Interval-valued Intuitionistic Fuzzy Multiple Attribute Decision Making Based on Gray Relational Projection Method for Project Selection

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Abstract: ***First, this paper presents a new methodology for determining the comprehensive weights of attributes under the interval-valued intuitionistic fuzzy (IVIF) environment based on some method for determining the subject and objective attribute weights. Second, based on the concept of the grey relative closeness coefficients, a pair of fractional programming models is constructed. This model can be used to calculate the relative closeness coefficient intervals of alternatives to the IVIF positive ideal solution, which can be employed to generate ranking order of alternatives based on the concept of likelihood of interval numbers. The proposed method is illustrated for project selection.***

Keywords: Multi-attribute decision making, Interval-valued intuitionistic fuzzy set, Fractional programming, Gray relation analysis

1. Introduction

MADM problem is an important research topic in decision theory. Because the objects are fuzzy and uncertain, the attributes involved in decision problems are not always expressed as real numbers, and some are better suited to be denoted by fuzzy numbers, such as interval numbers [1-3], triangular fuzzy numbers [4], trapezoidal fuzzy numbers [3,5,6], linguistic variables [7,8] or uncertain linguistic variables [9-12], and intuitionistic fuzzy numbers [13-17]. Many ranking methods of MADM, such as the TOPSIS method [2, 10, 18], the grey relational analysis method [19-23], the VIKOR method [24-26], the PROMETHEE method [27, 28], and the ELECTRE method [29] have been extended to fit different attribute types, such as interval numbers, triangular fuzzy numbers, and trapezoidal fuzzy numbers. In [25, 26] a detailed comparison of VIKOR and TOPSIS and PROMETHEE and ELECTRE. have been presented. According to a comparative analysis of VIKOR and TOPSIS, the VIKOR method and the TOPSIS method use different aggregation functions and different normalization methods. The TOPSIS method is based on the principle that the optimal point should have the shortest distance from the positive ideal solution (PIS) and the farthest distance from the negative ideal solution (NIS). Therefore, this method is suitable for cautious (risk avoiding) decision makers. There are some results to associate TOPSIS and the other decision-making methods. For example, a hybrid methodology of DEA (data envelopment analysis) and TOPSIS is proposed for multiple criteria decision analysis in emergency management [30]. And a hybrid methodology integrating OWA (Ordered Weighted Averaging) aggregation into TOPSIS is proposed to tackle multiple criteria decision analysis (MCDA) problems [31]. Moreover, computing the optimal point using the VIKOR method is based on the particular measure of ''closeness'' to the PIS. Therefore, this method is suitable for those situations in which the decision maker intends to obtain maximum profit, and the risk in the decision is less important. Pertaining to comparing PROMETHEE and ELECTRE to the VIKOR method [26, 32, 33], the ranking result by PROMETHEE is based on the maximum group utility, whereas the VIKOR method integrates the maximum group utility and the minimal individual regret. The ranking result by ELECTRE is based on the minimum of the individual regret, whereas the compromise solution by the VIKOR method provides a balance between the maximum group utility of the majority, obtained by measuring S, which represents concordance (agreement), and the minimum of the individual regret of the opponent, obtained by measuring R, which represents discordance (disagreement). In addition, the grey relational analysis method is based on the relevance of the data curve from all of the attribute values of each alternative. It is also suitable for the cautious (risk avoider) decision maker. Because the basic model of fuzzy decision making based on the theory of fuzzy mathematics has been proposed initially in [34], fuzzy MADM has been receiving more and more attention. The intuitionistic fuzzy (IF) set has been introduced by [14, 15], which is a generalization of the fuzzy set. For the concept of the vague set is introduced by [35], which is another generalization of the fuzzy set. But, it was proven that the vague set is the same as the IF set [36]. The IF set has received more and more attention and has been applied to many fields since its appearance. The theory of the IF set has been found to be more useful to deal with vagueness and uncertainty in decision situations than that of the fuzzy set [37-39]. Over the last decades, the IF set theory has been successfully applied to solve decision making problems [40-51]. Further the IF set in the spirit of ordinary interval-valued fuzzy (IVF) sets is generalized and defined the notion of an interval-valued intuitionistic fuzzy (IVIF) set by [52] . The relations, operations and operators related to IF sets and IVIF sets are systematically studied in [38].

Recently, some researchers studied the intuitionistic fuzzy multi-attribute decision making problem under the information of attribute weights. A new method for solving intuitionistic fuzzy multi-attribute decision making problem is proposed, in which the information of attribute weights is incompletely known [53]. An intuitionistic fuzzy group decision- making methodology based on entropy and similarity measures is studied by [54]. A hybrid IFS-TOPSIS method in evaluation of project and portfolio management information systems is studied by [55]. Some researchers proposed several aggregation operators such as the IF weighted averaging operator, the IVIF weighted averaging operator, the IF ordered weighted averaging operator, the IVIF ordered weighted averaging operator and the IF ordered weighted geometric operator as well as the IVIF ordered weighted geometric operator, and employed them to deal with MADM with IF and IVIF information [48, 49, 51]. The IVIF weighted arithmetic average operator, the IVIF weighted geometric average operator and a novel accuracy function of IVIF values are introduced by [56]. An methodology to multiattribute decision making with interval-valued intuitionistic fuzzy assessments and incomplete weights is given by [16]. A fractional programming methodology for multi-attribute group decision making using IFS is presented by [45]. Linear programming method for MADM with interval- valued intuitionistic fuzzy sets is presented by [57] , [58].. A TOPSIS-based nonlinear programming method for multiattribute decision making in which both the ratings of alternatives with respect to the attributes and the weights of attributes are represented by interval-valued intuitionistic fuzzy values is presented by [59]. Interval-valued intuitionistic fuzzy mathematical programming method for hybrid multi-criteria group decision making with interval-valued intuitionistic fuzzy truth degrees is presented by [60]. However, there is little investigation of gray relational fractional programming on MADM problems in the case which ratings of alternatives on attributes are given by IVIF sets and weights of attribute are expressed as interval grey numbers.

In this paper, a methodology for determining the comprehensive weights of attribute is given, in which have been used the method for determining the subjective weight (AHP method) and some methods for determining the objective weight (optimization method, entropy method, weighted average deviation method with least membership degree, mean-squared deviation method and grey relational degree method) using interval-valued intuitionistic fuzzy sets. Then, based on the concept of the grey relative closeness coefficients, a pair of nonlinear fractional programming models is constructed to calculate the relative closeness coefficient intervals of alternatives with respect to the IVIF positive ideal solution (IVIFPIS), which can be used to generate ranking order of the alternatives. The paper is organized as follows. Section 2 briefly introduces the concept of the IF set and the IVIF set. The ranking between interval numbers by likelihood, the operations of interval grey numbers and GRA method are also presented in Section 2. In Section 3, a methodology to obtain the comprehensive weights of attributes with interval grey number is given. A pair of nonlinear fractional programming models and solving method for MADM problems with IVIF sets is constructed based on GRA are given in section 4. And the method estimating the ranking order of alternatives is given by using the likelihood for interval numbers. A real example and short remark are given in Sections 5 and 6, respectively.

2. Preliminaries

[Definition 1] ([13, 14]). Let $X=\{x_1,x_2,\cdots,x_n\}$ be a finite universal set. An IF set A in X is an object having the following form: $A=\{<x_j,\mu_A(x_j),\nu_A(x_j)>|x_j\in X\}$,where the functions $\mu_A:X\to[0,1]$, $x_j\to\mu_A(x_j)\in[0,1]$ and $\nu_A:X\to$, $[0,1]$, $x_j\to\nu_A(x_j)\in[0,1]$ define the degree of membership and degree of non- membership of the element $x_j\in X$ to the set $A\subseteq X$, respectively, and for every $x_j\in X$, $0\le\mu_A(x_j)+\nu_A(x_j)\le 1$.

Let $\pi_A(x_j)=1-\mu_A(x_j)-\nu_A(x_j)$, which is called the IF index of the element x_j in the set A . It is the degree of indeterminacy membership of the element x_j to the set A. Obviously, $0\le\pi_A(x_j)\le 1$.

[Definition 2] ([52]). Let $X=\{x_1,x_2,\cdots,x_n\}$ be a finite universal set and l be the set of all closed subintervals of the interval [0, 1]. An IVIF set $\tilde{A}$ in X is an object having the following form: $\tilde{A}=\{<x_j,\tilde{\mu}_{\tilde{A}}(x_j),\tilde{\nu}_{\tilde{A}}(x_j)>|x_j\in X\}$ where the functions $\tilde{\mu}_{\tilde{A}}:X\to l$, $x_j\to\tilde{\mu}_{\tilde{A}}(x_j)\subseteq[0,1]$ and $\tilde{\nu}_{\tilde{A}}:X\to l$, $x_j\to\tilde{\nu}_{\tilde{A}}(x_j)\subseteq[0,1]$ define the intervals of the degree of membership and degree of nonmembership of the element $x_j\in X$ to the set $\tilde{A}\subseteq X$, respectively, and for every $x_j\in X$, $0\le\sup(\tilde{\mu}_{\tilde{A}}(x_j))+\sup(\tilde{\nu}_{\tilde{A}}(x_j))\le 1$.

Obviously, $\tilde{\mu}_{\tilde{A}}(x_j)$ and $\tilde{\nu}_{\tilde{A}}$ are closed intervals. Their lower and upper bounds are denoted by $\mu^l_{\tilde{A}}(x_j)$, $\mu^u_{\tilde{A}}(x_j)$, $\nu^l_{\tilde{A}}(x_j)$, $\nu^u_{\tilde{A}}(x_j)$ respectively. Thus, the IVIF set $\tilde{A}$ may be expressed as follows;

$$\tilde{A}=\{<x_j,[\mu^l_{\tilde{A}}(x_j),\mu^u_{\tilde{A}}(x_j)],[\nu^l_{\tilde{A}}(x_j),\nu^u_{\tilde{A}}(x_j)]>|x_j\in X\}$$

The degree of indeterminacy membership is denoted by $\tilde{\pi}_{\tilde{A}}(x_j)=[\pi^l_{\tilde{A}}(x_j),\pi^u_{\tilde{A}}(x_j)]$, $\tilde{\pi}^l_{\tilde{A}}(x_j)=1-\mu^u_{\tilde{A}}(x_j)-\nu^u_{\tilde{A}}(x_j)$, $\tilde{\pi}^u_{\tilde{A}}(x_j)=1-\mu^l_{\tilde{A}}(x_j)-\nu^l_{\tilde{A}}(x_j)$.

Some operations on intuitionistic fuzzy sets are given by [61]. To rank order among interval numbers, the concept of likelihood is introduced in the following. Denote $a\ge b$, which means " a being not smaller b ". Assume that $a=[a^-,a^+]$ and $b=[b^-,b^+]$ be any two interval numbers, denote their interval lengths by $L(a)=a^+-a^-$ and $L(b)=b^+-b^-$, respectively. Obviously, $a=[a^-,a^+]$ may degenerate to a real number $\bar{a}$ if $a^+=a^-$, where $\bar{a}=a^+=a^-$.

[Definition 3] ([62]). For any two real numbers a and b, the likelihood of $a>b$ is defined as follows;

$$p(a>b)=\begin{cases}1 \ ; \ a<b\\ 0 \ ; \ a\ge b\end{cases}$$

[Definition 4] ([62]). For any two interval numbers $a=[a^l,a^u]$ and $b=[b^l,b^u]$, the likelihood of $a\ge b$ is defined as follows

$$p(a \ge b) = \max\left\{1 - \max\left\{\frac{b^u - a^l}{L(a) + L(b)}, 0\right\}, 0\right\} \quad (1)$$

where $L(a) = a^u - a^l$ and $L(b) = b^u - b^l$.

[Definition 5]([19]). A grey number with both a lower limit $\underline{a}$ and a upper limit $\bar{a}$ is called an interval grey number, denoted as $a(\otimes) \in [\underline{a}, \bar{a}]$.

Operations of interval grey numbers are defined as follows:
Assume that $a(\otimes) \in [\underline{a}, \bar{a}]$, $\underline{a} < \bar{a}$ and $b(\otimes) \in [\underline{b}, \bar{b}]$, $\underline{b} < \bar{b}$. The sum of $a(\otimes)$ and $b(\otimes)$, written $a(\otimes) + b(\otimes)$, is defined as follows; $a(\otimes) + b(\otimes) \in [\underline{a} + \underline{b}, \bar{a} + \bar{b}]$

The negative inverse of $a(\otimes)$, written $-a(\otimes)$, is defined as follows; $-a(\otimes) \in [-\bar{a}, -\underline{a}]$

The difference of $a(\otimes)$ with $b(\otimes)$ is defined as follows;
$a(\otimes) - b(\otimes) = a(\otimes) + (-b(\otimes)) \in [\underline{a} - \bar{b}, \bar{a} - \underline{b}]$
Assume $a(\otimes) \in [\underline{a}, \bar{a}]$, $\underline{a} < \bar{a}$, and $\underline{a} \cdot \bar{a} > 0$. The reciprocal of $a(\otimes)$, written $a(\otimes)^{-1}$, is defined as follows;
$a(\otimes)^{-1} \in [\frac{1}{\bar{a}}, \frac{1}{\underline{a}}]$

The product of $a(\otimes)$ and $b(\otimes)$ is defined as follows;
$a(\otimes) \cdot b(\otimes) \in [\min\{\underline{a}\underline{b}, \underline{a}\bar{b}, \bar{a}\underline{b}, \bar{a}\bar{b}\}, \max\{\underline{a}\underline{b}, \underline{a}\bar{b}, \bar{a}\underline{b}, \bar{a}\bar{b}\}]$
Assume $a(\otimes) \in [\underline{a}, \bar{a}]$, $\underline{a} < \bar{a}$, and $b(\otimes) \in [\underline{b}, \bar{b}]$, $\underline{b} < \bar{b}$, and $\underline{b} \cdot \bar{b} > 0$. The quotient of $a(\otimes)$ divided by $b(\otimes)$ is as defined as follows; $a(\otimes) / b(\otimes) = a(\otimes) \cdot b(\otimes)^{-1}$

Assume that k is a positive real number. The scalar multiplication of k and $a(\otimes)$ is defined as follows;
$k \cdot a(\otimes) \in [k\underline{a}, k\bar{a}]$

The calculation process for GRA ([19]) is expressed as follows.
Suppose X be a factor set of grey relation, $X = \{X_0, X_1, \cdots, X_m\}$, where X_0 represents the referential sequence, X_i $(i = 1,2,\cdots,m)$ denotes the comparative sequence. X_0 and X_i consist of n elements and can be expressed as follows; $X_0 = (x_0(1), x_0(2), \cdots, x_0(n))$ $X_i = (x_i(1), x_i(2), \cdots, x_i(n))$ ($i = 1,2,\cdots,m$).
$x_0(k)$ and $x_i(k)$ ($k = 1,2,\cdots,n$) are the numbers of referential sequences and comparative sequences at point k, respectively. In practical applications of decision making, the referential sequence can be an ideal solution and the comparative sequences are alternatives.

[Definition 6] ([19]). If the grey relational coefficient of the referential sequences and comparative sequences at point k is $\tau(x_0(k), x_i(k))$, then the degree of grey relation for X_0 and X_i will be defined $\tau(X_0, X_i)$ subject to the four conditions:
1) The property of normality:
$0 < \tau(X_0, X_i) \le 1$, $\tau(X_0, X_i) = 1 \Leftrightarrow X_0 = X_i$
2) The property of wholeness:
$\forall X_i, X_j \in X = \{X_s | s = 1,2,\cdots,m; m \ge 2\}$, $\tau(X_i, X_j) \ne \tau(X_j, X_i), (i \ne j)$.
3) The property of pair symmetry: For $X_i, X_j \in X$, $\tau(X_i, X_j) = \tau(X_j, X_i) \Leftrightarrow X = \{X_i, X_j\}$.
4) The property of closeness: The smaller $|x_0(k) - x_i(k)|$ is the larger $\tau(x_0(k), x_i(k))$.

The essential condition and quantitative model for grey relation are produced based on the above four prerequisites. The grey relational coefficient of the referential sequences and comparative sequences at point k can be expressed as follows:

$$\tau(x_0(k), x_i(k)) = \frac{\min_i \min_k |x_0(k) - x_i(k)| + \rho \max_i \max_k |x_0(k) - x_i(k)|}{|x_0(k) - x_i(k)| + \rho \max_i \max_k |x_0(k) - x_i(k)|} \quad (2)$$

The symbol ρ represents the equation's "contrast control", sometimes also be referred to as the "environmental coefficient" or the "distinguishing coefficient". This coefficient is a free parameter. Its value, over a broad appropriate range of values, $\rho \in [0,1]$, does not affect the ordering of the grey relational grade values, but a good value of the contrast control is needed for clear identification of key system factors. For the end points 0 and 1, i.e. for $\rho = 1$, the comparison environment is unaltered and for $\rho = 0$, the comparison environment disappears. In cases where data variation is large, ρ usually ranges from 0.1 to 0.5 for reducing the influence of extremely large $\max_i \max_k |x_0(k) - x_i(k)|$. In general, ρ is set as 0.5.

3. A Methodology for Determining the Comprehensive Weights of Attribute

Let $A = \{A_1, A_2, \cdots, A_m\}$ be a discrete set of alternatives and $S = \{S_1, S_2, \cdots, S_n\}$ be the set of attributes. Assume that $[\mu_{ij}^l, \mu_{ij}^u]$ and $[\nu_{ij}^l, \nu_{ij}^u]$ be intervals of the degrees of memb-ership and the degrees of non-membership of alternatives $A_i \in A$ on attributes $S_j \in S$ with respect to the concept ''excellence", respectively, where $0 \le \mu_{ij}^l \le \mu_{ij}^u \le 1$, $0 \le \nu_{ij}^l \le \nu_{ij}^u \le 1$ and $\mu_{ij}^u + \nu_{ij}^u \le 1$. In other words, ratings of the alternatives $A_i \in A$ on attributes $S_j \in S$ are IVIF sets, denoted by $X_{ij} = \{< x_i, [\mu_{ij}^l, \mu_{ij}^u][\nu_{ij}^l, \nu_{ij}^u] >\}$. For short, denote $X_{ij} = \{< [\mu_{ij}^l, \mu_{ij}^u][\nu_{ij}^l, \nu_{ij}^u] >\}$.

Thus, a MADM problem with IVIF sets can be expressed concisely in the interval-valued matrix format as follows

$$X=(<[\mu_{ij}^l,\mu_{ij}^u][\nu_{ij}^l,\nu_{ij}^u]>)=\begin{array}{c} \\ A_1 \\ A_2 \\ \vdots \\ A_m \end{array}\begin{array}{c} \begin{array}{cccc} S_1 & S_2 & \cdots & S_n \end{array} \\ \begin{bmatrix} [\mu_{11}^l,\mu_{11}^u],[\nu_{11}^l,\nu_{11}^u]> & [\mu_{12}^l,\mu_{12}^u],[\nu_{12}^l,\nu_{12}^u]> & \cdots & [\mu_{1n}^l,\mu_{1n}^u],[\nu_{1n}^l,\nu_{1n}^u]> \\ [\mu_{21}^l,\mu_{21}^u],[\nu_{21}^l,\nu_{21}^u]> & [\mu_{22}^l,\mu_{22}^u],[\nu_{22}^l,\nu_{22}^u]> & \vdots & [\mu_{2n}^l,\mu_{2n}^u],[\nu_{2n}^l,\nu_{2n}^u]> \\ \vdots & \vdots & \ddots & \vdots \\ [\mu_{m1}^l,\mu_{m1}^u],[\nu_{m1}^l,\nu_{m1}^u]> & [\mu_{m2}^l,\mu_{m2}^u],[\nu_{m2}^l,\nu_{m2}^u]> & & [\mu_{mn}^l,\mu_{mn}^u],[\nu_{mn}^l,\nu_{mn}^u]> \end{bmatrix} \end{array} \quad (3)$$

which is usually referred to an IVIF decision matrix represented the MADM problem with IVIF sets. Usually, alternatives $A_i \in A$ may be interchangely expressed with

$A_i=(<[\mu_{ij}^l,\mu_{ij}^u],[\nu_{ij}^l,\nu_{ij}^u]>)_{1\times n}$

3.1. A method for determining the subjective weight of attribute

Analytic Hierarchy Process (AHP), developed by [63], addresses how to determine the relative importance of a set of activities in a multi-criteria decision problem. Therefore, AHP method is usually used in subjective weight determination of attribute.

We assume that L persons decision-experts take part in attribute weight determination.

Let $\alpha_1=[\alpha_1^1,\alpha_1^2,\cdots,\alpha_1^j,\cdots,\alpha_1^n]$ is the attribute weight determined by AHP from decision-maker 1, and $\alpha_2=[\alpha_2^1,\alpha_2^2,\cdots,\alpha_2^j,\cdots,\alpha_2^n]$ is the attribute weight determined by AHP from decision-maker 2.

Generally, let $\alpha_L=[\alpha_K^1,\alpha_K^2,\cdots,\alpha_L^n]$ is the attribute weight determined by AHP from decision- maker L.

By using the weights $\alpha_1,\alpha_2,\cdots,\alpha_L$ determined by L persons of decision- experts, the subject weight of attribute S_j is composed with interval grey number $\alpha_j(\otimes)$ $(j=1,2,\cdots,n)$ as follows ; $\alpha_j(\otimes)\in[\underline{\alpha}_j,\overline{\alpha}_j]$ $(0\le\underline{\alpha}_j\le\overline{\alpha}_j,\ j=1,2,\cdots,n)$.

Where $\underline{\alpha}_j=\min_{1\le l\le L}\{\alpha_l^j\}$, $\underline{\alpha}_j=\max_{1\le l\le L}\{\alpha_l^j\}$.

3.2 Some methods for determining the objective weight of attribute

1) Method for determining the objective weight by optimization method.

We consider the method for determining the objective weight by optimization method in the case which attribute values are given as interval-valued intuitionistic fuzzy numbers. Generally, the smaller attribute value under the attribute S_j processed for given alternatives, the smaller the decision-making action for alternative under this attribute. On the contrary, if the attribute value of the given decision alternative under the attribute S_j have great preference comparatively, then this attribute is the decision that take the importance action for decision alternative. Therefore, considering from the standpoint of decision-making from the importance degree of attribute value of alternative itself, the greater deviation in attribute value, the greater weight and reversely, the smaller deviation, the smaller weight. Especially, if there is no the difference for attribute value of given alternative on attribute S_j, then there is no the action of attribute S_j to decision alternative and its weight can be taken with zero. We define the deviation of alternative A_j between others all alternative for attribute S_j in decision matrix $X=(<[\mu_{ij}^l,\mu_{ij}^u][\nu_{ij}^l,\nu_{ij}^u]>)$ as follows;

$$D_{ij}(\beta^{opt})=\sum_{k=1}^{m}\beta_j^{opt}d(x_{ij},x_{kj})=$$

$$=\sum_{k=1}^{m}\beta_j^{opt}\sqrt{(\mu_{kj}^l-\mu_{ij}^l)^2+(\mu_{kj}^u-\mu_{ij}^u)^2+(\nu_{kj}^l-\nu_{ij}^l)^2+(\nu_{kj}^u-\nu_{ij}^u)^2}$$

or $D_j(\beta^{opt})=\sum_{i=1}^{m}D(\beta_j^{opt})=\sum_{i=1}^{m}\sum_{k=1}^{m}\beta_j^{opt}d(x_{ij},x_{kj})$ $(i=1,2,\cdots,m;$ $j=1,2,\cdots,n)$

Then, $D_j(\beta^{opt})$ denote the total deviation of alternative A_j between others all alternative on attribute S_j.

It is ought to so choice weight vector β^{opt} as the sum of total deviation for the decision-making alternative become maximum. Thus, we compose the deviation function as follows.

$$D(\beta^{opt})=\sum_{j=1}^{n}\sum_{i=1}^{m}\sum_{k=1}^{m}\beta_j^{opt}d(x_{ij},x_{kj})$$

Then, we can stand the nonlinear programming problem as follows;

$$\textbf{[P1]}\quad \max D(\beta^{opt})=\sum_{j=1}^{n}\sum_{i=1}^{m}\sum_{k=1}^{m}\beta_j^{opt}d(x_{ij},x_{kj})$$

$$s.t.\begin{cases}\sum_{j=1}^{n}(\beta_j^{opt})^2=1\\ \beta_j^{opt}\ge 0,\ j=1,2,\cdots,n\end{cases}$$

[Theorem 1] The solution of problem P1 is given by

$$\beta_j^{opt}=\frac{\sum_{i=1}^{m}\sum_{k=1}^{m}d(x_{ij},x_{kj})}{\sum_{j=1}^{n}\sum_{i=1}^{m}\sum_{k=1}^{m}d(x_{ij},x_{kj})}\quad (j=1,2,\cdots,n)$$

Proof. Let's construct Lagrange function as follows

$$F(\beta_j^{opt},\lambda)=\sum_{j=1}^{n}\sum_{i=1}^{m}\sum_{k=1}^{m}\beta_j^{opt}d(x_{ij},x_{kj})+\lambda[\sum_{j=1}^{n}(\beta_j^{opt})^2-1]$$

Finding the partial derivated functions for β_j^{opt} and λ, putting those are zero, we can obtain such as

$$\frac{\partial F(\beta_j^{opt},\lambda)}{\partial\beta_j^{opt}}=\sum_{i=1}^{m}\sum_{k=1}^{m}d(x_{ij},x_{kj})+2\lambda\beta_j^{opt}=0\quad (j=1,2,\cdots,n)$$

$$\frac{\partial F(\beta_j^{opt},\lambda)}{\partial \beta_j^{opt}} = \sum_{j=1}^{n} (\beta_j^{opt})^2 - 1 = 0$$

From here we obtain

$$\beta_j^{opt} = \frac{1}{2\lambda}\sum_{i=1}^{m}\sum_{k=1}^{m} d(x_{ij}, x_{kj}) .$$

Substituting this formula to above expression

$$\frac{1}{4\lambda^2}\sum_{j=1}^{n}[\sum_{i=1}^{m}\sum_{k=1}^{m} d(x_{ij}, x_{kj})]^2 - 1 = 0 .$$

That is

$$\lambda = \pm\frac{1}{2}\sqrt{\sum_{j=1}^{n}[\sum_{i=1}^{m}\sum_{k=1}^{m} d(x_{ij}, x_{kj})]^2} .$$

Consequently,

$$\beta_j^{opt} = \mp \frac{\sum_{i=1}^{m}\sum_{k=1}^{m} d(x_{ij}, x_{kj})}{\sqrt{\sum_{j=1}^{n}[\sum_{i=1}^{m}\sum_{k=1}^{m} d(x_{ij}, x_{kj})]^2}} \quad (j = 1, 2, \cdots, n),$$

where the positive value is chosen. (end of proof).

Standardizing this, we can obtain

$$\beta_j^{opt} = \frac{\sum_{i=1}^{m}\sum_{k=1}^{m} d(x_{ij}, x_{kj})}{\sum_{j=1}^{n}\sum_{i=1}^{m}\sum_{k=1}^{m} d(x_{ij}, x_{kj})} \quad (j = 1, 2, \cdots, n)$$

Then, the vector $\beta^{opt} = (\beta_1^{opt}, \beta_2^{opt}, \cdots, \beta_n^{opt})$ is called the weight by the method of optimization.

2) Method for determining the objective weight by Entropy method

Next, we consider the objective weight of attribute by entropy method in the case which attribute values are given as interval-valued intuitionistic fuzzy numbers.

The entropy is the most important basic concept in information theory. It denotes whether amount of information offered from indeterminacy phenomenon are many or little. The information entropy is a measure reflecting the disorder degree of system. Commonly, if the change degree of attribute value for some item is very large, then the information entropy is very small and the amount of information offered by this attribute is very large. That is, the influence which this attribute cause the action in the alternative arrangement is larger. Conversely, if the change degree of attribute value for this attribute is smaller, then the information entropy is larger and the amount of information offered by this attribute is smaller. That is, the influence which this attribute cause the action in the alternative arrangement is smaller, then the weight of this attribute is smaller. This is just the principle of entropy method. In order to obtain weight, a set of grades of importance, intuitionistic fuzzy entropy may be used due to [64] as follows;

$$E_j = -\frac{1}{2}\sum_{i=1}^{m}[\mu_{ij}\ln\mu_{ij} + \nu_{ij}\ln\nu_{ij} - (1-\pi_{ij})\ln(1-\pi_{ij}) - \pi_{ij}\ln 2]$$

Here if μ_{ij} =0, $\nu_{ij} = 0$, $\pi_{ij} = 1$, then $\mu_{ij}\ln\mu_{ij} = 0$, $\nu_{ij}\ln\nu_{ij} = 0$, $(1-\pi_{ij})\ln(1-\pi_{ij}) = 0$ respectively. The entropy weight β_j^{ent} of the j -th attribute is defined as follows; $\beta_j^{ent} = \dfrac{1-E_j}{n - \sum_{j=1}^{n} E_j}$

3) Method for determining the objective weight by weighted average deviation method with least membership degree.

An IVIF positive ideal solution (IVIFPIS) and an IVIF negative ideal solution (IVIFNIS) may be defined as x^+ and x^-, respectively.

IVIF sets of x^+ and x^- on attributes $S_j \in S$ may be chosen as $\{A_j, <[1,1],[0,0]>\}$, $\{A_j, <[0,0],[1,1]>\}$, respectively. Namely, the degree of membership and the degree of non-membership of x^+ on $A_i \in A$ is 1 and 0, respectively. Denoted by $\{<[1,1],[0,0]>\}$ for short. Thus, the IVIF set vector of the IVIFPIS x^+ on all attributes is expressed concisely in the vector format as follows;

$$(<[\mu_j^{l+}, \mu_j^{u+}],[\nu_j^{l+}, \nu_j^{u+}]>)_{1\times n} = (<[1,1],[0,0]>)_{1\times n} .$$

Similarly, the IVIF set vector of the IVIFNIS x^- on all attributes is expressed as follows

$$(<[\mu_j^{l-}, \mu_j^{u-}],[\nu_j^{l-}, \nu_j^{u-}]>)_{1\times n} = (<[0,0],[1,1]>)_{1\times n} .$$

Deviation between each alternative attribute vector x_i and the positive ideal solution x^+ by considering object weight β_j^{ave} is defined as

$$f_i(\beta_j^{ave}) = \sum_{j=1}^{n}\beta_j^{ave}\cdot\left[(1-\mu_{ij}^l) + (1-\mu_{ij}^u) + (\nu_{ij}^l - 0) + (\nu_{ij}^u - 0)\right]$$

That is, $f_i(\beta_j^{ave}) = \sum_{j=1}^{n}\beta_j^{ave}\cdot\left[(1-\mu_{ij}^l) + (1-\mu_{ij}^u) + \nu_{ij}^l + \nu_{ij}^u)\right]$

Evidently, the smaller $f_i(\beta_j^{ave})$ is, the better it. Therefore, we can construct an optimization problem as following.

$$\min Z = \sum_{i=1}^{m}\sum_{j=1}^{n}\beta_j^{ave}\cdot\left[(1-\mu_{ij}^l) + (1-\mu_{ij}^u) + \nu_{ij}^l + \nu_{ij}^u\right]$$

$$s.t.\ \begin{cases}\sum_{j=1}^{n}(\beta_j^{ave})^2 = 1\\ \beta_j^{ave} \geq 0\end{cases}$$

The solution of this problem is given by

$$\beta_j^{ave} = \frac{\sum_{i=1}^{m}\left[(1-\mu_{ij}^l) + (1-\mu_{ij}^u) + \nu_{ij}^l + \nu_{ij}^u\right]}{\sqrt{\sum_{j=1}^{n}\left[\sum_{i=1}^{m}\left[(1-\mu_{ij}^l) + (1-\mu_{ij}^u) + \nu_{ij}^l + \nu_{ij}^u\right]\right]^2}} \quad (j = 1, 2, \cdots, n) .$$

By the normalization of β_j^{ave}, we obtain

$$\beta_j^{ave} = \frac{\sum_{i=1}^{m}\left[(1-\mu_{ij}^l)+(1-\mu_{ij}^u)+\nu_{ij}^l+\nu_{ij}^u\right]}{\sum_{j=1}^{n}\left[\sum_{i=1}^{m}\left[(1-\mu_{ij}^l)+(1-\mu_{ij}^u)+\nu_{ij}^l+\nu_{ij}^u\right]\right]} \quad (j=1,2,\cdots,n)$$

4) Method for determining the objective weight by mean-squared deviation method

The objective weights of mean-squared deviation method β_j^{dev} are obtained as follows.

$$\beta_j^{dev} = \frac{\sigma(S_j)}{\sum_{j=1}^{n}\sigma(S_j)} \quad (j=1,2,\cdots,n)$$

where

$$\sigma(S_j) = \sqrt{\sum_{i=1}^{m}[(\mu_{ij}^l - E_\mu^l(S_j))^2 + (\mu_{ij}^u - E_\mu^u(S_j))^2 + (\nu_{ij}^l - E_\nu^l(S_j))^2 + (\nu_{ij}^u - E_\nu^u(S_j))^2}$$

$$E_\mu^l(S_j) = \frac{1}{m}\sum_{i=1}^{m}\mu_{ij}^l \ , \ E_\mu^u(S_j) = \frac{1}{m}\sum_{i=1}^{m}\mu_{ij}^u \ , \ E_\nu^l(S_j) = \frac{1}{m}\sum_{i=1}^{m}\nu_{ij}^l \ ,$$

$$E_\nu^u(S_j) = \frac{1}{m}\sum_{i=1}^{m}\nu_{ij}^u \ .$$

5) Method for determining the objective weight by grey relational degree method.

Definition 7. The grey relational coefficients ξ_{ij}^+ of each alternative from IVIFPIS x^+ is defined as follows

$$\xi_{ij}^+ = \frac{\min_i \min_j d_{ij}^+ + \rho \max_i \max_j d_{ij}^+}{d_{ij}^+ + \rho \max_i \max_j d_{ij}^+} \qquad (4)$$

where Euclidian distance between each alternative attribute vector x_i and the positive ideal solution x^+ is denoted by

$$d_{ij}^+ = \frac{1}{\sqrt{4}}\sqrt{(\mu_{ij}^l - 1)^2 + (\mu_{ij}^u - 1)^2 + (\nu_{ij}^l - 0)^2 + (\nu_{ij}^u - 0)^2} \ ,$$

$\rho \in [0,1]$ is a discriminative coefficient or resolving factor, commonly, it is taken $\rho = 0.5$.

We can compute the weighted grey relational degree between each alternative A_i, which it regards as the intuitionistic attribute values x_i and ideal solutions x^+ by the following formulae:

$$r(x_i, x_i^+) = \xi_i^+ = \sum_{j=1}^{n} w_j \xi_{ij}^+ \ , \ i = 1,2,\cdots,m$$

The grey relational coefficient between ideal solutions and itself is $(1,1,\cdots,1)$, so the comprehensive gray relational coefficient deviation sum is

$$d_i(w) = \sum_{j=1}^{n}(1-\xi_{ij}^+)w_j$$

So, we can establish the following multiple objective optimization models to obtain the weight information:

[P2]
$$\begin{cases} \min d_i(w) = \sum_{j=1}^{n}(1-\xi_{ij}^+)w_j, \ i = 1,2,\cdots,m \\ s.t.: \ w \in H \end{cases},$$

where H is a set of the given weight information.
Since each alternative is non-inferior, so there exists no preference relation on the all the alternatives. Then, we may regard the above multiple objective optimization model as problem to find the weights in the following single objective optimization model:

[P3]
$$\begin{cases} \min d(w) = \sum_{i=1}^{m} d_i(w) = \sum_{i=1}^{m}\sum_{j=1}^{n}(1-\xi_{ij}^+)w_j \\ s.t.: \ w \in H \end{cases}$$

By solving the model P3 we get the optimal solution $w = (w_1, w_2, \cdots, w_n)$, which can be used as the weight vector of attributes. If the information about attribute weights is completely unknown, we can establish another multiple objective programming model as follows:

[P4]
$$\begin{cases} \min d_i(w) = \sum_{j=1}^{n}[(1-\xi_{ij}^+)w_j]^2 \\ s.t.: \ \sum_{j=1}^{n} w_j = 1, w_j \geq 0 (j = 1,2,\cdots,n) \end{cases}$$

Similarly, may regard the above multiple objective optimization model as problem to find the weights in the following single objective optimization model:

[P5]
$$\begin{cases} \min d(w) = \sum_{i=1}^{m} d_i(w) = \sum_{i=1}^{m}\sum_{j=1}^{n}[(1-\xi_{ij}^+)w_j]^2 \\ s.t.: \ \sum_{j=1}^{n} w_j = 1, w_j \geq 0 (j = 1,2,\cdots,n) \end{cases}$$

[Theorem 2] The solution of problem P5 is given by

$$w_j = [\sum_{i=1}^{m}(\sum_{j=1}^{n}(1-\xi_{ij}^+)^2)^{-1}]^{-1} / \sum_{i=1}^{m}(1-\xi_{ij}^+)^2 \quad (j = 1,2,\cdots,n)$$

Proof. Let's construct Lagrange function as follows

$$L(w,\lambda) = \sum_{i=1}^{m}\sum_{j=1}^{n}[(1-\xi_{ij}^+)w_j]^2 + \lambda(\sum_{j=1}^{n} w_j - 1)$$

where λ is the Lagrange multiplier.
Differentiating Lagrange function $L(w,\lambda)$ with respect to w_j $(j = 1,2,\cdots,n)$ and λ, and setting these partial derivatives equal to zero, we get a simple and exact formula for determining the attribute weights as follows:

$$w_j = [\sum_{i=1}^{m}(\sum_{j=1}^{n}(1-\xi_{ij}^+)^2)^{-1}]^{-1} / \sum_{i=1}^{m}(1-\xi_{ij}^+)^2$$

(end of proof).
We call this value w_j obtained from above optimization method is objective weight by grey relational degree method and it denote β_j^{grey}

3.3. Method for determining the comprehensive objective weights of attribute

The comprehensive objective weight is determined by the interval grey number $\beta(\otimes)=(\beta_1(\otimes), \beta_2(\otimes),\cdots,\beta_j(\otimes),\cdots,\beta_n(\otimes))$, where $\beta_j(\otimes)\in[\underline{\beta}_j(\otimes),\overline{\beta}_j(\otimes)]$ and $\underline{\beta}_j(\otimes)=\min\{\beta_j^{opt}, \beta_j^{ent}, \beta_j^{avee},\beta_j^{dev},\beta_j^{grey}\}$, $\overline{\beta}_j(\otimes)=\max\{\beta_j^{opt}, \beta_j^{ent}, \beta_j^{avee},\beta_j^{dev},\beta_j^{grey}\}$.

3.4. Method for determining the final comprehensive weights of attribute

When the subjective weight $\alpha(\otimes)=(\alpha_1(\otimes),\alpha_2(\otimes),\cdots,\alpha_n(\otimes))$, $\alpha_j(\otimes)\in[\underline{\alpha}_j,\overline{\alpha}_j]$ and objective weight $\beta(\otimes)=$ $\beta(\otimes)=(\beta_1(\otimes),\beta_2(\otimes),\cdots,\beta_n(\otimes))$, $\beta_j(\otimes)\in[\underline{\beta}_j,\overline{\beta}_j]$ are given, in order to consider the overall importance degree among the attributes, we assume that the comprehensive weights $w(\otimes)=(w_1(\otimes),w_2(\otimes),\cdots,w_n(\otimes))$ are determined by the geometric mean as following;

$$w_j(\otimes)=\frac{\alpha_j(\otimes)\times\beta_j(\otimes)}{\sum_{j=1}^{n}\alpha_j(\otimes)\times\beta_j(\otimes)}$$

where $\alpha_j(\otimes)$ and $\beta_j(\otimes)$ are the subjective weight and the objective weight for j-th attribute.

Thus, the weight of the attribute S_j is given by the interval grey number $w_j(\otimes)\in[\underline{w}_j,\overline{w}_j]$ $(j=1,2,\cdots,n)$. That is, the weight information is known partially and is unknown partially.

Therefore, we can assume that weights w_j of attributes $S_j\in S$ given by the decision maker are interval grey number $w_j\in[w_j^l,w_j^u]=[\underline{w}_j,\overline{w}_j]$. Then a weight vector of all attributes can be expressed in the interval-valued format as follows;
$w(\otimes)=([w_1^l,w_1^u],[w_2^l,w_2^u],\cdots,[w_n^l,w_n^u])$

4. Fractional Programming for MADM using Interval-Valued Intuitionistic Fuzzy Sets based on GRA

Similarly to definition 7, the grey relational coefficients ξ_{ij}^- of each alternative from IVIFNIS x^- is defined as follows

$$\xi_{ij}^-=\frac{\min_i\min_j d_{ij}^-+\rho\max_i\max_j d_{ij}^-}{d_{ij}^-+\rho\max_i\max_j d_{ij}^-} \tag{5}$$

where

$$d_{ij}^-=\frac{1}{\sqrt{4}}\sqrt{(\mu_{ij}^l-0)^2+(\mu_{ij}^u-0)^2+(\nu_{ij}^l-1)^2+(\nu_{ij}^u-1)^2}$$

By considering the weight $w_j\in[w_j^l,w_j^u]$ ($0\le w_j^l\le w_j^u$) of attribute $S_j\in S$, the degree of grey relational coefficient of each alternative from IVIFPIS x^+ is defined as follows;

$$\gamma_i^+=\gamma(x_i,x^+)=\sum_{j=1}^{n}w_j\xi_{ij}^+$$

Similarly to definition 7, the degree of grey relational coefficient of each alternative from IVIFNIS x^- is defined as follows;

$$\gamma_i^-=\gamma(x_i,x^-)=\sum_{j=1}^{n}w_j\xi_{ij}^- \tag{7}$$

Using Eqs. (6) and (7), the relative closeness coefficients of alternatives $A_i\in A$ with respect to the IVIFPIS x^+ are defined as follows

$$C_i=\frac{\gamma_i^-}{\gamma_i^++\gamma_i^-}=\frac{\sum_{j=1}^{n}w_j\xi_{ij}^-}{\sum_{j=1}^{n}w_j(\xi_{ij}^++\xi_{ij}^-)}\quad(i=1,2,\cdots,m)$$

Obviously, C_i is different for different value $w_j\in[w_j^l,w_j^u]$. Values of C_i should be in some range when w_j take all values in the interval $[w_j^l,w_j^u]$. In other words, C_i is an interval, denoted by $[c_i^l,c_i^u]$.The lower and upper bounds c_i^l and c_i^u of $[c_i^l,c_i^u]$ can be captured solving the following pair of nonlinear fractional programming models

$$\textbf{[P6]}\quad c_i^l=\min\left\{\frac{\sum_{j=1}^{n}\xi_{ij}^-w_j}{\sum_{j=1}^{n}w_j(\xi_{ij}^++\xi_{ij}^-)}\right\}(i=1,2,\cdots,m)$$

$$s.t.\quad S=\{w_j^l\le w_j\le w_j^u, w_j\ge 0,\sum_{j=1}^{n}w_j=1\}$$

$$\textbf{[P7]}\quad c_i^u=\max\left\{\frac{\sum_{j=1}^{n}\xi_{ij}^-w_j}{\sum_{j=1}^{n}w_j(\xi_{ij}^++\xi_{ij}^-)}\right\}(i=1,2,\cdots,m)$$

$$s.t.\quad S=\{w_j^l\le w_j\le w_j^u, w_j\ge 0,\sum_{j=1}^{n}w_j=1\}$$

respectively.
Problem P6 and P7 can be very easily solved using the existing fractional programming. Then m relative closeness coefficient intervals $[c_i^l,c_i^u]$ $(i=1,2,\cdots,m)$ are obtained.
Thus, comprehensive evaluations of alternatives $A_i\in A$ are described as interval numbers $C_i=[c_i^l,c_i^u]$
''Alternative A_i being not inferior to A_k'' is denoted by $A_i\succeq A_k$. The likelihood of $A_i\succeq A_k$ is measured by that of $C_i\ge C_k$, where C_i and C_k are corresponding interval numbers of alternatives A_i and A_k $(i,k=1,2,\cdots,m)$, respectively. Then using Eq. (1) the likelihood of $A_i\succeq A_k$ for alternatives A_i and A_k $(i,k=1,2,\cdots,m)$ can be determined as follows;

$$p(A_i \succeq A_k) = p(c_i \geq c_k) = \left\{1 - \max\left\{\frac{c_k^u - c_i^l}{L(c_i) + L(c_k)}, 0\right\}, 0\right\} \tag{8}$$

where

$C_i = [c_i^l, c_i^u]$, $C_k = [c_k^l, c_k^u]$, $L(C_i) = c_i^u - c_i^l$, $L(C_k) = c_k^u - c_k^l$.

Thus, the likelihood matrix can be obtained and expressed as follows;

$$P = (p_{ik})_{m \times m} = \begin{bmatrix} p_{11} & p_{12} & \cdots & p_{1m} \\ p_{21} & p_{22} & \cdots & p_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ p_{m1} & p_{m2} & \cdots & p_{mm} \end{bmatrix}$$

where $p_{ik} = p(A_i \succeq A_k)$ $(i, k = 1, 2, \cdots, m)$. Optimal degrees of membership for alternatives A_i $(i = 1, 2, \cdots, m)$ are defined as follows ([62])

$$\theta_i = \frac{1}{m(m-1)}\left(\sum_{k=1}^{m} p_{ik} + \frac{m}{2} - 1\right) \tag{9}$$

respectively.

Ranking order of all alternatives A_i $(i = 1, 2, \cdots, m)$ is generated according to the decreasing order of all values θ_i.

In sum, fractional programming models and method for MADM problems with IVIF sets on gray relational analysis may be given as follows.

Step 1. Identify the evaluation attributes and alternatives.

Step 2. Make a collection of the decision maker's opinion to get ratings of alternatives on attributes, i.e., IVIF decision matrix $X = (<[\mu_{ij}^l, \mu_{ij}^u], [\nu_{ij}^l, \nu_{ij}^u]>)_{m \times n}$.

Step 3. Make a collection of the decision maker's opinion to get weights of attributes and composite the subjective weight presented by AHP and the objective weight presented by optimization method, entropy method, weighted average deviation method with least membership degree, mean-squared deviation method and grey relational degree method for each attribute, i.e., obtain the interval- valued weight vector $W = ([w_j^l, w_j^u])_{1 \times n}$.

Step 4. Decide an IVIF positive ideal solution (IVIFPIS) and an IVIF negative ideal solution (IVIF NIS) and find the grey relational coefficients ξ_{ij}^+ of each alternative from IVIFPIS and the grey relational coefficients ξ_{ij}^- of each alternative from IVIFNIS, respectively.

Step 5. Find the degrees of grey relational coefficient γ_i^+ and γ_i^- of each alternative from IVIFPIS and IVIFNIS, respectively. Then obtain the relative closeness coefficients C_i of alternatives $A_i \in A$ with respect to the IVIFPIS.

Step 6. Construct fractional programming models for alternatives $A_i \in A$ using problem P6 and P7 to determine lower bounds c_i^l upper bounds c_i^u of the relative closeness coefficients C_i.

Step 7. Solve the fractional programming models and obtain relative closeness coefficient intervals $[c_i^l, c_i^u]$.

Step 8. Determine the best alternative from alternatives set A and generate the ranking order of all alternatives $A_i \in A$ according to the decreasing order of all values θ_i $(i = 1, 2, \cdots, m)$.

5. Illustrative Example for Project Selection

We illustrate the approach by using a numerical example. Suppose that a high-technology manufacturing company desires to select an information system(IS) project to develop a new product. After initial screening, four candidates (i.e., alternatives) A_1, A_2, A_3 and A_4 remain for further evaluation. The investment company must take a decision according to the following three attributes: S_1 (the risk analysis), S_2 (the growth analysis), S_3 (the social–political impact analysis), S_4 (the environmental impact analysis). The four possible alternatives are evaluated using the IVIF sets by the decision maker under the above three attributes, as listed in the following IVIF matrix

$$X = \begin{bmatrix} ([0.4,0.5],[0.3,0.4]) & ([0.4,0.6],[0.2,0.4]) & ([0.3,0.4],[0.4,0.5]) & ([0.5,0.6],[0.1,0.3]) \\ ([0.5,0.6],[0.2,0.3]) & ([0.6,0.7],[0.2,0.3]) & ([0.5,0.6],[0.3,0.4]) & ([0.4,0.7],[0.1,0.2]) \\ ([0.3,0.5],[0.3,0.4]) & ([0.1,0.3],[0.5,0.6]) & ([0.2,0.5],[0.4,0.5]) & ([0.2,0.3],[0.4,0.6]) \\ ([0.2,0.5],[0.3,0.4]) & ([0.5,0.7],[0.1,0.2]) & ([0.4,0.5],[0.3,0.5]) & ([0.5,0.8],[0.1,0.2]) \\ ([0.3,0.4],[0.1,0.3]) & ([0.7,0.8],[0.1,0.2]) & ([0.5,0.6],[0.2,0.4]) & ([0.6,0.7],[0.1,0.2]) \end{bmatrix}$$

By using the weights $\alpha_1, \alpha_2, \alpha_3$ determined by 3 persons of decision- experts, the subject weight of attributes is composed with interval grey number $\alpha_j(\otimes)$ $(j = 1, 2, 3, 4)$ as follows;

$\alpha(\otimes) = ([\underline{\alpha}_1, \overline{\alpha}_1], [\underline{\alpha}_2, \overline{\alpha}_2], [\underline{\alpha}_3, \overline{\alpha}_3], [\underline{\alpha}_4, \overline{\alpha}_4])$ = ([0.3375, 0.530876], [0.20625, 0.435186], [0.0966017, 0.29375], [0.0731692, 0.1625])

Now, we evaluated the object weight of attribute.

Object weight of attribute by the method of optimization is obtained as follow;

$\beta^{opt} = (\beta_1^{opt}, \beta_2^{opt}, \beta_3^{opt}, \beta_4^{opt})$ = (0.173459, 0.350689, 0.183847, 0.292006)

Object weight of attribute by the by entropy method is obtained as follow;

$\beta^{ent} = (\beta_1^{ent}, \beta_2^{ent}, \beta_3^{ent}, \beta_4^{ent})$ = (0.105126, 0.359764, 0.0479697, 0.48714)

Objective weight of attribute by weighted average deviation method with least membership degree is obtained as follow;

$\delta = (\delta_1, \delta_2, \delta_3, \delta_4)$ = (0.142857, 0.380952, 0.126984, 0.349206)

Objective weight of attribute by mean-squared deviation method is obtained as follow;

$\eta = (\eta_1, \eta_2, \eta_3, \eta_4)$ = (0.160035, 0.357848, 0.173282, 0.308835)

Objective weight of attribute by grey relational degree method is obtained as follow;

$w^{grey} = (w_1^{grey}, w_2^{grey}, w_3^{grey}, w_4^{grey})$ = (0.37847, 0.0612837, 0.491781, 0.0684654)

The objective comprehensive weights of attribute is given as follow;

$w^{obj}(\otimes) = ([\underline{w}_1, \overline{w}_1], [\underline{w}_2, \overline{w}_2], [\underline{w}_3, \overline{w}_3], [\underline{w}_4, \overline{w}_4])$ = ([0.105126, 0.37847], [0.0612837, 0.380952], [0.0479697, 0.491781], [0.0684654, 0.48714])

The final comprehensive weights of attribute is given as follow;

$w(\otimes) = ([\underline{w}_1, \overline{w}_1], [\underline{w}_2, \overline{w}_2], [\underline{w}_3, \overline{w}_3], [\underline{w}_4, \overline{w}_4])$ = ([0.221313, 0.454673], [0.133767, 0.408069], [0.0722857, 0.392765],

According to Eqs. (8) and (9), two fractional programming models can be constructed for the alternative $A_i \in A$ $(i = 1,2,3,4,5)$ as follows;

$$c_1^l = \min\left\{\frac{0.69w_1 + 0.56w_2 + 0.77w_3 + 0.59w_4}{1.69w_1 + 1.15w_2 + 1.56w_3 + 1.2w_4}\right\} \quad (10)$$
$$s.t.\ w \in S$$

Where

$$S = \{\ 0.11 < w_1 < 0.38,\ 0.06 < w_2 < 0.38,\ 0.05 < w_3 < 0.49,$$
$$0.07 < w_4 < 0.49,\ w_j \geq 0 (j = 1,2,3,4),\ \sum_{j=1}^{4} w_j = 1\}$$

and

$$c_1^u = \max\left\{\frac{0.69w_1 + 0.56w_2 + 0.77w_3 + 0.59w_4}{1.69w_1 + 1.15w_2 + 1.56w_3 + 1.2w_4}\right\} \quad (11)$$
$$s.t.\ w \in S$$

respectively. Optimal objective values of Eqs. (10) and (11) are obtained using the solving method of fractional programming, i.e., c_1^l =0.4795, c_1^u =0.5734.

Similarly,

$$c_2^l = \min\left\{\frac{0.64w_1 + 0.47w_2 + 0.64w_3 + 0.59w_4}{1.64w_1 + 1.11w_2 + 1.64w_3 + 1.15w_4}\right\} \quad (12)$$
$$s.t.\ w \in S$$

and

$$c_2^u = \max\left\{\frac{0.64w_1 + 0.47w_2 + 0.64w_3 + 0.59w_4}{1.64w_1 + 1.11w_2 + 1.64w_3 + 1.15w_4}\right\} \quad (13)$$
$$s.t.\ w \in S$$

respectively. Optimal objective values of Eqs. (12) and (13) are obtained using the solving method of fractional programming, i.e., c_2^l =0.4286, c_2^u =0.4908.

Similarly,

$$c_3^l = \min\left\{\frac{0.75w_1 + 0.51w_2 + 0.77w_3 + 0.55w_4}{1.61w_1 + 0.97w_2 + 1.52w_3 + 1.07w_4}\right\} \quad (14)$$
$$s.t.\ w \in S$$

and

$$c_3^u = \max\left\{\frac{0.75w_1 + 0.51w_2 + 0.77w_3 + 0.55w_4}{1.61w_1 + 0.97w_2 + 1.52w_3 + 1.07w_4}\right\} \quad (15)$$
$$s.t.\ w \in S$$

respectively. Optimal objective values of Eqs. (14) and (15) are obtained using the solving method of fractional programming, i.e., c_3^l =0.5843, c_3^u =0.6055.

Similarly,

$$c_4^l = \min\left\{\frac{w_1 + 0.42w_2 + 0.65w_3 + 0.51w_4}{1.71w_1 + 0.97w_2 + 1.52w_3 + 1.1w_4}\right\} \quad (16)$$
$$s.t.\ w \in S$$

and

$$c_4^u = \max\left\{\frac{w_1 + 0.42w_2 + 0.65w_3 + 0.51w_4}{1.71w_1 + 0.97w_2 + 1.52w_3 + 1.1w_4}\right\} \quad (17)$$
$$s.t.\ w \in S$$

respectively. Optimal objective values of Eqs. (16) and (17) are obtained using the solving method of fractional programming, i.e., c_4^l =0.3692, c_4^u =0.3899.

Similarly,

$$c_5^l = \min\left\{\frac{0.77w_1 + 0.51w_2 + 0.75w_3 + 0.5w_4}{1.52w_1 + 1.05w_2 + 1.69w_3 + 1.09w_4}\right\} \quad (18)$$
$$s.t.\ w \in S$$

and

$$c_5^u = \max\left\{\frac{0.77w_1 + 0.51w_2 + 0.75w_3 + 0.5w_4}{1.52w_1 + 1.05w_2 + 1.69w_3 + 1.09w_4}\right\} \quad (19)$$
$$s.t.\ w \in S$$

respectively. Optimal objective values of Eqs. (18) and (19) are obtained using the solving method of fractional programming, i.e., c_5^l =0.4128, c_5^u =0.5027.

Using Eq. (1) and the properties in Section 2, likelihoods of pair-wise comparisons of alternatives $A_i (i = 1,2,3,4,5)$ can be obtained and expressed in the matrix as follows.

$$P = \begin{bmatrix} 0.5 & 0.9276 & 0 & 1 & 0.874048 \\ 0.0723996 & 0.5 & 0 & 1 & 0.51283 \\ 1 & 1 & 0.5 & 1 & 1 \\ 0 & 0 & 0 & 0.5 & 0 \\ 0.125952 & 0.48717 & 0 & 1 & 0.5 \end{bmatrix}$$

Using Eq. (9), optimal degrees of membership for alternatives A_i $(i = 1,2,3,4,5)$ can be calculated as follows

$\theta_1 = 0.2401$, $\theta_2 = 0.1793$, $\theta_3 = 0.3000$, $\theta_4 = 0.1000$, $\theta_5 = 0.1807$

respectively. Then the best alternative is A_3 and the ranking order of the five alternatives is given as

$$A_3 \succ A_1 \succ A_5 \succ A_2 \succ A_4$$

Each of methods has its advantages and disadvantages and none of them can always perform better than the others in any situations. It all depends on how we look at things, and not on how they are themselves.

The traditional TOPSIS has solved a MADM problem with exact real number weight. But actuality, to obtain the exact real number weight is difficult of handling. In general, the information of weights on attributes in the practical process of decision making is obtained by the interval grey number. As far as the methodology for determining weights of attribute in this paper is concerned, its biggest advantage is that the information of weights on attributes are comprehensible, so it is a clear, convenient and practical methodology for dealing with decision problems.

6. Conclusion

IVIF sets are a useful tool to deal with uncertainty such as fuzziness and greyness in MADM problems. This paper focused on developing a methodology for determining the comprehensive weights of attribute and the solving method (fractional programming) for MADM problems, in which ratings of alternatives on attributes are given with IVIF sets. In this methodology, based on the concept of the degree of grey relational coefficients, we construct a pair of nonlinear fractional programming models to calculate the relative closeness coefficient intervals of alternatives to the IVIFPIS, which can be used to generate ranking order of alternatives based on the concept of likelihood of interval numbers. It is easily seen that the method proposed in this paper can be applied to MADM problems with interval-valued fuzzy soft sets.

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