Common Fixed Point in Cone Banach Space

Buthainah A.A. Ahmed[1], **Essraa A.Hasan**[2]

Departmentof Mathematics -College of Science -University of Baghdad

Abistract: ***In this paper we prove that if A and B are two mappings defined on a closed subset G of a cone Banach space $(X,||.||_c)$ and A,B satisfied either the condition $||Ax - By| \leq a_1||x - Ax| + a_2||y - By| + a_3||x - y| + a_4||x - By|$ for all $x.y \in G$ or A and B satisfied condition B then A and B have common fixed point***

Keywords: Condition B

1. Introduction

In(2007) Huang and Zhang introduced cone metric space be means of partial ordering $((\leq))$ on real Banach space $(E,||.||)$ also they proved some fixed point theorems, while [5] gave the definition of cone Banach space . In [3]Golubovic′, Zorana and Kadelburg, Zoran and Radenovic′, Stojan proved common fixed point theorem of weak contractive mapping in cone metric space while [2] Akbar Azam, Muhammad Arshad, and Ismat Beg, gave the sufficient conditions for existence of points of coincidence and common fixed point of three self mapping satisfying a contractive type conditions in cone metric space To defined the cone Banach space [4] define the partial ordering $\leq$ with respect to p by $x \leq y$ if and only if $y - x \in p$. We shall write $x < y$ to indicate that $x \leq y$ but $x \neq y$, while $x \ll y$ will stand for $y - x \in intp$ (interior of p)
The following definitions and result will be needed in the sequel.

Definition 1.1 *Let E be a real Banach space. A subset p of E is called cone if*
(1)p is closed , nonempty and $p \neq \{0\}$
(2)$a, b \in R, a, b \geq 0\ andx, y \in p implyax + by \in p$
(3)$p \cap (-p) = \{0\}$

Definition 1.2 *[4] Let X be a vector space over $\mathbb{R}$, suppose the mapping $||.||_p: X \to E$satisfies*
(1) $||x||_p > 0$ for all $x \in X$
(2) $||x||_p = 0$ if and only if $x = 0$
(3) $||x + y||_p \leq ||x||_p + ||y||_p$ for all $x, y \in X$
(4) $||kx||_p = |k|||x||_p$ for all $k \in \mathbb{R}$
then $||.||_p$ is called a cone norm on X and the pair $(X,||.||_p)$ is called a cone normed space .(CNS).

Definition 1.3 *[4] Let $(X,||.||_p)$ be a cone normed space $x \in X$ and $\{x_n\}_{n\geq1}$ a sequence in X, then*
1) $\{x_n\}_{n\geq1}$ converge to x whenever for each $c \in E$ with $0 \ll c$ there is natural number n such that $||x_n \to x||_p \ll c$ for all $n \in N$ it is denoted by $\lim_{n\to\infty} x_n = xorx_n \to x$
2) $\{x_n\}_{n\geq1}$ is a Cauchy sequence whenever for every $c \in E$ with $0 \ll c$ there is a natural number n, such that $||x_n - x_m||_p \ll c$ for all $n, m \in N$
3) $(X,||.||_p)$ is complete cone normed space if every Cauchy sequence is convergent. Complete cone normed space will be called cone Banach space.

Definition 1.4 [1] Let A, B be self mapping on a cone metric space (X, d) a point $z \in X$ is called a common fixed point of A, B if $Az = z = Bz$ moreover a pair of self mapping (A, B) is called weakly compatible on X if they commute at their coincidence point , in other words
$z \in X, Az = Bz \Leftarrow ABz = BAz$

Definition 1.5 Let $(X,||.||)$ be a cone Banach space a map $B: X \to X$ is said to be satisfy condition B if there exists $0 < \delta < 1$ and $L > 0$such that for all $x, y \in X$ we have
$$||Bx - By|| < \delta||x - y|| + L.u where u \in \{||x - Bx||, ||y - By||, ||x - By||, ||y - Bx||\}$$

2. Main Result

Now we give the first main result

Theorem 2.1 *Let G be a closed subset of cone Banach space $(X.||.||)$ and let A, B be a mapping on G into it self satisfying $||Ax - By|| \leq a_1||x - Ax|| + a_2||y - By|| + a_3||x - y|| + a_4||x - By||$ for all $x.y \in G$ where a_1, a_2, a_3, a_4 are non-negative real with $a_1, a_2, a_3 < 1$ then A and B have a unique common fixed point in G.*

Proof. Let $z_0 \in G$ we define a sequence $\{z_n\}$ as follows
$z_{2n+1} = Az_{2n}$

$z_{2n+2} = Bz_{2n+1} n = 0,1,2, ..$
we have
$$||z_{2n+1} - z_{2n}|| = ||Az_{2n} - Bz_{2n-1}|| \leq a_1||z_{2n} - Az_{2n}|| + a_2||z_{2n-1} - Bz_{2n-1}|| + a_3||z_{2n} - z_{2n-1}||2 + a_4||z_{2n} - Bz_{2n-1}|| \leq a_1||z_{2n} - z_{2n+1}|| + a_2||z_{2n-1} - z_{2n}|| + a_3||z_{2n} - z_{2n-1}|| + a_4||z_{2n} - z_{2n}||$$

hence $||z_{2n+1} - z_{2n}|| \leq \frac{a_2+a_3}{1-a_1}||z_{2n} - z_{2n-1}||$

let $m = \frac{a_2+a_3}{(1-a_1)} < 1$

we have $||z_{2n+1} - z_{2n}||^2 \leq m||z_{2n} - z_{2n-1}||$

processing in this way
$$||z_{2n+1} - z_{2n}|| \leq m^n||z_1 - z_0||$$
for any positive integer k we get

$$||z_n - z_{n+k}|| \le ||z_n - z_{n+1}|| + ||z_{n+1} - z_{n+2}|| + \ldots + \le (m^n + m^{n+1} + m^{n+2} + \ldots + m^{n+k-1})||z_0 - z_1||$$

$||z_n - z_{n+k}|| \le \frac{\alpha^n}{1-\alpha}||z_0 - z_1||$

thus $||z_n - z_{n+k}|| \to 0\ as n \to \infty$ hence $\{z_n\}$ is Cauchy sequence in G so it is a Cauchy sequence in X but X is cone Banach space so $\{z_n\}$ converge to z but G is closed subset of X so $f \in G$ such that

$z_n \to f$

we have

$$||f - Bf|| \le a_1||z_{2n} - z_{2n+1}|| + a_2||f - Bf||^2 + a_3||z_{2n} - f||^2 + a_4||z_{2n} - Bf||$$

as $n \to \infty, z_{2n} \to f, z_{2n+1} \to f$

we have

$||f - Bf|| \le ||a_2||f - Bf||$ then implies that f, Bf, since $a_2 < 1$

similarly we get $f = Af$ then f is a Common fixed point of A and B let $p \ne f$ then $||p - f|| = ||Af - Bp||$

$a_1||f - Af|| + a_2||p - Bp|| + a_3||p - f|| + a_4||f - Bf||$

since $a_1 + a_2 + a_3 < 1$ then $p = f$ that is Common fixed point is unique.

Theorem 2.2 LetG be a closed and convex subset of a cone Banach space (X, ||.||) with a cone p and let A and B are self mapping , satisfy condition B and B(G) ⊆ A(G), A(G) is complete subspace, then A and B have common coincidence point furthermore, if A and B are weakly compatible, then they have unique common fixed point in G

Proof. Let $y_0 \in G$ be arbitrary ,by $(B(G) \subset A(G))$we can fixed a point in G . say y_n such that $By_0 = Ay_1$ since A, B are self mapping there exists a point z_0 in G such that $z_0 = By_0 = Ay_1$ inductively we can define a sequence $\{z_n\}$ and a sequence $\{y_n\} \subset G$ in the following way

$z_n = Ay_{n+1} = By_n$

$z_{n-1} = Ay_n = By_{n-1}$

$z_{n+1} = Ay_{n+2} = By_{n+1}$

$z_{n+2} = Ax_{n+3} = By_{n+2}$

Let $||z_n - z_{n+1}|| \le \delta||Ay_n - Ay_{n+1}|| + L.U$ when $u \in \{||Ay_n - By_n||, ||Ay_{n+1} - By_{n+1}||, ||Ay_n - By_{n+1}||, ||Ay_{n+1} - By_n||$

then

(1) $||z_n - z_{n+1}|| \le \delta||Ay_n - Ay_{n+1}|| + L||Ay_n - By_n||$
$\le \delta||z_{n-1} - z_n|| = L||z_{n-1} - z_n||$
$\le (\delta + L)||z_{n-1} - z_n|| \ldots\ldots\ldots (1)$
$||z_{n-1} - z_n|| \le \alpha||z_{n-2} - z_{n-1}|| \ldots\ldots\ldots\ldots\ldots (2)$

where $\alpha = (\delta + l)$

take in (1) and (2) we have

$||z_n - z_{n+1}|| \le \alpha||z_{n-1} - z_n|| \le \alpha^2||z_{n-2} - z_{n-1}||$

by routine calculation

$||z_n - z_{n+1}|| \le \alpha^n||z_0 - z_1||$

By routine calculation

$$||z_n - z_{n+1}|| \le \alpha^n||z_0 - z_1||$$

(2) $||z_n - z_{n+1}|| \le \delta||Ay_n - Ay_{n+1}|| + L||Ay_{n+1} - By_{n+1}||$

$$\le \delta||z_{n-1} - z_n|| + L||z_n - z_{n-1}|| + L||z_{n-1} - z_{n+1}||$$
$$\le (\delta + L)||z_{n-1} - z_n|| + L||z_{n-1} - z_n|| + L||z_n - z_{n+1}||$$
$$\le (\delta + 2L)||z_{n-1} - z_n|| + L||z_n - z_{n-1}||$$
$$\le \frac{(\delta + 2L)}{1 - L}||z_{n-1} - z_n|| \ldots\ldots\ldots (1)$$

onecannoticethat

$$||z_{n-1} - z_n|| \le \alpha||z_{n-2} - z_{n-1}|| \ldots\ldots\ldots (2) \text{ where}\alpha = \frac{(\delta + 2L)}{(1 + L)}$$

take in (1) and (2) we have

$||z_n - z_{n+w1}|| \le K||z_{n-1} - z_n|| \le K^2||z_{n-2} - z_{n-1}||$

by routine calculation

$$||z_n - z_{n+1}|| \le \alpha^n||z_0 - z_1||$$

(3) $||z_n - z_{n+1} \le \delta||Ay_n - Ay_{n+1}|| + L||Ay_n - By_{n+1}||$
$\le \delta||z_{n-1} - z_n|| + L||z_{n-1} - z_{n+1}||$
$\le \delta||z_{n-1} - z_n|| + L||z_{n-1} - z_n|| + L||z_n - z_{n-+}||$
$\le (\delta + L)||z_{n-1} - z_n|| + L||z_n - z_{n+1}||$

$$||z_n - z_{n+1}|| \le \frac{\delta + L}{1 - L}||z_{n-1} - z_n|| \ldots\ldots (1)$$

onecannoticethat

$$||z_{n-1} - z_n|| \le \alpha||z_{n-2} - z_{n-1}|| \ldots\ldots\ldots (2)\text{where}\alpha = \frac{\delta + L}{1 - L}$$

take in (1)and (2) we have

$||z_n - z_{n-1}|| \le \alpha||z_{n-1} - z_n|| \le \alpha^2||z_{n-2} - n_{n-1}||$

by routine calculations

$||z_n - z_{n+1}|| \le \alpha^n||z_0 - z_1||$

(4) $||z_n - z_{n+1}|| \le \delta||Ay_n - Ay_{n+1}|| + L||Ay_0 - By_n||$
$\le \delta||z_{n-1} - z_n|| + L||z_n - z_n||$
$\le \delta||z_{n-1} - z_n|| \ldots\ldots (1)$

onecannoticethat

$||z_{n-1} - z_n|| \le \alpha||z_{n-2} - z_{n-1}|| \ldots\ldots\ldots (2)\text{where}\alpha = \delta$

take in (1)and(2) we have

$||z_n - z_{n-1}|| \le \alpha||z_{n-1} - z_n|| \le \alpha^2||z_{n-2} - z_{n-1}||$

by routine calculations

$||z_n - z_{n+1}|| \le \alpha^n||z_0 - z_1||$

to show $\{z_n\}$ is Cauchy sequence let $n > m$ then by $[||z_n - z_{n+1}|| \le \alpha^n||z_0 - z_1||$ and triangle inequality one can obtain

$||z_n - z_m|| \le ||z_n - z_{n-1}|| =$
$||z_{n-1} - z_{n-2}|| + \ldots + ||z_{m+1} - z_m||$
$\le \alpha^{n-1}||z_0 - z_1|| = \alpha^{n-2}||z_0 - z_1|| + \ldots + \alpha^m||z_0 - z_1||$
$\le \frac{\alpha^m}{1-\alpha}||z_0 - z_1||$

with concludes the proof that $\{z_n\}$ is Cauchy sequence since $A(z)$ is complete then $\{z_n = Ay_{n+1} = By_n\}$ converge to some point in $A(z)$, say y in other words there is point $q \in G$ such that $Aq = y$ now by replacing y with q and z with y_{n+1} in the condition B we get

(1) $||Bq - By_{n+1}|| \le \delta||Aq - Ay_{n+1}|| + L||Aq - Bq||$
(2) $||Bq - By_{n+1}|| \le \delta||Aq - Ay_{n+1}|| + L||Ay_{n+1} - By_{n+1}||$
(3) $||Bq - By_{n+1}|| \le \delta||Aq - Ay_{n+1}|| + L||Aq - By_{n+1}||$
(4) $||Bq - By_{n+1}|| \le \delta||Aq - Ay_{n+1}|| + L||Ay_{n+1} - Bq||$

whichisequivalentto(1) $||Bq - z_{n+1}|| \le \delta||y - z_n|| + L||y - Bq||$
(2) $||Bq - z_{n+1}|| \le \delta||y - z_n|| + L||z_n - z_{n+1}||$

(3) $||Bq - z_{n+1}|| \leq \delta||y - z_n|| + L||y - z_{n+1}||$
(4) $||Bq - z_{n+1}|| \leq \delta||y - z_n|| + L||z_n - Bq||$
as $n \to \infty$ it becomes
$||Bq - y|| \leq L\{||y - Bq||, ||y - Bq||\} \leq 0$

$||Bq - y|| - L||y - Bq||$
then $Bq = y$ hence $Bq = y = Aq$ another word , q is a coincidence point of A and B. If A and B are weakly compatible then they commute at coincidence point . then for
$Bq = y = Aq \Rightarrow ABq = BAq$
for some $q \in G$ that is $By = Ay$

Claim that X is common fixed point of A and B, to show this, substitute $y = q$ and $z = Bq = z$ in the ((condition B)) to give
(1) $||Bq - BBq|| \leq \delta||Aq - ABq|| + L||Aq - Bq||$
(2) $||Bq - BBq|| \leq \delta||Aq - ABq|| + L||ABq - BBq||$
(3) $||Bq - BBq|| \leq \delta||Aq - ABq|| + L||Aq - BBq||$
(4) $||Bq - BBq|| \leq \delta||Aq - ABq|| + L||ABq - Bq||$
whichisequivalentto(1) $||x - Bx|| \leq \delta||x - Bx|| + L||x - x||$
(2) $||x - Bx|| \leq \delta||x - Bx|| + L||Ax - Bx||$
(3) $||x - Bx|| \leq \delta||x - Bx|| + L||x - Bx||$
(4) $||x - Bx|| \leq \delta||x - Bx|| + L||Ax - x||$
so we have
$||x - Bx|| \leq 0 \; then x = Bx = Ax$
we use reduction and absurdum to prove uniqueness, suppose the contrary, that u is another common fixed point of A and B substituting y by x and z by u in the contrition B we get

$||Bx - Bu|| \leq \delta||Ax - Au|| + L||Ax - Bx||$
$\leq \delta||Ax - Au|| + L||Au - Bu||$
$\leq \delta||Ax - Au|| + L||Ax - Bu|| \leq \delta||Ax - Au|| + L||Au - Bx|$
which is equivalent to
$||x - u|| \leq \delta||x - u|| \Leftarrow ||x - u|| \leq 0$
which is contradiction, therefore the common fixed point of A and B is unique

References

[1] Thabet Abdeljawad, Erdal Karapinar, and Kenan Tas, *Common fixed point theorems in cone banach spaces*, Hacet. J. Math. Stat **40** (2011), no. 2, 211–217.
[2] Akbar Azam, Muhammad Arshad, and Ismat Beg, *Common fixed point theorems in cone metric spaces.*, Journal of Nonlinear Sciences & Applications (JNSA) **2** (2009), no. 4.
[3] Zorana Golubovi c′ , Zoran Kadelburg, and Stojan Radenovic′, *Common fixed points of weak contractions in cone metric spaces*, Abstract and Applied Analysis, vol. 2012, Hindawi Publishing Corporation, 2012, p. 18.
[4] Erdal Karapnar and Juan Jose Nieto, *Fixed point theorems in cone banach spaces*, Fixed Point Theory and Applications **2009** (2009), 18.
[5] Duran Turkoglu, Muhib Abuloha, and Thabet Abdeljawad, *Some theorems in cone metric spaces*, Journal of Concrete and Applicable matheatics **10** (2012), no. 1-2, 106–116.