

Strong and Δ -Convergence of a New Iteration for Nonexpansive Mapping in CAT (0) Space

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Abstract: In this paper, we study strong convergence and Δ -convergence theorems of a new iteration scheme for nonexpansive mappings in CAT(0) space. Our results extend and improve several recent results in the current literature.

Keywords: CAT(0) space, strong convergence, Δ - convergence, non-expansive mapping, fixed point.

MSC: 47H09; 47H10

1. Introduction

A metric space X is a CAT(0) space if it is geodesically connected and if every geodesic triangle in X is at least as “thin” as its comparison triangle in the Euclidean plane [5]. It is well known that any complete, simply connected Riemannian manifold having nonpositive sectional curvature is a CAT(0) space. Complete CAT(0) spaces are often called Hadamard spaces[12]. A CAT(0) space plays a fundamental role in various branches of mathematics, see Bridson and Haefliger [5] or Burago et al.[2].

In 1976, the concept of Δ -convergence in a general metric space was introduced by Lim[9]. In 2004, W.A. Kirk[12] introduced the fixed point theory for CAT(0) spaces and showed that every nonexpansive (single-valued) mapping defined on a bounded closed convex subset of a complete CAT(0) space always has a fixed point. Since then the fixed point theory for single-valued and multivalued mappings in CAT(0) spaces has been rapidly developed [1,6,7,8,11,13,14,15]. Then in 2008, Kirk and Panyanak [11] specialized Lim's concept to CAT(0) spaces and proved that it is very similar to weak convergence in the Banach space setting. In this sequel, Dhompongsa and Panyanak[6] used the concept of Δ -convergence introduced by Lim to prove the CAT(0) space analogs and obtained Δ -convergence theorems for the Picard, Mann and Ishikawa iterations in the CAT(0) space setting. In 2013, Sahin and Besarir [1] proved Δ -convergence and strong convergence theorems of SP-iterative schemes for non-expansive mappings in CAT(0) spaces.

In 2014, Thakur et al. [3] introduced the following new iteration scheme for approximation of fixed points of nonexpansive mappings: for any arbitrary $x_0 \in C$ construct a sequence $\{x_n\}$ by

$$\begin{aligned} x_{n+1} &= (1 - \alpha_n)Tx_n + \alpha_nTy_n \\ y_n &= (1 - \beta_n)z_n + \beta_nTz_n \\ z_n &= (1 - \gamma_n)x_n + \gamma_nTx_n \end{aligned} \tag{1.1}$$

where $\{\alpha_n\}, \{\beta_n\}$ and $\{\gamma_n\}$ are the real sequences in $(0,1)$. Using a numerical example, they compared the behavior of iteration (1.1) with Picard, Mann, Ishikawa, Noor, the

Agarwal et al. and the Abbas et al. iterations processes for contractions in the sense of Berinde [10] and proved that the iteration process (1.1) converges faster than all other iterations.

In this paper, we apply this new iteration (1.1) to prove Δ -convergence and strong convergence theorems in CAT(0) space for nonexpansive mappings. The iteration (1.1) in the setting of CAT(0) space is as follows:

$$\begin{aligned} x_{n+1} &= (1 - \alpha_n)Tx_n \oplus \alpha_nTy_n \\ y_n &= (1 - \beta_n)z_n \oplus \beta_nTz_n \\ z_n &= (1 - \gamma_n)x_n \oplus \gamma_nTx_n \end{aligned} \tag{1.2}$$

for all $n \geq 1$, where K is a nonempty convex subset of a CAT(0) space, $\{\alpha_n\}, \{\beta_n\}$ and $\{\gamma_n\}$ are sequences in $(0,1)$ and $x_1 \in K$.

2. Preliminaries

Firstly, we give some definitions and known results in the existing literature of CAT(0) space.

Let C be a nonempty subset of a CAT(0) space X . Then, a mapping: $C \rightarrow C$ is said to be nonexpansive if,

$$d(Tx, Ty) \leq d(x, y), \quad \forall x, y \in C.$$

A point $x \in C$ is called a fixed point of T if $Tx = x$. We will denote the set of fixed points of T by $F(T)$.

Let (X, d) be a metric space. A geodesic path joining $x \in X$ to $y \in X$ (or, more briefly, a geodesic from x to y) is a mapping c from a closed interval $[0, l] \subseteq \mathbb{R}$ to X such that

$$\begin{aligned} c(0) &= x; c(l) = y, \\ \text{and } d(c(t), c(t_0)) &= |t - t_0| \text{ for all } t, t_0 \in [0, l]. \end{aligned}$$

In particular, c is an isometry and $d(x, y) = l$. The image of c is called a geodesic (or metric) segment joining x and y . When it is unique, this geodesic is denoted by $[x, y]$. The space (X, d) is said to be a geodesic space if every two points of X are joined by a geodesic and X is said to be uniquely geodesic if there is exactly one geodesic joining x and y for each $x, y \in X$. A subset $Y \subseteq X$ is said to be convex if Y includes every geodesic segment joining any two of its points.

A geodesic triangle $\Delta(x_1, x_2, x_3)$ in a geodesic metric space (X, d) consists of three points in X (the vertices of Δ) and a geodesic segment between each pair of vertices (the edges of Δ). A comparison triangle for a geodesic triangle $\Delta(x_1, x_2, x_3)$ in (X, d) is a triangle $\bar{\Delta}(x_1, x_2, x_3) := \Delta(\bar{x}_1, \bar{x}_2, \bar{x}_3)$ in the Euclidean plane \mathbb{R}^2 such that $d_{\mathbb{R}^2}(\bar{x}_i, \bar{y}_j) = d(x_i, y_j)$ for $i, j \in \{1, 2, 3\}$. Such a triangle always exists [5].

A geodesic metric space is said to be a CAT(0) space if all geodesic triangles of appropriate size satisfy the following CAT(0) comparison axiom.

Let Δ be a geodesic triangle in X and let $\bar{\Delta} \subseteq \mathbb{R}^2$ be a comparison triangle for Δ . Then Δ is said to satisfy the CAT(0) inequality if for all $x, y \in \Delta$ and all the comparison points $\bar{x}, \bar{y} \in \bar{\Delta}$,

$$d(x, y) \leq d_{\mathbb{R}^2}(\bar{x}, \bar{y})$$

If x_1, y_1, z_1 are points in a CAT(0) space and if y_0 is the midpoint of the segment $[y_1, z_1]$, which we will denote by $\frac{y_1 \oplus z_1}{2}$, then the CAT(0) inequality implies

$$d^2\left(x, \frac{y_1 \oplus z_1}{2}\right) \leq \frac{1}{2}d^2(x, y_1) + \frac{1}{2}d^2(x, z_1) - \frac{1}{4}d^2(y_1, z_1) \quad \text{(CN)}$$

This is the (CN) inequality if Bruhat and Tits [4]. In fact, a geodesic space is CAT(0) space if and only if it satisfies the (CN) inequality [5].

Lemma 2.1 [6]: Let X be a CAT(0) space. Then $D((1-t)x \oplus ty, z) \leq (1-t)d(x, z) + td(y, z)$ for all $x, y, z \in X$ and $t \in [0, 1]$.

Lemma 2.2 [6]: Let (X, d) be a CAT(0) space. Then $d((1-t)x \oplus ty, z)^2 \leq (1-t)d(x, z)^2 + td(y, z)^2 - t(1-t)d(x, y)^2$ for all $t \in [0, 1]$ and $x, y, z \in X$.

Let $\{x_n\}$ be a bounded sequence in a complete CAT(0) space X . For $x \in X$, define a continuous functional

$$r(\cdot, \{x_n\}): X \rightarrow [0, \infty) \text{ by } r(x, \{x_n\}) = \lim_{n \rightarrow \infty} \sup d(x, x_n).$$

The asymptotic radius $r(\{x_n\})$ of $\{x_n\}$ is given by $r(\{x_n\}) = \inf\{r(x, \{x_n\}) : x \in X\}$.

The asymptotic center of $A\{x_n\}$ of $\{x_n\}$ is the set $A(\{x_n\}) = \{x \in X : r(x, \{x_n\}) = r(\{x_n\})\}$.

It is known (see, e.g., [7], Proposition 7) that in a CAT(0) space, $A(\{x_n\})$ consists of exactly one point.

Also, every CAT(0) space has the Opial property, i.e., if $\{x_n\}$ is a sequence in C and $\Delta - \lim_{n \rightarrow \infty} x_n = x$, then for each $y (\neq x) \in C$,

$$\lim_{n \rightarrow \infty} \sup d(x_n, x) < \lim_{n \rightarrow \infty} \sup d(x_n, y)$$

Definition 2.3 [6]: A sequence $\{x_n\}$ in a CAT(0) space X is said to be Δ -convergent to $x \in X$ if x is the unique asymptotic center of $\{x_n\}$ for every subsequence $\{x_{n_j}\}$ of

$\{x_n\}$. In this case we write $\Delta - \lim_{n \rightarrow \infty} x_n = x$ and called the Δ -limit of $\{x_n\}$.

Lemma 2.4 [6]: Every bounded sequence in a complete CAT(0) space always has a Δ -convergent subsequence.

Lemma 2.5 [6]: If C is a closed convex subset of a complete CAT(0) space and if $\{x_n\}$ is a bounded sequence in C , then the asymptotic center of $\{x_n\}$ is in C .

Lemma 2.6 [6]: Let C be a closed convex subset of a complete CAT(0) space and let $T : C \rightarrow X$ be a non-expansive mapping. Then the conditions, $\{x_n\}$ Δ -converges to x and $d(x_n, Tx_n) \rightarrow 0$, imply $x \in C$ and $Tx = x$.

3. Main Results

In this section, we first prove a lemma useful for the main result.

Lemma 3.1 Let C be a nonempty closed convex subset of a complete CAT(0) space X and let $T : C \rightarrow C$ be a non-expansive mapping such that $F(T)$ is nonempty. Then for the sequence $\{x_n\}$ defined in (1.2), we have

- (i) $\lim_{n \rightarrow \infty} d(x_n, p)$ exists for each $p \in F(T)$.
- (ii) $\lim_{n \rightarrow \infty} d(x_n, Tx_n) = 0$.

Proof. (i) Let $p \in F(T)$. By (1.2) and lemma 2.1, we have

$$\begin{aligned} d(z_n, p) &= d((1-\gamma_n)x_n \oplus \gamma_n Tx_n, p) \\ &\leq (1-\gamma_n)d(x_n, p) + \gamma_n d(Tx_n, p) \\ &\leq (1-\gamma_n)d(x_n, p) + \gamma_n d(x_n, p) \\ &= d(x_n, p) \\ d(z_n, p) &\leq d(x_n, p) \end{aligned} \quad (3.1)$$

using (3.1) we get

$$\begin{aligned} d(y_n, p) &= d((1-\beta_n)z_n \oplus \beta_n Tz_n, p) \\ &\leq (1-\beta_n)d(z_n, p) + \beta_n d(Tz_n, p) \\ &\leq (1-\beta_n)d(z_n, p) + \beta_n d(z_n, p) \\ &= d(z_n, p) \\ \Rightarrow d(y_n, p) &\leq d(z_n, p) \end{aligned} \quad (3.2)$$

Using (3.1) and (3.2), we have

$$d(y_n, p) \leq d(z_n, p) \leq d(x_n, p) \quad (3.3)$$

Now using (3.3) and lemma 2.1, we have

$$\begin{aligned} d(x_{n+1}, p) &= d((1-\alpha_n)Tx_n \oplus \alpha_n Ty_n, p) \\ &\leq (1-\alpha_n)d(Tx_n, p) + \alpha_n d(Ty_n, p) \\ &\leq (1-\alpha_n)d(x_n, p) + \alpha_n d(y_n, p) \\ &\leq (1-\alpha_n)d(x_n, p) + \alpha_n d(x_n, p) \\ &= d(x_n, p) \\ \Rightarrow d(x_{n+1}, p) &\leq d(x_n, p) \end{aligned}$$

This shows that the sequence $\{d(x_n, p)\}$ is non-increasing and bounded below, and so $\lim_{n \rightarrow \infty} d(x_n, p)$ exists for each $p \in F(T)$. This completes the proof of part (i).

- (ii). Assume that $\lim_{n \rightarrow \infty} d(x_n, p) = c$. From (3.3), we have

$$d(z_n, p) \leq d(x_n, p) \text{ and } d(y_n, p) \leq d(x_n, p)$$

Now, taking lim sup on both sides of both inequalities, we get

$$\lim_{n \rightarrow \infty} \sup d(z_n, p) \leq c \quad (3.4)$$

$$\text{and } \lim_{n \rightarrow \infty} \sup d(y_n, p) \leq c \quad (3.5)$$

By the nonexpansivity of T , it follows that

$$d(Tx_n, p) \leq d(x_n, p) \text{ and } d(Ty_n, p) \leq d(y_n, p)$$

Taking lim sup on both sides, we get

$$\lim_{n \rightarrow \infty} \sup d(Tx_n, p) \leq c, \quad (3.6)$$

$$\lim_{n \rightarrow \infty} \sup d(Ty_n, p) \leq c \quad (3.7)$$

Since

$$\begin{aligned} c &= \lim_{n \rightarrow \infty} d(x_{n+1}, p) \\ &= \lim_{n \rightarrow \infty} ((1 - \alpha_n)d(Tx_n, p) + \alpha_n d(Ty_n, p)) \end{aligned}$$

Now by Lemma 3.2 [3], we have

$$\lim_{n \rightarrow \infty} d(Tx_n, Ty_n) = 0 \quad (3.8)$$

Now,

$$\begin{aligned} d(x_{n+1}, p) &= d((1 - \alpha_n)Tx_n \oplus \alpha_n Ty_n, p) \\ &\leq d(Tx_n, p) + \alpha_n d(Tx_n, Ty_n) \end{aligned}$$

$$\text{yields } c \leq \lim_{n \rightarrow \infty} \inf d(Tx_n, p) \quad (3.9)$$

from (3.6) and (3.9), we get

$$\lim_{n \rightarrow \infty} d(Tx_n, p) = c \quad (3.10)$$

Also, we have

$$\begin{aligned} d(Tx_n, p) &\leq d(Tx_n, Ty_n) + d(Ty_n, p) \\ &\leq d(Tx_n, Ty_n) + d(y_n, p) \end{aligned}$$

which gives

$$c \leq \lim_{n \rightarrow \infty} \inf d(y_n, p) \quad (3.11)$$

from (3.5) and (3.11), we get

$$\lim_{n \rightarrow \infty} d(y_n, p) = c$$

By nonexpansivity of T and (3.1), we have

$$\lim_{n \rightarrow \infty} \sup d(Tz_n, p) \leq c \quad (3.12)$$

Now using (3.4), (3.12) and Lemma 3.2 [3], we have

$$\lim_{n \rightarrow \infty} d(Tz_n, z_n) = 0 \quad (3.13)$$

Since

$$d(y_n, p) \leq d(z_n, p) + \beta_n d(Tz_n, z_n),$$

we can write

$$c \leq \lim_{n \rightarrow \infty} \sup d(z_n, p)$$

$$\text{then } d(z_n, p) = c \quad (3.14)$$

so

$$\begin{aligned} c &= \lim_{n \rightarrow \infty} d(z_n, p) = \lim_{n \rightarrow \infty} d((1 - \gamma_n)x_n \oplus \gamma_n Tx_n, p) \\ &= \lim_{n \rightarrow \infty} ((1 - \gamma_n)d(x_n, p) + \gamma_n d(Tx_n, p)) \end{aligned}$$

Now using lemma 3.2 [3], we have $\lim_{n \rightarrow \infty} d(x_n, Tx_n) = 0$.

This completes the proof of (ii).

Now, we will prove the main result.

Theorem 3.2 Let $X, C, T, \{\alpha_n\}, \{\beta_n\}$ and $\{\gamma_n\}$ satisfy the hypotheses of Lemma 3.1. Then the sequence $\{x_n\}$ Δ -converges to a fixed point of T .

Proof. Since the sequence $\{x_n\}$ is bounded by Lemma 3.1. Now we suppose that $W_\Delta(x_n) = \bigcup A(\{u_n\})$, where the union

is taken over all subsequences $\{u_n\}$ of $\{x_n\}$. We claim that $W_\Delta(x_n) \subseteq F(T)$. Let $u \in W_\Delta(x_n)$. Then there exists a subsequence $\{u_n\}$ of $\{x_n\}$ such $A(\{u_n\}) = \{u\}$. By Lemmas 2.4 and 2.5, there exists a subsequence $\{v_n\}$ of $\{u_n\}$ such that $\Delta - \lim_{n \rightarrow \infty} v_n = v \in C$. By lemma 2.6, $v \in F(T)$. By lemma 3.1 $\lim_{n \rightarrow \infty} d(x_n, v)$ exists. Now we claim that $u = v$.

On the contrary, assume that $u \neq v$. Then by the uniqueness of asymptotic centers, we have

$$\begin{aligned} \lim_{n \rightarrow \infty} \sup d(v_n, v) &< \lim_{n \rightarrow \infty} \sup d(v_n, u) \\ &\leq \lim_{n \rightarrow \infty} \sup d(u_n, u) \\ &< \lim_{n \rightarrow \infty} \sup d(u_n, v) \\ &= \lim_{n \rightarrow \infty} \sup d(x_n, v) \\ &= \lim_{n \rightarrow \infty} \sup d(v_n, v) \end{aligned} \quad (3.15)$$

This is a contradiction.

Hence $u = v \in F(T)$ and $W_\Delta(x_n) \subseteq F(T)$. To show that $\{x_n\}$ Δ -converges to a fixed point of T , it suffices to show that $W_\Delta(x_n)$ consists of exactly one point. Let $\{u_n\}$ be a subsequence of $\{x_n\}$ with $A(\{u_n\}) = \{u\}$ and let $A(\{x_n\}) = \{x\}$.

We have already seen that $u = v \in F(T)$.

Finally we show that $x = v$. If not, then by the existence of $\lim_{n \rightarrow \infty} d(x_n, v)$ and by the uniqueness of asymptotic centers, we get a contradiction as (3.15) and hence $x = v \in F(T)$. Hence the result.

We now establish strong convergence of the iteration (1.2) in the setting of CAT(0) space.

Theorem 3.3 Let $X, C, T, \{\alpha_n\}, \{\beta_n\}$ and $\{\gamma_n\}$ satisfy the hypotheses of Lemma 3.1. Then the sequence $\{x_n\}$ converges strongly to a fixed point of T if and only if

$$\lim_{n \rightarrow \infty} d(x_n, F(T)) = 0$$

where $d(x, F(T)) = \inf \{d(x, p) : p \in F(T)\}$

Proof: Necessity is obvious. Conversely, assume that

$$\lim_{n \rightarrow \infty} \inf d(x_n, F(T)) = 0.$$

We have from the proof of Lemma 3.1(i) that

$$d(x_{n+1}, p) \leq d(x_n, p), \text{ for all } p \in F(T).$$

It follows that

$$d(x_{n+1}, F(T)) \leq d(x_n, F(T)),$$

which implies that the sequence $\{d(x_n, F(T))\}$ is nonincreasing and bounded below.

Thus, $\lim_{n \rightarrow \infty} d(x_n, F(T))$ exists.

By the hypothesis, we can conclude that

$$\lim_{n \rightarrow \infty} d(x_n, F(T)) = 0.$$

Next, we will show that $\{x_n\}$ is a Cauchy sequence in C .

Let $\epsilon > 0$ be arbitrarily chosen.

Since $\lim_{n \rightarrow \infty} d(x_n, F(T)) = 0$, there exists a positive integer n_0 such that for all $n \geq n_0$, we have

$$d(x_n, F(T)) < \frac{\epsilon}{4}.$$

In particular, $\inf \{d(x_{n_0}, p) : p \in F(T)\} < \frac{\epsilon}{4}$.

Thus there exists $p^* \in F(T)$ such that

$$d(x_{n_0}, p^*) < \frac{\epsilon}{2}$$

Now, for all $m, n \geq n_0$, we have

$$d(x_{n+m}, x_n) \leq d(x_{n+m}, p^*) + d(x_n, p^*)$$

$$\leq 2d(x_{n_0}, p^*) < 2\left(\frac{\varepsilon}{2}\right) = \varepsilon$$

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Hence $\{x_n\}$ is a Cauchy sequence in C .

Since C is closed in a complete $CAT(0)$ space X , the sequence $\{x_n\}$ must be convergent to a point in C .

Let $\lim_{n \rightarrow \infty} x_n = q \in C$.

Since $\lim_{n \rightarrow \infty} d(x_n, F(T)) = 0$, give that
 $d(q, F(T)) = 0$.

Moreover, $q \in F(T)$ because $F(T)$ is closed.

Therefore, the sequence $\{x_n\}$ converges strongly to a fixed point q of T .

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