

Common Fixed Point in Cone Banach Space

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Abstract: In this paper we prove that if A and B are two mappings defined on a closed subset G of a cone Banach space $(X, \|\cdot\|_c)$ and A, B satisfied either the condition $\|Ax - By\| \leq a_1\|x - Ax\| + a_2\|y - By\| + a_3\|x - y\| + a_4\|x - By\|$ for all $x, y \in G$ or A and B satisfied condition B then A and B have common fixed point

Keywords: Condition B

1. Introduction

In (2007) Huang and Zhang introduced cone metric space by means of partial ordering (\leq) on real Banach space $(E, \|\cdot\|)$ also they proved some fixed point theorems, while [5] gave the definition of cone Banach space. In [3] Golubovic', Zorana and Kadelburg, Zoran and Radenovic', Stojan proved common fixed point theorem of weak contractive mapping in cone metric space while [2] Akbar Azam, Muhammad Arshad, and Ismat Beg, gave the sufficient conditions for existence of points of coincidence and common fixed point of three self mapping satisfying a contractive type conditions in cone metric space. To define the cone Banach space [4] define the partial ordering \leq with respect to p by $x \leq y$ if and only if $y - x \in p$. We shall write $x < y$ to indicate that $x \leq y$ but $x \neq y$, while $x \ll y$ will stand for $y - x \in \text{int}p$ (interior of p). The following definitions and result will be needed in the sequel.

Definition 1.1 Let E be a real Banach space. A subset p of E is called cone if

- (1) p is closed, nonempty and $p \neq \{0\}$
- (2) $a, b \in \mathbb{R}, a, b \geq 0$ and $x, y \in p$ imply $ax + by \in p$
- (3) $p \cap (-p) = \{0\}$

Definition 1.2 [4] Let X be a vector space over \mathbb{R} , suppose the mapping $\|\cdot\|_p: X \rightarrow E$ satisfies

- (1) $\|x\|_p > 0$ for all $x \in X$
- (2) $\|x\|_p = 0$ if and only if $x = 0$
- (3) $\|x + y\|_p \leq \|x\|_p + \|y\|_p$ for all $x, y \in X$
- (4) $\|kx\|_p = |k|\|x\|_p$ for all $k \in \mathbb{R}$

then $\|\cdot\|_p$ is called a cone norm on X and the pair $(X, \|\cdot\|_p)$ is called a cone normed space (CNS).

Definition 1.3 [4] Let $(X, \|\cdot\|_p)$ be a cone normed space $x \in X$ and $\{x_n\}_{n \geq 1}$ a sequence in X , then

- 1) $\{x_n\}_{n \geq 1}$ converge to x whenever for each $c \in E$ with $0 \ll c$ there is natural number n such that $\|x_n - x\|_p \ll c$ for all $n \in \mathbb{N}$ it is denoted by $\lim_{n \rightarrow \infty} x_n = x$ or $x_n \rightarrow x$
- 2) $\{x_n\}_{n \geq 1}$ is a Cauchy sequence whenever for every $c \in E$ with $0 \ll c$ there is a natural number n , such that $\|x_n - x_m\|_p \ll c$ for all $n, m \in \mathbb{N}$
- 3) $(X, \|\cdot\|_p)$ is complete cone normed space if every Cauchy sequence is convergent. Complete cone normed space will be called cone Banach space.

Definition 1.4 [1] Let A, B be self mapping on a cone metric space (X, d) a point $z \in X$ is called a common fixed point of A, B if $Az = z = Bz$ moreover a pair of self mapping (A, B) is called weakly compatible on X if they commute at their coincidence point, in other words

$$z \in X, Az = Bz \Leftrightarrow ABz = BAZ$$

Definition 1.5 Let $(X, \|\cdot\|)$ be a cone Banach space a map $B: X \rightarrow X$ is said to be satisfy condition B if there exists $0 < \delta < 1$ and $L > 0$ such that for all $x, y \in X$ we have

$$\|Bx - By\| < \delta\|x - y\| + L \text{ where } u \in \{\|x - Bx\|, \|y - By\|, \|x - By\|, \|y - Bx\|\}$$

2. Main Result

Now we give the first main result

Theorem 2.1 Let G be a closed subset of cone Banach space $(X, \|\cdot\|)$ and let A, B be a mapping on G into itself satisfying $\|Ax - By\| \leq a_1\|x - Ax\| + a_2\|y - By\| + a_3\|x - y\| + a_4\|x - By\|$ for all $x, y \in G$ where a_1, a_2, a_3, a_4 are non-negative real with $a_1, a_2, a_3 < 1$ then A and B have a unique common fixed point in G .

Proof. Let $z_0 \in G$ we define a sequence $\{z_n\}$ as follows

$$z_{2n+1} = Az_{2n}$$

$$z_{2n+2} = Bz_{2n+1} \quad n = 0, 1, 2, \dots$$

we have

$$\begin{aligned} \|z_{2n+1} - z_{2n}\| &= \|Az_{2n} - Bz_{2n-1}\| \\ &\leq a_1\|z_{2n} - Az_{2n}\| + a_2\|z_{2n-1} - Bz_{2n-1}\| + a_3\|z_{2n} - z_{2n-1}\| \\ &\quad + a_4\|z_{2n} - Bz_{2n-1}\| \\ &\leq a_1\|z_{2n} - z_{2n+1}\| + a_2\|z_{2n-1} - z_{2n}\| \\ &\quad + a_3\|z_{2n} - z_{2n-1}\| + a_4\|z_{2n} - z_{2n}\| \end{aligned}$$

$$\text{hence } \|z_{2n+1} - z_{2n}\| \leq \frac{a_2 + a_3}{1 - a_1} \|z_{2n} - z_{2n-1}\|$$

$$\text{let } m = \frac{a_2 + a_3}{(1 - a_1)} < 1$$

$$\text{we have } \|z_{2n+1} - z_{2n}\|^2 \leq m\|z_{2n} - z_{2n-1}\|$$

processing in this way

$$\|z_{2n+1} - z_{2n}\| \leq m^n \|z_1 - z_0\|$$

for any positive integer k we get

$$\begin{aligned} \|z_n - z_{n+k}\| &\leq \|z_n - z_{n+1}\| + \|z_{n+1} - z_{n+2}\| + \dots + \\ &\leq (m^n + m^{n+1} \\ &\quad + m^{n+2} + \dots + m^{n+k-1}) \|z_0 - z_1\| \end{aligned}$$

$$\|z_n - z_{n+k}\| \leq \frac{\alpha^n}{1-\alpha} \|z_0 - z_1\|$$

thus $\|z_n - z_{n+k}\| \rightarrow 0$ as $n \rightarrow \infty$ hence $\{z_n\}$ is Cauchy sequence in G so it is a Cauchy sequence in X but X is cone Banach space so $\{z_n\}$ converge to z but G is closed subset of X so $f \in G$ such that

$$z_n \rightarrow f$$

we have

$$\|f - Bf\| \leq a_1 \|z_{2n} - z_{2n+1}\| + a_2 \|f - Bf\|^2 + a_3 \|z_{2n} - f\|^2 + a_4 \|z_{2n} - Bf\|$$

as $n \rightarrow \infty, z_{2n} \rightarrow f, z_{2n+1} \rightarrow f$

we have

$$\|f - Bf\| \leq \|a_2\| \|f - Bf\| \text{ then implies that } f, Bf, \text{ since } a_2 < 1$$

similarly we get $f = Af$ then f is a Common fixed point of A and B let $p \neq f$ then $\|p - f\| = \|Af - Bp\|$

$a_1 \|f - Af\| + a_2 \|p - Bp\| + a_3 \|p - f\| + a_4 \|f - Bf\|$ since $a_1 + a_2 + a_3 < 1$ then $p = f$ that is Common fixed point is unique.

Theorem 2.2 Let G be a closed and convex subset of a cone Banach space $(X, \|\cdot\|)$ with a cone p and let A and B are self mapping, satisfy condition B and $B(G) \subseteq A(G), A(G)$ is complete subspace, then A and B have common coincidence point furthermore, if A and B are weakly compatible, then they have unique common fixed point in G

Proof. Let $y_0 \in G$ be arbitrary, by $(B(G) \subset A(G))$ we can fixed a point in G . say y_n such that $By_0 = Ay_1$ since A, B are self mapping there exists a point z_0 in G such that $z_0 = By_0 = Ay_1$ inductively we can define a sequence $\{z_n\}$ and a sequence $\{y_n\} \subset G$ in the following way

$$z_n = Ay_{n+1} = By_n$$

$$z_{n-1} = Ay_n = By_{n-1}$$

$$z_{n+1} = Ay_{n+2} = By_{n+1}$$

$$z_{n+2} = Ay_{n+3} = By_{n+2}$$

Let $\|z_n - z_{n+1}\| \leq \delta \|Ay_n - Ay_{n+1}\| + L \|U$ when

$u \in \{\|Ay_n - By_n\|, \|Ay_{n+1} - By_{n+1}\|, \|Ay_n - By_{n+1}\|, \|Ay_{n+1} - By_n\|\}$ then

$$\begin{aligned} (1) \|z_n - z_{n+1}\| &\leq \delta \|Ay_n - Ay_{n+1}\| + L \|Ay_n - By_n\| \\ &\leq \delta \|z_{n-1} - z_n\| = L \|z_{n-1} - z_n\| \\ &\leq (\delta + L) \|z_{n-1} - z_n\|, \dots \dots (1) \end{aligned}$$

$$\|z_{n-1} - z_n\| \leq \alpha \|z_{n-2} - z_{n-1}\|, \dots \dots (2)$$

where $\alpha = (\delta + L)$

take in (1) and (2) we have

$$\|z_n - z_{n+1}\| \leq \alpha \|z_{n-1} - z_n\| \leq \alpha^2 \|z_{n-2} - z_{n-1}\|$$

by routine calculation

$$\|z_n - z_{n+1}\| \leq \alpha^n \|z_0 - z_1\|$$

By routine calculation

$$\|z_n - z_{n+1}\| \leq \alpha^n \|z_0 - z_1\|$$

$$(2) \|z_n - z_{n+1}\| \leq \delta \|Ay_n - Ay_{n+1}\| + L \|Ay_{n+1} - By_{n+1}\|$$

$$\leq \delta \|z_{n-1} - z_n\| + L \|z_n - z_{n-1}\| + L \|z_{n-1} - z_{n+1}\|$$

$$\begin{aligned} &\leq (\delta + L) \|z_{n-1} - z_n\| + L \|z_{n-1} - z_n\| + L \|z_n - z_{n+1}\| \\ &\leq (\delta + 2L) \|z_{n-1} - z_n\| + L \|z_n - z_{n+1}\| \\ &\leq \frac{(\delta + 2L)}{1-L} \|z_{n-1} - z_n\|, \dots \dots (1) \end{aligned}$$

one can notice that

$$\|z_{n-1} - z_n\| \leq \alpha \|z_{n-2} - z_{n-1}\|, \dots \dots (2) \text{ where } \alpha = \frac{(\delta + 2L)}{(1 + L)}$$

take in (1) and (2) we have

$$\|z_n - z_{n+w}\| \leq K \|z_{n-1} - z_n\| \leq K^2 \|z_{n-2} - z_{n-1}\|$$

by routine calculation

$$\|z_n - z_{n+1}\| \leq \alpha^n \|z_0 - z_1\|$$

$$\begin{aligned} (3) \|z_n - z_{n+1}\| &\leq \delta \|Ay_n - Ay_{n+1}\| + L \|Ay_n - By_{n+1}\| \\ &\leq \delta \|z_{n-1} - z_n\| + L \|z_{n-1} - z_{n+1}\| \\ &\leq \delta \|z_{n-1} - z_n\| + L \|z_{n-1} - z_n\| + L \|z_n - z_{n+1}\| \\ &\leq (\delta + L) \|z_{n-1} - z_n\| + L \|z_n - z_{n+1}\| \end{aligned}$$

$$\|z_n - z_{n+1}\| \leq \frac{\delta + L}{1-L} \|z_{n-1} - z_n\|, \dots \dots (1)$$

one can notice that

$$\|z_{n-1} - z_n\| \leq \alpha \|z_{n-2} - z_{n-1}\|, \dots \dots (2) \text{ where } \alpha = \frac{\delta + L}{1-L}$$

take in (1) and (2) we have

$$\|z_n - z_{n-1}\| \leq \alpha \|z_{n-1} - z_n\| \leq \alpha^2 \|z_{n-2} - z_{n-1}\|$$

by routine calculations

$$\|z_n - z_{n+1}\| \leq \alpha^n \|z_0 - z_1\|$$

$$\begin{aligned} (4) \|z_n - z_{n+1}\| &\leq \delta \|Ay_n - Ay_{n+1}\| + L \|Ay_0 - By_n\| \\ &\leq \delta \|z_{n-1} - z_n\| + L \|z_{n-1} - z_n\| \\ &\leq \delta \|z_{n-1} - z_n\|, \dots \dots (1) \end{aligned}$$

one can notice that

$$\|z_{n-1} - z_n\| \leq \alpha \|z_{n-2} - z_{n-1}\|, \dots \dots (2) \text{ where } \alpha = \delta$$

take in (1) and (2) we have

$$\|z_n - z_{n-1}\| \leq \alpha \|z_{n-1} - z_n\| \leq \alpha^2 \|z_{n-2} - z_{n-1}\|$$

by routine calculations

$$\|z_n - z_{n+1}\| \leq \alpha^n \|z_0 - z_1\|$$

to show $\{z_n\}$ is Cauchy sequence let $n > m$ then by $\|z_n - z_{n+1}\| \leq \alpha^n \|z_0 - z_1\|$ and triangle inequality one can obtain

$$\begin{aligned} \|z_n - z_m\| &\leq \|z_n - z_{n-1}\| + \dots + \|z_{m+1} - z_m\| \\ &\leq \alpha^{n-1} \|z_0 - z_1\| + \dots + \alpha^m \|z_0 - z_1\| \\ &\leq \frac{\alpha^m}{1-\alpha} \|z_0 - z_1\| \end{aligned}$$

with concludes the proof that $\{z_n\}$ is Cauchy sequence since $A(z)$ is complete then $\{z_n = Ay_{n+1} = By_n\}$ converge to some point in $A(z)$, say y in other words there is point $q \in G$ such that $Aq = y$ now by replacing y with q and z with y_{n+1} in the condition B we get

$$(1) \|Bq - By_{n+1}\| \leq \delta \|Aq - Ay_{n+1}\| + L \|Aq - Bq\|$$

$$(2) \|Bq - By_{n+1}\| \leq \delta \|Aq - Ay_{n+1}\| + L \|Ay_{n+1} - By_{n+1}\|$$

$$(3) \|Bq - By_{n+1}\| \leq \delta \|Aq - Ay_{n+1}\| + L \|Aq - By_{n+1}\|$$

$$(4) \|Bq - By_{n+1}\| \leq \delta \|Aq - Ay_{n+1}\| + L \|Ay_{n+1} - Bq\|$$

which is equivalent to (1) $\|Bq - z_{n+1}\| \leq \delta \|y - z_n\| + L \|y - Bq\|$

$$(2) \|Bq - z_{n+1}\| \leq \delta \|y - z_n\| + L \|z_n - z_{n+1}\|$$

$$(3) \|Bq - z_{n+1}\| \leq \delta \|y - z_n\| + L \|y - z_{n+1}\|$$

$$(4) \|Bq - z_{n+1}\| \leq \delta \|y - z_n\| + L \|z_n - Bq\|$$

as $n \rightarrow \infty$ it becomes

$$\|Bq - y\| \leq L\{\|y - Bq\|, \|y - Bq\|\} \leq 0$$

$\|Bq - y\| - L\|y - Bq\|$
 then $Bq = y$ hence $Bq = y = Aq$ another word , q is a coincidence point of A and B . If A and B are weakly compatible then they commute at coincidence point . then for
 $Bq = y = Aq \Rightarrow ABq = BAq$
 for some $q \in G$ that is $By = Ay$

Claim that X is common fixed point of A and B , to show this, substitute $y = q$ and $z = Bq = z$ in the ((condition B)) to give

$$(1) \|Bq - BBq\| \leq \delta \|Aq - ABq\| + L \|Aq - Bq\|$$

$$(2) \|Bq - BBq\| \leq \delta \|Aq - ABq\| + L \|ABq - BBq\|$$

$$(3) \|Bq - BBq\| \leq \delta \|Aq - ABq\| + L \|Aq - BBq\|$$

$$(4) \|Bq - BBq\| \leq \delta \|Aq - ABq\| + L \|ABq - Bq\|$$

which is equivalent to (1) $\|x - Bx\|$
 $\leq \delta \|x - Bx\| + L \|x - x\|$
 (2) $\|x - Bx\| \leq \delta \|x - Bx\| + L \|Ax - Bx\|$
 (3) $\|x - Bx\| \leq \delta \|x - Bx\| + L \|x - Bx\|$
 (4) $\|x - Bx\| \leq \delta \|x - Bx\| + L \|Ax - x\|$

so we have
 $\|x - Bx\| \leq 0$ then $x = Bx = Ax$
 we use reduction and absurdum to prove uniqueness, suppose the contrary, that u is another common fixed point of A and B substituting y by x and z by u in the condition B we get

$$\|Bx - Bu\| \leq \delta \|Ax - Au\| + L \|Ax - Bx\|$$

$$\leq \delta \|Ax - Au\| + L \|Au - Bu\|$$

$$\leq \delta \|Ax - Au\| + L \|Ax - Bu\| \leq \delta \|Ax - Au\| + L \|Au - Bx\|$$

which is equivalent to
 $\|x - u\| \leq \delta \|x - u\| \Leftrightarrow \|x - u\| \leq 0$
 which is contradiction, therefore the common fixed point of A and B is unique

References

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