = 2L^2C_3C_{3|0}h_0\varsigma_{(j,k)}\{2h_{mn}m_jn_k + 2h_{nk}m_jn_m + h_{nj}m_mn_k + h_{km}m_jn_n\}.$$

Remark: It is known [6], that if $C_3=0$, the Finsler space reduces to a Riemannian space and in such case the curvature tensor R_{jkmn} vanishes as is visible from (4.17).

As a special case if $C_{3|0} = 0$, the tensors P_{jkh} and R_{jkmn} satisfy

(4.19) $$P_{jkh} = L\, C_3\, h_0\,(h_{jk}n_h + h_{kh}n_j + h_{hj}n_k)$$

(4.20) $$R_{jkmn} = (LC_3h_0)^2\varsigma_{(j,k)}\{h_{nj}L_{km} + h_{km}L_{nj}\}$$

and

(4.21) $$\Sigma_{(k,m,n)}\{R_{jkmn}\} = 0,$$

where $L_{km} = (1/2)h_{km} + n_mn_k$. Hence we have :

Theorem 4.7: In a three-dimensional P-reducible Finsler space F^3 of R-K type satisfying $C_{3|0} = 0$ and admitting a B-concurrent vector field, curvature tensor R_{jkmn} satisfies (4.21).

References

[1] **H. Kawaguchi :** On Finsler spaces with the vanishing second curvature tensor, *Tensor, N.S.*, **26** (1972), 250-254.

[2] **M. Matsumoto :** On Finsler spaces with curvature tensors of some special forms, *Tensor, N.S.*, **22** (1971), 201-204.

[3] **M. Matsumoto :** On C-reducible Finsler spaces, *Tensor, N.S.*, **24** (1972), 29-37.

[4] **M. Matsumoto :** Finsler spaces admitting con-current vector field, *Tensor, N.S.*, **28** (1974), 239-249.

[5] **M. Matsumoto :** Foundations of Finsler geometry and special Finsler spaces, *Kaiseisha Press*, Otsu, Japan, 1986.

[6] **S.C. Rastogi and H. Kawaguchi :** A geometrical meaning of the P-reducible condition in Finsler spaces, *Tensor, N.S.*, **51** (1992), 251-256.

[7] **S.C. Rastogi and A.K. Dwivedi :** On the existence of concurrent vector field in a Finsler space, *Tensor, N.S.* **65** (2004), 48-54.

[8] **B.K. Singh, P. Chandra and A. Dwivedi:** On the existence of concurrent vector field in a Finsler space I, IJMTT, 2017.

[9] **H. Rund :** The differential geometry of Finsler spaces, *Springer-Verlag*, 1959.

[10] **U.P. Singh :** On concurrent vector fields in Finsler spaces, *Rev. Univ. Nac. Tucuman, A. mat. fis. teor. Tucuman*, **28** (1978), 141-146.

[11] **S. Tachibana :** On Finsler spaces which admit concurrent vector field, *Tensor, N.S.*, **1** (1950), 1-5.