On The Homogeneous Bi-quadratic Equation with Five Unknowns \( x^4 - y^4 = 37 (z^2-w^2)T^2 \)

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Abstract: The Bi-quadratic equation with 5 unknowns given by \( x^4 - y^4 = 37 (z^2-w^2)T^2 \) is analyzed for its patterns of non-zero distinct integral solutions. A few interesting relations between the solutions and special polygonal numbers are exhibited.

Keywords: Bi-Quadratic equation, Integral solutions, Special polygonal numbers, Pyramidal numbers.

1. Introduction

Bi-quadratic Diophantine Equations (homogeneous and non-homogeneous) have aroused the interest of numerous mathematicians since ambiguity as can be seen from [1-7]. In the context one may refer [8-20] for varieties of problems on the Diophantine equations with two, three and four variables. This communication concerns on the Diophantine equations with two, three and four homogeneous\(^\text{1}\). A few interesting relations between the solutions and special polygonal numbers are exhibited.

Notations used
- \( T_{m,n} \) - Polynomial number of rank n with size m.
- \( Pr_n \) - Pronic number of rank n.
- \( SO_n \) - Stella Octagonal number of rank n.
- \( Obl_n \) - Oblong number of rank n.
- \( OH_n \) - Octahedral number of rank n.
- \( GnO_n \) - Gnomic number of rank n.
- \( PP_n \) - Pentagonal Pyramidal number of rank n

2. Method of Analysis

The Diophantine equation representing the bi-quadratic equation with five unknowns under consideration is

\[ x^4 - y^4 = 37 \left(z^2-w^2\right)T^2 \]  

(1)

Properties
1) \( x(a,2a-1) - z(a,2a-1) + T(a,2a-1) - 2T_{17,a} - 2 GnO_n - Pr_n \equiv 0 \) (mod 4)
2) \( y(a,4a-3) - w(a,4a-3) - T(a,4a-3) - 12T_{10,a} + T_{62,a} - 32 Pr_n \equiv 0 \) (mod 61)
3) \( x(11b-9,b) - y(11b-9,b) - 3z(11b-9,b) - 148T_{13,b} \equiv 0 \)
4) \( w(6b-5,b) - 8T(6b-5,b) + 22 T_{14,n} - 16 Obl_b \equiv 0 \) (mod 16)

Pattern-2
Assume \( T = T(a,b) = a^2 + b^2 \) where a and b are non-zero distinct integers.

\[ u + iv = (1 + 6i)(a + ib)^2 \]

(5)

Equating the real and imaginary parts, we get

\[ u = u(a,b) = a^2 - b^2 - 12ab \]

\[ v = v(a,b) = 6a^2 - 6b^2 + 2ab \]

\[ u + iv \]
Hence in view of (2) the corresponding solutions of (1) are
\[ x = x(a,b) = 7a^2 - 7b^2 - 10ab \]
\[ y = y(a,b) = -5a^2 + 5b^2 - 14ab \]
\[ z = z(a,b) = 8a^2 - 8b^2 - 22ab \]

Properties
1) \( x(2b^2 + 1, b) - z(2b^2 + 1, b) + T(2b^2 + 1, b) - t_{26,b} + t_{14,b} + t_{10,b} - 36OH_B \equiv 0 \) (mod 3)
2) \( w(a,a(a+1)) - y(a,a(a+1)) + T(a,a(a+1)) - 32Pr_a + t_{62,a} + 24PP_a \equiv 0 \) (mod 61)
3) \( z(a,2a^2 - 1) + 2w(a,2a^2 - 1) + 74SO_a \equiv 0 \)
4) \( z(a,2a^2 + 1) + 2w(a,2a^2 + 1) + 222OH_a \equiv 0 \)
5) \( z(a,12a - 11) + 2w(a,12a - 11) + 74t_{26,a} \equiv 0 \)
6) \( 5\{x(1,b) + 7y(1,b) + GnO_a \equiv -1 \) (mod 146)
7) \( 5\{x(b,b+1) + 7y(b,b+1) + 148Pr_b \equiv 0 \)
8) \( 5\{x(6b - 5,b) + 7y(6b - 5,b) + 148t_{14,b} \equiv 0 \)

9) Each of the following expression represents a nasty number
   i) \( x(a,a) + y(a,a) \)
   ii) \( z(b,b) + w(b,b) \)
   iii) \( 3T(a,a) \)

Pattern-3
Instead of (5) write 37 as
\[ 37 = (6 + i)(6 - i) \]

Pattern-4
Consider (3) as
\[ u^2 + v^2 = 37T^2 \times 1 \]
write 1 as
\[ 1 = \frac{(m^2 - n^2) + 2imn}{(m^2 + n^2)^2} \cdot \frac{(m^2 - n^2) - 2imn}{(m^2 + n^2)^2} \]
Substituting (5) and (10) in (9) and employing method of factorization, define
\[ u + iv = \frac{(m^2 - n^2 + 2imn)(1 + 6i)(a + ib)^2}{m^2 + n^2} \]
Equating the real and imaginary parts in the above equation, we get
\[ u = \frac{(m^2 - n^2)(a^2 - b^2 - 12ab) - 2mn(6a^2 - 6b^2 + 2ab)}{m^2 + n^2} \]
\[ v = \frac{(m^2 - n^2)(6a^2 - 6b^2 + 2ab) + 2mn(a^2 - b^2 - 12ab)}{m^2 + n^2} \]
The corresponding integer solutions of (1) are given by
\[ x = (m^2 + n^2) \left[ (m^2 - n^2) (7A^2 - 7B^2 - 10AB) + 2mn (-5A^2 + 5B^2 - 14AB) \right] \]
\[ y = (m^2 + n^2) \left[ (m^2 - n^2) (-5A^2 + 5B^2 - 14AB) - 2mn (7A^2 - 7B^2 - 10AB) \right] \]
\[ z = (m^2 + n^2) \left[ (m^2 - n^2) (8A^2 - 8B^2 - 22AB) + 2mn (-11A^2 + 11B^2 - 16AB) \right] \]
\[ w = (m^2 + n^2) \left[ (m^2 - n^2) (-4A^2 + 4B^2 - 26AB) - 2mn (13A^2 - 13B^2 - 8AB) \right] \]
\[ T = (m^2 + n^2)^2 \left[ A^2 + B^2 \right] \]

For simplicity and clear understanding, taking \(m=2, n=1\) in the above equations, the corresponding integer solutions of (1) are given by
\[ x(A, B, 2, 1) = 5A^2 - 5B^2 - 430AB \]
\[ y(A, B, 2, 1) = -215A^2 + 215B^2 - 10AB \]

Properties
1) \(43x(A, A + 1, 2, 1) - y(A, A + 1, 2, 1) + 1859Pr_A = 0 \)
2) \(32z(2B^2 - 1, B, 2, 1) - 10w(2B^2 - 1, B, 2, 1) + 18500SO_B = 0 \)
3) \(5x(B(B + 1), B, 2, 1) + T(B(B + 1), B, 2, 1) - 2t_{17, B} - t_{22, B} + 4300PP_B = 0 \) (mod 46)
4) \(x(A, 1, 2, 1) - y(A, 1, 2, 1) - t_{472, A} + 2t_{17, A} = -220 \) (mod 196)
5) \(z(A, 1, 2, 1) + t_{180, A} + 2t_{13, A} = 0 \) (mod 747)

Substituting (5) and (10) in (9) and employing method of factorization, define

\[ u = \frac{(m^2 - n^2)(6a^2 - 6b^2 - 2ab) - 2mn(a^2 - b^2 + 12ab)}{m^2 + n^2} \]
\[ v = \frac{(m^2 - n^2)(a^2 - b^2 + 12ab) + 2mn(6a^2 - 6b^2 - 2ab)}{m^2 + n^2} \]

Equating the real and imaginary parts in the above equation, we get

For simplicity and clear understanding, taking \(m=2, n=1\) in the above equations, the corresponding integer solutions of (1) are given by
\[ x(A, B, 2, 1) = 205A^2 - 205B^2 - 130AB \]
\[ y(A, B, 2, 1) = -65A^2 + 65B^2 - 410AB \]
\[ z(A, B, 2, 1) = 275A^2 - 275B^2 - 400AB \]

Properties
\[ w(A, B, 2, 1) = 5A^2 - 5B^2 + 28AB \]
\[ T(A, B) = 25A^2 - 25B^2 \]
1) \(x(A,2A^2-1,2,1) - y(A,2A^2-1,2,1) - z(A,2A^2-1,2,1) + w(A,2A^2-1,2,1) - 708SO = 0\)
2) \(y(B+1,B,2,1) - 13w(B+1,B,2,1) + 92PP_y = 0\)
3) \(5w(A, A+1,2,1) + T(A, A+1) - 140Pr - t_{72,a} + 2t_{17,a} \equiv 0 \text{ (mod 7)}\)
4) \(28w(8B - 7,B,2,1) - x(8B - 7,B,2,1) - y(8B - 7,B,2,1) - 1324t_{18,b} \equiv 0\)
5) \(y(A,-1,2,1) + t_{102,a} + 2t_{17,a} = -65\text{ (mod 1348)}\)

**Pattern-6**

In view of (2) & (14), the solutions of (1) are obtained as

\[
x = x(a,b) = 259a^2 + 5b^2 + 74ab
\]
\[
y = y(a,b) = -185a^2 - 7b^2 - 74ab
\]
\[
z = z(a,b) = 296a^2 + 4b^2 + 74ab
\]
\[
w = w(a,b) = -148a^2 - 8b^2 - 74ab
\]
\[
T = T(a,b) = 37a^2 + b^2 + 12ab
\]

**Properties**

1) \(x(b+1,b) - 7T(b+1,b) + 10Pr + 38Obl = t_{74,b} \equiv 0 \text{ (mod 73)}\)
2) \(y(2b^2 - 1,b) + z(2b^2 - 1,b) + w(2b^2 - 1,b) + T(2b^2 - 1,b) + 62SO + 1t_{18,b} + 2Pr = 0 \text{ (mod 5)}\)
3) \(y(2b^2 + 1,b) + 5T(2b^2 + 1,b) + 42OH - 36Obl + t_{28,b} \equiv 0 \text{ (mod 74)}\)
4) \(w(a,a(a+1)) + 8T(a,a(a+1))44PP_a - t_{298,a} \equiv 0 \text{ (mod 147)}\)

**Pattern-7**

Introduction of the linear transformations

\[
v = X + 37R \quad T = X + R \quad u = 6U \quad \text{(15)}
\]

In (3) leads to \(X^2 = 37R^2 + U^2\) which is satisfied by

\[
X = r^2 + 37s^2
\]
\[
R = 2rs
\]
\[
U = r^2 - 37s^2
\]

Substituting the above values of \(X, U\) and \(R\) in (15), the corresponding non-zero distinct integral solutions of (3) are given by

\[
v = v(r,s) = r^2 + 37s^2 + 74rs
\]
\[
u = u(r,s) = 6r^2 - 222s^2
\]
\[
T = T(r,s) = r^2 + 37s^2 + 2rs
\]

Thus the corresponding solutions of (1) are found to be
1) \( x(r_{11}r - 9) + y(r_{11}r - 9) - z(r_{11}r - 9) + T(r_{11}r - 9) + 144r_{13,r} \equiv 0 \)
2) \( x(2s^2 - 1,s) + y(2s^2 - 1,s) - w(2s^2 - 1,s) - T(2s^2 - 1,s) - 72SO_s \equiv 0 \)
3) \( w(s + 1,s) - 11T(s + 1,s) + t_{178,s} + 98P_r \equiv 0 \text{ (mod 887)} \)
4) \( x(2s^2 - 1,s) - 7T(2s^2 - 1,s) - 60SO_s + t_{980_s} \equiv \text{(mod 443)} \)
5) \( y(s + 1,s) + 7T(s + 1,s) + 60Ob_l - t_{26_s} \equiv 0 \text{ (mod 11)} \)

4. Conclusion

To conclude, one may search for other patterns of solutions and their corresponding properties.

References

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