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S_n - Primary Submodules

Nuhad Salim Al-Mothafar Iman A. Athab

Department of Mathematics, College of Science, Baghdad University, Baghdad, Iraq

Abstract: ***Let R be a commutative ring with identity. In this paper we introduce the concept of S_n- Primary Submodules, where n is a fixed positive integer for this notion, let N be a proper submodule of an R-module M. Nis called S_n- Primary Submodules if whenever $f(x) \in N$ for $x \in M$and $f \in S = End(M)$ implies either $x \in N$or $f^n(M) \subseteq N$. We study this concept and give some of its properties.***

Keywords: prime submodule, n-prime submodule, n- primary submodule, S - primary submodule

1. Introduction

Let R be a commutative ring with identityand M be an R-module. Recall that a proper submodule$N$ of an R- module$M$ is called primary, if whenever $rx \in N$; $r \in R$ and $x \in M$ imphes that either $x \in N$ or $r^n \in [N:M] = \{r \in R : rM \subseteq N\}$ for some $n \in Z^+$, [5]. In [6] the concept of n – primary submodule where n is a fixed positive integer, was introduce as follows: we say that a proper submodule$N$ of an R-module$M$ is n- primary submodule if whenever $rx \in N$; $r \in R$ and $x \in M$ imphes that either $x \in N$ or $r^n \in [N:M]$. In [7] the notion of S - primary submodule was given as follows: a proper submodule$N$ of an R- module$M$ is said to be S - primary submodule of M, if whenever $f(m) \in N$ for some $f \in End(M)$ and $m \in M$, then either $m \in N$ or there exists a positive integer n, such that $f^n(M) \subseteq N$. In this paper we introduce a new class of sub modules where n a fixed positive integer, a proper submodule$N$ of an R-module$M$ is said to be S_n- primary if whenever $f(x) \in N$, for $x \in M$ and $f \in S = End(M)$ imphes that either $x \in N$or $f^n(M) \subseteq N$. We prove some results for this class. Also we introduce a new characterization for S_n- primary Submodules in a multiplication module.

2. S_n- PrimarySubmodules:

Recall that a proper submodule$N$ of an R- module$M$ is called primary submodule, if whenever$f(m) \in N$ for some $f \in End(M)$ and $m \in M$, implies that either $m \in N$or $f^n(M) \subseteq N$, for some $n \in Z^+$,[7]. Also recall that a proper submodule$N$of an R- module$M$ is called n- primary submodule where n is a fixed positive integer if whenever $rx \in N$; $r \in R$and $x \in M$, implies that either $x \in N$ or $r^n \in [N:M]$, [6].
We introduce the following definition:

Definition (2.1):
A proper submodule N of an R- module$M$ is said to be S_n- primary where n is a fixed positive integer,if whenever$f(m) \in N$ for some $f \in End(M)$ and $m \in M$, implies that either $m \in N$or $f^n(M) \subseteq N$.

Remarks and examples (2.2):
1) It is clear that every S_n- primary sub modules is S-primary. But the converse is not true in general for example: The $\{\bar{0}\}$ submodule of Z_4 as Z-module is an S-primary submodule of Z_4, [7], which is not S_n- primary Sub module of Z_4
2) Every S_n- primary submodule is n- primary.

Proof:
Let N be an S_n- primary submoduleof an R- module$M$. Suppose that $rx \in N$; $r \in R$and $x \in M$, assume that$x \notin N$. Define $f: M \to M$ by $f(m) = rm$; $m \in M$. Now, $rx = f(x) \in N$, but N is an S_n- primary submodule, thus for a fixed integer n, we have $f^n(M) \subseteq N$, hence $r^n M \subseteq N$, therefore $r^n \in [N:M]$, this implies that N is n- primary submodule.

The converse of the previous remark is not true for example:
Let $M = Z_8$as Z-module, $N = \{\bar{0}, \bar{4}\}$, since N is a prime submodule of M, then N is a 1-primary submodule of M.
But N is not S_1- primary.
3) Recall that a proper submodule$N$ of an R- module$M$ is called$S$-prime, if whenever $f(x) \in N$, for $m \in M$ and $f \in S = End(M)$implies that either $m \in N$ or $f^n(M) \subseteq N$,[4], It is clear that every S-prime submodule is an S-primary submodule, since every S-prime submodule is an S-primary submodule ,[7], hence be (1) is an S_n-primary submodule.
4) Let P be a prime number, the module Z_P over Z has no S_n- primary submodule.

Now, we introduce the notion of $\sqrt[n]{Ann(M)}$ as follows:

Definition (2.3):
Let N be a submodule of an R_-module$M$a fixed positive integer n, define $\sqrt[n]{Ann(M)}$ as follows, $\sqrt[n]{Ann(M)} = \{r \in R : r^n \in Ann(M)\}$. It is clear thus $\sqrt[n]{Ann(M)} \subseteq \sqrt{Ann(M)}$.
Let us introduce the following characterization:

Proposition (2.4):
Let M be a non $zero R$- module and n is a fixed positive integer, then$\{0_M\}$ is n- primary submodule of M if and only if $Ann(N) \subseteq \sqrt[n]{Ann(M)}$, for all non $zero$ submodule $N$of M.

Proof:
Suppose that $N \neq 0$ and $\{0_M\}$ is n- primary submodule of M, we must show that $Ann(N) \subseteq \sqrt[n]{Ann(M)}$. For this direction, let $r \in Ann(N)$. Since $N \neq 0$, so there exists a non $zero$element $x \in N$. Now $rx = 0$ and N is n- primary, therefore for a fixed positive integer n we have $r^n \in$

$[0:M] = Ann(M)$, thus $r \in \sqrt[n]{Ann(M)}$. This implies that $Ann(N) \subseteq \sqrt[n]{Ann(M)}$.

Conversely, let $rx = 0$, foe $r \in R$and $x \in M$. Suppose that $x \neq 0$, then $< x > \neq 0$, hence by assumption we get that $Ann(< x >) \subseteq \sqrt[n]{Ann(M)}$ since $rx = 0$,therefore $r \in Ann(< x >)$, and hence $r \in \sqrt[n]{Ann(M)}$. This implies that $r^n \in [0:M]$. This implies that $\{0\}_M$ is n- primary submodule of M.

Recall that an$R$- module$M$ is called multiplication if for each submodule$N$ of M there exists an idea I of R such that $N = IM$, [3].

Now we are ready to give the following characterization:

Proposition (2.5):
Let M be a non $zero$ multiplication R- module and n is a fixed positive integer, then $\{0_M\}$ is n- primary submodule of M if and only if it is S_n- primary submodules.

Proof:
Suppose that $f(m) = 0$ where $f \in End(M)$ and $m \in M$. Assume that $m \neq 0$, we have to prove that for a fixed positive integer $n, f^n(M) = 0$. Since$m \neq 0$, thus$< m > \neq 0$, but M is multiplication, so there exist an ideal I of R such that $< m >= IM$. Now, if $f(M) = 0$, then we are done. Thus suppose that $f(M) \neq 0$ hence there exists a non $zero$ idea J of R such that $f(M) = JM$.
$0 = f(< m >) = f(IM) = If(M) = I(JM) = J(IM)$, therefore $J(IM) = 0$. This implies that $J \subseteq Ann(IM)$. Since $\{0_M\}$ is n- primary, therefore by proposition (2.4) we have $Ann(IM) \subseteq \sqrt[n]{Ann(M)}$, hence $J \subseteq \sqrt[n]{Ann(M)}$, this implies that $J \subseteq Ann(M)$, this implies that $f^n(M) = 0$, hence$\{0_M\}$ is S_n- primary submodulesof M. The converse side from (remark(2) in (2.2)).

Compare the following definition with[7, Definition (2.3.6)]

Definition (2.6):
Let M be a non $zeroR$- module and n is a fixed positive integer. If $\{0_M\}$ is S_n- primary submodule .of M, then M is called S_n- primary module.

Now we can give the following an important theorem.

Theorem (2.7):

Let M be a multiplication R- module, then N is n-primary submodule of M, if and only if it is S_n- primary submodule.

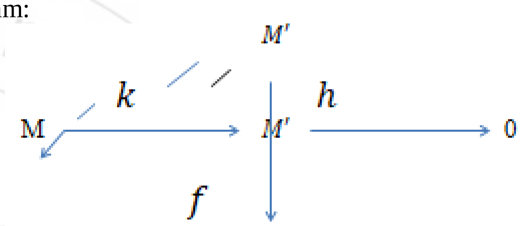
Proof:

Since M is a multiplication R- module, then $\frac{M}{N}$ is also multiplication R- module by [1, corollary (3.22)]. By the previous proposition, N is n- primary submodules .of M, if and only if it is an S_n- primary submodules .of M. Let us introduce the definition of M' is M-projective where both M and M' are R- module, see[2]. Let M and M' be R- modules, the module M is called M'-projective, if for every homomorphism $f: M' \to \frac{M}{K}$, where K is a submodule of M, can be lifted to a homomorphism $g: M' \to M$.

Now, we can give the following proposition:

Proposition (2.8):
Let $f: M \to M'$be and R- moduleepimorphism. If N is an S_n-primary submodules of M such that $kerf \subseteq N$, then $f(N)$ is an S_n- primary submodulesof M'. Whenever M' is M-projective module.

Proof:
First, we must prove that $f(N)$ is a proper submodule of a module M'. Suppose that $f(N) = M'$, the $f(N) = f(M)$ and hence $M = N + kerf$. This implies that $M = N$, which is a contradiction. Now, let $h(m') \in f(N)$, where $h \in End(M')$ and $m' \in M'$ and suppose that $m' \notin f(N)$, we have to show that for a fixed positive integer n we have $h^n(M) \subseteq f(N)$. Since f is an epimorphism and $m' \in M'$, then there exists $m \in M$ such that $f(m) = m'$. Now, consider the following diagram:



SinceM' is M-projective, then there exists a homomorphism k such that $fok = h$. But $h(m') \in f(N)$, this implies that $(fok)(m') \in f(N)$and hence $(fok)(f(m)) \in f(N)$. but $kerf \subseteq N$, thus $(kof)(m) \in N$, but N is S_n- primary submodules of M and $m \notin N$, then for a fixed position integer n we have $(kof)^n(M) \subseteq N$. therefore $f[(kof)^n(M)] \subseteq f(N)$. By a simple calculation and since $fok = h$ we conclude that $h^n(M) \subseteq f(N)$. This means that $f(N)$ is S_n- primary submodulesof M'.

Corollary (2.9):
If N is S_n- primary submodules of M and K is a submodule of $M; K \subseteq N$, then $\frac{N}{K}$ is S_n- primary submodules of $\frac{M}{K}$, whenever $\frac{M}{K}$ is an M- projective module.
From the remark (1) in (2.2) and the previous proposition we can prove the following corollary which introduce in [7].

Corollary (2.10):
Let $f: M \to M'$be an epimorphism. If N is and S-primary submodule of M such that $kerf \subseteq N$, then $f(N)$ is an S-primary sub module of M'.Whenever M' is M projective module.

Recall that a proper submodule$N$ of an R-module M is called and S-semiprime if whenever $f^2(m) \in N; m \in M$ and $f \in S = End(M)$ implies that $f(m) \in N$, [7]. Also every S-prime module is S-semiprime, [7].

The following proposition was provide in [7].

Proposition (2.11):

Let N be a proper submodule of an R-module M, then N is an S-primary and is S-prime of M if and only if it is an S-prime submodule of M.

Now, from remarks (1) and (3) is (2.2) and proposition (2.11) we can prove the following corollary easily:

Corollary (2.12):

Let N be a proper submodule of an R-module M, then N is an S_n- primary and S- semiprime submodule of Mif and only if it is an S-prime submodule of M.

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