Common Fixed Point in Cone Banach Space

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Abstract: In this paper we prove that if A and B are two mappings defined on a closed subset G of a cone Banach space \((X, ||.||_C)\) and \(A,B\) satisfied either the condition \(|Ax-By| \leq a_1||x-Ax|| + a_2||y-By|| + a_3||x-y|| + a_4||x-By||\) for all \(x,y \in G\) or A and B satisfied condition B then A and B have common fixed point.

Keywords: Condition B

1. Introduction

In (2007) Huang and Zhang introduced cone metric space be means of partial ordering (\((\leq)\)) on real Banach space \((E, ||.||_E)\) also they proved some fixed point theorems, while [5] gave the definition of cone Banach space. In [3] Golubović, Zorana and Kadelburg, Zoran and Radenovic', Stojan proved common fixed point theorem of weak contractive mapping in cone metric space while [2] Akbar Azam, Muhammad Arshad, and Ismat Beg, gave the sufficient conditions for existence of points of coincidence and common fixed point of three self mapping satisfying a contractive type conditions in cone metric space To defined the cone Banach space [4] define the partial ordering \(\leq\) with respect to \(p\) by \(x \leq y\) if and only if \(y-x \in p\). We shall write \(x < y\) to indicate that \(x \leq y\) but \(x \neq y\), while \(x \ll y\) will stand for \(y-x \in int p\) (interior of \(p\)).

Definition 1.1 Let \(E\) be a real Banach space. A subset \(p\) of \(E\) is called cone if

1. \(p\) is closed, nonempty and \(p \neq \{0\}\)
2. \(a,b \in R, a,b \geq 0\) and \(x \in p\), \(y \in p\)
3. \(p \cap (-p) = \{0\}\)

Definition 1.2 [4] Let \(X\) be a vector space over \(R\), suppose the mapping \(||.||_p: X \rightarrow \mathbb{R}\) satisfies

1. \(||x||_p > 0\) for all \(x \in X\)
2. \(||x||_p = 0\) if and only if \(x = 0\)
3. \(||x + y||_p \leq ||x||_p + ||y||_p\) for all \(x,y \in X\)
4. \(||kx||_p = ||k||_R ||x||_p\) for all \(k \in \mathbb{R}\)

then \(\langle X, ||.||_p \rangle\) is called a cone normed space (CNS).

Definition 1.3 [4] Let \((X, ||.||_p)\) be a cone normed space \(x \in X\) and \(\{x_n\}_{n \geq 1}\) a sequence in \(X\), then

1. \(\{x_n\}_{n \geq 1}\) converge to \(x\) whenever for each \(c \in E\) with \(0 < c\) there is natural number \(n\) such that \(||x_n - x||_p < c\) for all \(n \in N\) it is denoted by \(\lim_{n \to x} x_n = x\)
2. \(\{x_n\}_{n \geq 1}\) is a Cauchy sequence whenever for each \(c \in E\) with \(0 < c\) there is a natural number \(n\), such that \(||x_n - x_m|| < c\) for all \(n,m \in N\)
3. \((X, ||.||_p)\) is complete cone normed space if every Cauchy sequence is convergent. Complete cone normed space is called Banach space.

Definition 1.4 [1] Let \(A, B\) be self mapping on a cone metric space \((X, d)\) a point \(z \in X\) is called a common fixed point of \(A, B\) if \(Az = Bz = z\) moreover a pair of self mapping \((A, B)\) is called weakly compatible on \(X\) if they commute at their coincidence point , in other words \(z \in X, Az = Bz \iff ABz = BAz\)

Definition 1.5 Let \((X, ||.||)\) be a cone Banach space a map \(B:X \rightarrow X\) is said to be satisfy condition B if there exists \(0 < \delta < 1\) and \(L > 0\)such that for all \(x,y \in X\) we have \(||Bx - By|| \leq \delta ||x - y|| + L \) where

Proof. Let \(z_0 \in G\) we define a sequence \((z_n)\) as follows

\[z_{n+1} = Az_n\]

\[z_{n+2} = Bz_{n+1} = 0,1,2,\ldots\]

we have

\[||z_{n+1} - 2z_n|| = ||Az_n - Bz_{n-1}|| \leq a_1||z_{n+1} - Az_n|| + a_2||z_{n+1} - Bz_{n-1}|| + a_3||z_{n+1} - z_{n-1}||\]

hence

\[||z_{n+1} - 2z_n|| \leq \frac{a_2+a_3}{1-a_1}||z_{n+1} - z_{n-1}||\]

we have

\[||z_{n+1} - 2z_n|| \leq m||z_{n+1} - z_{n-1}||\]

processing in this way

\[||z_{n+1} - 2z_n|| \leq m^k||z_1 - z_0||\]

for any positive integer \(k\) we get

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Let $A, B$ be a compatible mapping of a Banach space so $\{z_n\}$ converge to $z$ but $G$ is closed subset of $X$ such that $z_n \to f$ we have

$$||f - Bf|| \leq a_1 ||z_n - z_{n+1}|| + a_2 ||f - Bf||^2 + a_3 ||z_n - f||^2 + a_4 ||z_n - Bf||$$

as $n \to \infty$, $z_n \to f$, $z_{n+1} \to f$ we have

$$||f - Bf|| \leq ||a_2||||f - Bf||$$

then implies that $f, Bf$ since $a_2 < 1$ thereby get $f = Af$ then $f$ is a Common fixed point of $A$ and $B$ let $p \neq f$ then $||p - f|| = ||Af - Bp||$

$$a_1 ||p - A|| + a_2 ||p - Bp|| + a_3 ||p - f|| + a_4 ||p - Bf||$$

since $a_1 + a_2 + a_3 < 1$ then $p = f$ that is Common fixed point is unique.

**Theorem 2.2** Let $G$ be a closed and convex subset of a cone Banach space $(X, ||.||)$ with a cone $p$ and let $A$ and $B$ are self mapping , satisfy condition B and $B(G) \subseteq A(G)$, $A(G)$ is complete subspace, then $A$ and $B$ have common coincidence point furthermore, if $A$ and $B$ are weakly compatible, then they have unique common fixed point in $G$

**Proof.** Let $y_0 \in G$ be arbitrary ,by $(B(G) \subseteq A(G))$we can fixed a point in $G$, say $y_0$ such that $B y_0 = A y_1$ since $A,B$ are self mapping there exists a point $z_0$ in $G$ such that $z_0 = B y_0 = A y_1$ inductively we can define a sequence $\{z_n\}$ and a sequence $\{y_n\}$ in $G$ in the following way

$z_n = A y_{n+1} = B y_{n+1}$

$z_{n-1} = A y_n = B y_{n-1}$

$z_{n+1} = A y_{n+2} = B y_{n+1}$

$z_{n+2} = A y_{n+3} = B y_{n+2}$

Let $\delta||z_n - z_{n+1}|| \leq \delta||Ay_n - Ay_{n+1}|| + L||U u\|$ when $u \in \{||Ay_n - B y_n||, ||Ay_{n+1} - B y_{n+1}||, ||Ay_n - B y_{n+1}||\}$ then

$||z_n - z_{n+1}|| \leq \delta||Ay_n - Ay_{n+1}|| + L||Ay_n - B y_n||$

$\leq \delta||z_n - z_{n-1}|| + L||z_{n-1} - z_n||$

$\leq (\delta + L)||z_{n-1} - z_n||$ where $\alpha = \frac{1}{1 - \delta}$

$\leq (\delta + L)||z_{n-1} - z_n||$ take in (1) and (2) we have

$$||z_{n+1} - z_{n+2}|| \leq \alpha||z_{n-1} - z_n|| \leq \alpha^2||z_{n-2} - z_{n-1}||$$

by routine calculation

$$||z_n - z_{n+1}|| \leq \alpha^n||z_0 - z_1||$$

By routine calculation

$$||z_n - z_{n+1}|| \leq \alpha^n||z_0 - z_1||$$

(2) $||z_{n+1} - z_{n+2}|| \leq \delta||Ay_{n+1} - Ay_{n+2}|| + L||Ay_{n+1} - B y_{n+1}||$

$\leq \delta||z_{n-1} - z_n|| + L||z_{n-1} - z_n|| + L||z_{n-1} - z_n||$

$\leq (\delta + L)||z_{n-1} - z_n|| + L||z_{n-1} - z_n|| + L||z_{n-1} - z_n||$

$\leq (\delta + 2L)||z_{n-1} - z_n|| + L||z_{n-1} - z_n||$

(2) $||z_{n-1} - z_{n-2}|| \leq \delta||z_{n-2} - z_{n-1}|| + L||z_{n-2} - z_{n-1}||$

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$\leq (\delta + L)||z_{n-2} - z_{n-1}|| + L||z_{n-2} - z_{n-1}||$

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$\leq (\delta + 2L)||z_{n-2} - z_{n-1}|| + L||z_{n-2} - z_{n-1}||$
(3) \(|Bq - z_{n+1}|| \leq \delta||y - z_n|| + L||y - z_{n+1}||
(4) \)|Bq - z_{n+1}|| \leq \delta||y - z_n|| + L||z_n - Bq||
as \(n \to \infty\) it becomes
\(|Bq - y|| \leq L(||y - Bq||.||y - Bq||) \leq 0

\(|Bq - y|| - L||y - Bq||
then \(Bq = y\) hence \(Bq = y = Aq\) another word, \(q\) is a coincidence point of \(A\) and \(B\). If \(A\) and \(B\) are weakly compatible then they commute at coincidence point. then for \(Bq = y = Aq \Rightarrow ABq = BAq\)
for some \(q \in G\) that is \(By = Ay\)

Claim that \(X\) is common fixed point of \(A\) and \(B\), to show this, substitute \(y = q\) and \(z = Bq = z\) in the (condition \(B\)) to give

(1) \(|Bq - BBq|| \leq \delta||Aq - ABq|| + L||Aq - Bq||
(2) \)|Bq - BBq|| \leq \delta||Aq - ABq|| + L||ABq - BBq||
(3) \(|Bq - BBq|| \leq \delta||Aq - ABq|| + L||Aq - BBq||
(4) \)|Bq - BBq|| \leq \delta||Aq - ABq|| + L||ABq - Bq||
which is equivalent to (1) \(|x - Bx|| \leq \delta||x - Bx|| + L||x - x||
(2) \(|x - Bx|| \leq \delta||x - Bx|| + L||Ax - Bx||
(3) \)|x - Bx|| \leq \delta||x - Bx|| + L||x - Bx||
(4) \)|x - Bx|| \leq \delta||x - Bx|| + L||Ax - x||
so we have
\(|x - Bx|| \leq 0 then \(Bx = Ax\)
we use reduction and absurdum to prove uniqueness, suppose the contrary, that \(u\) is another common fixed point of \(A\) and \(B\) substituting \(y\) by \(x\) and \(z\) by \(u\) in the condition \(B\) we get
\(|Bx - Bu|| \leq \delta||Ax - Au|| + L||Ax - Bx||
\leq \delta||Ax - Au|| + L||Au - Bu||
\leq \delta||Ax - Au|| + L||Ax - Bu|| \leq \delta||Ax - Au|| + L||Au - Bx||
which is equivalent to
\(|x - u|| \leq \delta||x - u|| \Rightarrow ||x - u|| \leq 0
which is contradiction, therefore the common fixed point of \(A\) and \(B\) is unique

References