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On The Homogeneous Bi-quadratic Equation with Five Unknowns $x^4 - y^4 = 37 (z^2 - w^2)T^2$

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Abstract: ***The Bi-quadratic equation with 5 unknowns given by $x^4 - y^4 = 37 (z^2 - w^2)T^2$ is analyzed for its patterns of non-zero distinct integral solutions. A few interesting relations between the solutions and special polygonal numbers are exhibited.***

Keywords: Bi- Quadratic equation, Integral solutions, Special polygonal numbers, Pyramidal numbers.

1. Introduction

Bi-quadratic Diophantine Equations (homogeneous and non-homogeneous) have aroused the interest of numerous mathematicians since ambiguity as can be seen from [1-7]. In the context one may refer [8-20] for varieties of problems on the Diophantine equations with two, three and four variables. This communication concerns with the problems of determining non-zero integral solutions of yet another bi-quadratic equation in 5 unknowns represented by $x^4 - y^4 = 37 (z^2 - w^2)T^2$. A few interesting relations between the solutions and special polygonal numbers are presented.

Notations used

- $T_{m,n}$ -Polygonal number of rank n with size m.
- Pr_n - Pronic number of rank n.
- SO_n -Stella Octangular number of rank n.
- Obl_n -Oblong number of rank n.
- OH_n - Octahedral number of rank n.
- GnO_n -Gnomic number of rank n.
- PP_n - Pentagonal Pyramidal number of rank n

2. Method of Analysis

The Diophantine equation representing the bi-quadratic equation with five unknowns under consideration is

$$x^4 - y^4 = 37\left(z^2 - w^2\right)T^2 \quad (1)$$

The substitution of the linear transformations

$$x = u + v,\ y = u - v,\ z = 2u + v,\ w = 2u - v \quad (2)$$

in (1) leads to $u^2 + v^2 = 37T^2$ (3)

Different patterns of solutions of (1) are presented below

Pattern-1

Equation (3) can be written as

$$\frac{u+T}{6T+v} = \frac{6T-v}{u-T} = \frac{a}{b} \quad (4)$$

From equation (4),we get the values of u, v and T

$$u = -a^2 + b^2 - 12ab$$
$$v = 6a^2 - 6b^2 - 2ab$$
$$T = -(a^2 + b^2)$$

Hence in view of (2) the corresponding solutions of (1) are

$$x = x(a,b) = 5a^2 - 5b^2 - 14ab$$
$$y = y(a,b) = -7a^2 + 7b^2 - 10ab$$
$$z = z(a,b) = 4a^2 - 4b^2 - 26ab$$
$$w = w(a,b) = -8a^2 + 8b^2 - 22ab$$
$$T = -(a^2 + b^2)$$

Properties

1) $x(a,2a-1) - z(a,2a-1) + T(a,2a-1) - 2T_{17,a} - 2GnO_n - \Pr_a \equiv 0 \pmod 4$
2) $y(a,4a-3) - w(a,4a-3) - T(a,\ 4a-3) - 12T_{10,a} + T_{62,a} - 32\Pr_a \equiv 0 \pmod{61}$
3) $x(11b-9,b) - y(11b-9,b) - 3z(11b-9,b) - 148T_{13,b} \equiv 0$
4) $w(6b-5,b) - 8T(6b-5,b) + 22\ T_{14,n} - 16Obl_b \equiv 0 \pmod{16}$

Pattern-2

Assume $T = T(a,b) = a^2 + b^2$ where a and b are non-zero distinct integers. (5)

write 37 as $37 = (1+6i)\ (1-6i)$ (6)

using (5) & (6) in(3) and employing the method of factorization, define

$$u + iv = (1+6i)(a+ib)^2$$

Equating the real and imaginary parts, we get

$$u = u(a,b) = a^2 - b^2 - 12ab$$
$$v = v(a,b) = 6a^2 - 6b^2 + 2ab$$

Hence in view of (2) the corresponding solutions of (1) are

$$x = x(a,b) = 7a^2 - 7b^2 - 10ab$$
$$y = y(a,b) = -5a^2 + 5b^2 - 14ab$$
$$z = z(a,b) = 8a^2 - 8b^2 - 22ab$$
$$w = w(a,b) = 4a^2 - 4b^2 - 26ab$$
$$T = a^2 + b^2$$

Properties

1) $x(2b^2+1,b) - z(2b^2+1,b) + T(2b^2+1,b) - t_{26,b} + t_{14,b} + t_{10,b} - 36OH_b \equiv 0 \pmod{3}$

2) $w(a,a(a+1)) - y(a,a(a+1)) + T(a,a(a+1)) - 32\Pr_a + t_{62,a} + 24PP_a \equiv 0 \pmod{61}$

3) $z(a,2a^2-1) + 2w(a,2a^2-1) + 74SO_a \equiv 0$

4) $z(a,2a^2+1) + 2w(a,2a^2+1) + 222OH_a \equiv 0$

5) $z(a,12a-11) + 2w(a,12a-11) + 74t_{26,a} \equiv 0$

6) $5x(1,b) + 7y(1,b) + GnO_b \equiv -1 \pmod{146}$

7) $5x(b,b+1) + 7y(b,b+1) + 148\Pr_b \equiv 0$

8) $5x(6b-5,b) + 7y(6b-5,b) + 148t_{14,b} \equiv 0$

9) *Each of the following expression represents a nasty number*

i) $x(a,a) + y(a,a)$

ii) $z(b,b) + w(b,b)$

iii) $3T(a,a)$

Pattern-3

Instead of (5) write 37 as

$$37 = (6+i)(6-i) \qquad (7)$$

Following a similar procedure as in pattern-2, the solutions for (3) are as follows

$$\left.\begin{aligned} u = u(a,b) &= 6a^2 - 6b^2 - 2ab \\ v = v(a,b) &= a^2 - b^2 + 12ab \end{aligned}\right\} \qquad (8)$$

In view of (2) and (8) the solutions of (1) are obtained as

$$x = x(a,b) = 7a^2 - 7b^2 + 10ab$$
$$y = y(a,b) = 5a^2 - 5b^2 - 14ab$$
$$z = z(a,b) = 13a^2 - 13b^2 + 8ab$$
$$w = w(a,b) = 11a^2 - 11b^2 - 16ab$$
$$T = a^2 + b^2$$

Properties

1) $x(8b-7,b) + y(8b-7,b) - z(8b-7,b) + 12t_{18,b} - 47\Pr_b + t_{92,b} \equiv 0 \pmod{91}$

2) $x(a,2a^2-1) + y(a,2a^2-1) - w(a,2a^2-1) + T(a,2a^2-1) - 2t_{17,a} + t_{28,a} - 12SO_a \equiv 0 \pmod{1}$

3) $x(a,1) + y(a,1) - t_{18,a} - t_{10,a} \equiv -12 \pmod{6}$

4) $z(a,-a) + w(a,-a) - 2t_{17,a} + t_{16,a} \equiv 0 \pmod{7}$

Pattern-4

Consider (3) as

$$u^2 + v^2 = 37T^2 * 1 \qquad (9)$$

write 1 as

$$1 = \frac{\{(m^2-n^2) + 2imn\}\{(m^2-n^2) - 2imn\}}{(m^2+n^2)^2} \qquad (10)$$

Substituting (5) and (10) in (9) and employing method of factorization, define

$$u + iv = \frac{(m^2-n^2+2imn)(1+6i)(a+ib)^2}{m^2+n^2}$$

Equating the real and imaginary parts in the above equation, we get

$$u = \frac{(m^2-n^2)(a^2-b^2-12ab) - 2mn(6a^2-6b^2+2ab)}{m^2+n^2}$$

$$v = \frac{(m^2-n^2)(6a^2-6b^2+2ab) + 2mn(a^2-b^2-12ab)}{m^2+n^2}$$

Replacing 'a' by $(m^2+n^2)A$ and 'b' by $(m^2+n^2)B$ in the above equation, we get

$$u = (m^2+n^2)\,[(m^2-n^2)(A^2-B^2-12AB) - 2mn(6A^2-6B^2+2AB)]$$
$$v = (m^2+n^2)\,[(m^2-n^2)(6A^2-6B^2+2AB) + 2mn(A^2-B^2-12AB)]$$

The corresponding integer solutions of (1) are given by

$$x=(m^2+n^2)\,[(m^2-n^2)(7A^2-7B^2-10AB)+2mn(-5A^2+5B^2-14AB)]$$
$$y=(m^2+n^2)\,[(m^2-n^2)(-5A^2+5B^2-14AB)-2mn\,(7A^2-7B^2-10AB)]$$
$$z=(m^2+n^2)\,[(m^2-n^2)(8A^2-8B^2-22AB)+2mn(-11A^2+11B^2-16AB)]$$
$$w=(m^2+n^2)\,[(m^2-n^2)(-4A^2+4B^2-26AB)-2mn\,(13A^2-13B^2-8AB)]$$
$$T=(m^2+n^2)^2\,[A^2+B^2]$$

For simplicity and clear understanding, taking m=2, n=1 in the above equations, the corresponding integer solutions of (1) are given by

$$x(A,B,2,1)=5A^2-5B^2-430AB$$
$$y(A,B,2,1)=-215A^2+215B^2-10AB$$
$$z(A,B,2,1)=-100A^2+100B^2-650AB$$
$$w(A,B,2,1)=-320A^2+320B^2-230AB$$
$$T(A,B)=25A^2-25B^2$$

Properties

$$1)\ 43x(A,A+1,2,1)-y(A,A+1,2,1)+1859\Pr_A\equiv 0$$
$$2)\ 32z(2B^2-1,B,2,1)-10w(2B^2-1,B,2,1)+18500SO_B\equiv 0$$
$$3)\ 5x(B(B+1),B,2,1)+T(B(B+1),B,2,1)-2t_{17,B}-t_{22,B}-t_{52,B}+4300PP_B\equiv 0 \pmod{46}$$
$$4)\ x(A,1,2,1)-y(A,1,2,1)-t_{472,A}+2t_{17,A}\equiv -220 \pmod{196}$$
$$5)\ z(A,1,2,1)+t_{180,A}+2t_{13,A}\equiv 0 \pmod{747}$$

Pattern-5

Substituting (5) and (10) in (9) and employing method of factorization, define

$$u+iv=\frac{(m^2-n^2+2imn)(6+i)(a+ib)^2}{m^2+n^2}$$

Equating the real and imaginary parts in the above equation, we get

$$u=\frac{(m^2-n^2)(6a^2-6b^2-2ab)-2mn(a^2-b^2+12ab)}{m^2+n^2}$$
$$v=\frac{(m^2-n^2)(a^2-b^2+12ab)+2mn(6a^2-6b^2-2ab)}{m^2+n^2}$$

Replacing 'a' by $(m^2+n^2)A$ and 'b' by $(m^2+n^2)B$ in the above equation, we get

$$u=(m^2+n^2)\,[(m^2-n^2)(6A^2-6B^2-2AB)-2mn(A^2-B^2+12AB)]$$
$$v=(m^2+n^2)\,[(m^2-n^2)(A^2-B^2+12AB)+2mn(6A^2-6B^2-2AB)]$$

The corresponding integer solutions of (1) are given by

$$x=(m^2+n^2)\,[(m^2-n^2)(7A^2-7B^2+10AB)+2mn(5A^2-5B^2-14AB)]$$
$$y=(m^2+n^2)\,[(m^2-n^2)(5A^2-5B^2-14AB)-2mn\,(7A^2-7B^2+10AB)]$$
$$z=(m^2+n^2)\,[(m^2-n^2)(13A^2-13B^2+8AB)+2mn(4A^2-4B^2-26AB)]$$
$$w=(m^2+n^2)\,[(m^2-n^2)(11A^2-11B^2-16AB)-2mn\,(8A^2-8B^2+22AB)]$$
$$T=(m^2+n^2)^2\,[A^2+B^2]$$

For simplicity and clear understanding, taking m=2, n=1 in the above equations, the corresponding integer solutions of (1) are given by

$$x(A,B,2,1)=205A^2-205B^2-130AB$$
$$y(A,B,2,1)=-65A^2+65B^2-410AB$$
$$z(A,B,2,1)=275A^2-275B^2-400AB$$
$$w(A,B,2,1)=5A^2-5B^2+28AB$$
$$T(A,B)=25A^2-+25B^2$$

Properties

1) $x(A,2A^2-1,2,1)-y(A,2A^2-1,2,1)-z(A,2A^2-1,2,1)+w(A,2A^2-1,2,1)-708SO_A \equiv 0$

2) $y(B+1,B,2,1)-13w(B+1,B,2,1)+92PP_B \equiv 0$

3) $5w(A,A+1,2,1)+T(A,A+1)-140\Pr_A - t_{72,A}+2t_{17,A} \equiv 0 \pmod 7$

4) $28w(8B-7,B,2,1)-x(8B-7,B,2,1)-y(8B-7,B,2,1)-1324t_{18,B} \equiv 0$

5) $y(A,-1,2,1)+t_{102,A}+2t_{17,A} \equiv -65 \pmod{1348} 9$

Pattern-6

Rewrite (3) as $37T^2-v^2=u^2*1$ (11)

Assume $u=37a^2-b^2$ (12)

Write 1as $1=(\sqrt{37}-6)(\sqrt{37}+6)$ (13)

Using (12) and (13) in (11) and employing the method of factorization, we write

$\sqrt{37}T+v=(\sqrt{37}+6)(\sqrt{37}a+b)^2$

Equating the rational and irrational parts, we have

$v=v(a,b)=222a^2+6b^2+74ab$

$T=T(a,b)=37a^2+b^2+12ab$

(14)

In view of (2) & (14), the solutions of (1) are obtained as

$x=x(a,b)=259a^2+5b^2+74ab$

$y=y(a,b)=-185a^2-7b^2-74ab$

$z=z(a,b)=296a^2+4b^2+74ab$

$w=w(a,b)=-148a^2-8b^2-74ab$

$T=T(a,b)=37a^2+b^2+12ab$

Properties

1) $x(b+1,b)-7T(b+1,b)+10\Pr_b+38Obl_b-t_{74,b} \equiv 0 \pmod{73}$

2) $y(2b^2-1,b)+z(2b^2-1,b)+w(2b^2-1,b)+T(2b^2-1,b)+62SO_b+t_{18,b}+2\Pr_b \equiv 0 \pmod 5$

3) $y(2b^2+1,b)+5T(2b^2+1,b)+42OH_b-36Obl_b+t_{78,b} \equiv 0 \pmod{74}$

4) $w(a,a(a+1))+8T(a,a(a+1))44PP_a-t_{298,a} \equiv 0 \pmod{147}$

Pattern-7

Introduction of the linear transformations

$v=X+37R \quad T=X+R \quad u=6U$ (15)

In (3) leads to $X^2=37R^2+U^2$ which is satisfied by

$X=r^2+37s^2$

$R=2rs$

$U=r^2-37s^2$

Substituting the above values of X, U and R in (15), the corresponding non-zero distinct integral solutions of(3) are given by

$v=v(r,s)=r^2+37s^2+74rs$

$u=u(r,s)=6r^2-222s^2$

$T=T(r,s)=r^2+37s^2+2rs$

Thus the corresponding solutions of (1) are found to be

$x=x(r,s)=7r^2-185s^2+74rs$

$y=y(r,s)=5r^2-259s^2-74rs$

$z=z(r,s)=13r^2-407s^2+74rs$

$w=w(r,s)=11r^2-481s^2+2rs$

$T=T(r,s)=r^2+37s^2+74rs$

3. Properties

1) $x(r,11r-9)+y(r,11r-9)-z(r,11r-9)+T(r,11r-9)+144t_{13,r}\equiv 0$

2) $x(2s^2-1,s)+y(2s^2-1,s)-w(2s^2-1,s)-T(2s^2-1,s)-72SO_s\equiv 0$

3) $w(s+1,s)-11T(s+1,s)+t_{1778,s}+98\Pr_s\equiv 0(\bmod 887)$

4) $x(2s^2-1,s)-7T(2s^2-1,s)-60SO_s+t_{890,s}\equiv(\bmod 443)$

5) $y(s+1,s)+7T(s+1,s)+60Obl_s-t_{26,s}\equiv 0(\bmod 11)$

4. Conclusion

To conclude, one may search for other patterns of solutions and their corresponding properties.

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