Stability and Data Dependence Results for the Jungck-T-CR Iterative Scheme

Jamil-Zeana, Zaki[1], Abdullateef-Assma, Khaldoon[2]

University of Baghdad-College of Science-Department of Math-Baghdad-Iraq
[1] zeana.zaki[at]scbaghdad.edu.iq, [2] asmakalidi[at]yahoo.com

Abstract: In this paper, we establish improved results about stability and data dependence for the Jungck-T-CR iterative scheme.

Keywords: Jungck-T-CR iterative scheme, stability, data dependency

1.Introduction and Preliminaries

In [8], we define Jungck-T-CR iteration as follows:

Definition (1. 1)[8]:

Let B be a Banach space and C be a nonempty subset of B. Let $T,S:C\to C$be two self mappings such that $T(C)\subseteq S(C)$. For $u_0\in C$ the Jungck- T-CR iterative scheme is the sequence $\{Su_n\}_{n=1}^{\infty}$ is defined by:

$$\begin{aligned} Su_{n+1} &= T[(1-\alpha_n)Sv_n+\alpha_n Tv_n] \\ Sv_n &= T[(1-\beta_n)Tu_n+\beta_n Tw_n] \\ Sw_n &= T[(1-\gamma_n)Su_n+\gamma_n Tu_n],\ n\in\mathbb{N} \end{aligned} \tag{1.1}$$

where $\{\alpha_n\}_{n=1}^{\infty}$, $\{\beta_n\}_{n=1}^{\infty}$ and $\{\gamma_n\}_{n=1}^{\infty}$ are real sequences in [0,1] such that $\sum_{n=1}^{\infty}\alpha_n=\sum_{n=1}^{\infty}\beta_n=\sum_{n=1}^{\infty}\beta_n\gamma_n=\infty$.

Thus in 1996 Jungck et. al. [6] introduced the concept of coincidence point and depending on it , in 1998, Jungck and Rhoades [7] defined the notion of weakly compatible and showed that compatible mappings are weakly compatible but the converse is not true.

Definition (1.2) [7]:

Let B be a Banach space and , $T,S:B\to B$. A point $u^*\in B$ is called a coincidence point of a pair of self mappings T,S if there exists a point z (called a point of coincidence) in B such that $z=Su^*=Tu^*$. Two self mappings S and T are weakly compatible if they commute at their coincidence points, that is if $Su^*=Tu^*$ for some $u^*\in B$ then $STu^*=TSu^*$. And the point $u^*\in B$ is called common fixed point of S and T if $u^*=Su^*=Tu^*$.

$C(S,T)$ denotes the set of coincidence points of S and T.

In 2005, Singh et. al. [5] significantly improved on the result of Jungck [2] when he proved the following result which is now called Jungck-contraction principle.

Theorem (1.3) [5]:

Let (X,d) be a metric space. Let $T,S:X\to X$ satisfying $d(Tx,Ty)\le\delta d(Sx,Sy)$, $0\le\delta<1$, for all $x,y\in X$. $T(X)\subseteq S(X)$ and $S(X)$ or $T(X)$ is a complete subspace of X, then S and T have a coincidence. Indeed, for any $x_1\in X$, there exists a sequence $\{x_n\}_{n=1}^{\infty}$ in X such that

1. $Sx_{n+1}=Tx_n,\ n=1,2,\cdots$
2. $\{Sx_n\}_{n=1}^{\infty}$ converges to Su^* for some u^* in X, and $Su^*=Tu^*$ that is S and T have a coincidence at u^*.

Further, if S,T commute (just) at u^* then S and T have a unique common fixed point.

Remark (1.4):

If $S=id$ (identity mapping), then the Jungck-contraction mapping

$$d(Tx,Ty)\le\delta d(Sx,Sy),\ 0\le\delta<1 \tag{1.2}$$

is the same as the well known the contraction mapping.

The following definition will be needed in the sequel.

Definition (1.5), [1]:

Let X be a Banach space, C be a nonempty closed convex subset of X. A self mapping $T: C \to C$ is said to be **nonexpansive** if for all x, y in C, we have

$$\|Tx - Ty\| \leq \|x - y\| \tag{1.3}$$

Furthermore T is called **quasi-nonexpansive** if $y = u^*$ provided T has a fixed point in C and if $u^* \in C$ is a fixed point of T, then

$$\|Tx - Tu^*\| \leq \|x - u^*\| \tag{1.4}$$

is true for all $x \in C$.

Lemma (1.6), [6]:

Let $\{a_n\}_{n=1}^{\infty}$ be a nonnegative sequence for which one assumes there exists $n_1 \in \mathbb{N}$, such that for all $n \geq n_1$ one has satisfied the inequality

$$a_{n+1} \leq (1 - \lambda_n) a_n + \lambda_n \rho_n$$

where $\lambda_n \in (0,1)$, for all $n \in \mathbb{N}$, $\sum_{n=1}^{\infty} \lambda_n = \infty$ and $\rho_n \geq 0$, for all $n \in \mathbb{N}$. Then the following inequality holds

$$\lim_{n \to \infty} sup a_n \leq \lim_{n \to \infty} sup \rho_n$$

The following lemma will be needed in the next theorem.

Lemma (1.7), [7]:

Let $\{\tau_n\}_{n=1}^{\infty}$ and $\{\rho_n\}_{n=1}^{\infty}$ be nonnegative real sequences satisfying the following inequality:

$$\tau_{n+1} \leq (1 - \lambda_n) \tau_n + \rho_n,$$

where $\lambda_n \in (0,1)$ for all $n \geq n_1$, $\sum_{n=1}^{\infty} \lambda_n = \infty$ and $\frac{\rho_n}{\lambda_n} \to 0$ as $n \to \infty$. Then $\lim_{n \to \infty} \tau_n = 0$.

2. Stability

In the next theorem, we prove that the Jungck- T-CR (1.1) is stable with respect to (S, T).

Theorem (2.1):

Let C be a nonempty closed convex subset of a Banach space B, $S, T: C \to C$ be two self-mappings satisfying Jungck-contraction condition (1.2) provided that S is quasi-nonexpansive mapping (1.4) as well, assume $T(C) \subseteq S(C)$, let $\{Su_n\}_{n=1}^{\infty}$ be the Jungck- T-CR iterative scheme (1.1) converges to u^* such that $\sum_{n=1}^{\infty} \alpha_n = \sum_{n=1}^{\infty} \beta_n = \sum_{n=1}^{\infty} \beta_n \gamma_n = \infty$.Then the Jungck- T-CR iterative scheme is stable with respect to (S, T).

Proof:

Let $\{Su_n\}_{n=1}^{\infty}$ converges to u^* and $\{Sa_n\}_{n=1}^{\infty}$ be an arbitrary sequence in C.

Define $\varepsilon_n = \|Sa_{n+1} - T[(1 - \alpha_n) Sb_n + \alpha_n Tb_n]\|$ where

$$Sb_n = T[(1 - \beta_n) Ta_n + \beta_n Tc_n]$$
$$Sc_n = T[(1 - \gamma_n) Sa_n + \gamma_n Ta_n]$$

Now for some $0 \leq \delta < 1$ such that

$\|Tx - Ty\| \leq \delta \|Sx - Sy\|$ then
Put $q_n = T[(1 - \alpha_n) Sb_n + \alpha_n Tb_n]$ we get

$$
\begin{aligned}
\|Sa_{n+1}-u^*\| &= \|Sa_{n+1}-q_n+q_n-u^*\| \\
&\le \|Sa_{n+1}-q_n\|+\|q_n-u^*\| \\
&= \|Sa_{n+1}-T[(1-\alpha_n)Sb_n+\alpha_n Tb_n]\| \\
&\quad +\|T[(1-\alpha_n)Sb_n+\alpha_n Tb_n]-u^*\| \\
&\le \varepsilon_n+\|T[(1-\alpha_n)Sb_n+\alpha_n Tb_n]-u^*\| \\
&\le \varepsilon_n+\delta\|S[(1-\alpha_n)Sb_n+\alpha_n Tb_n]-u^*\| \\
&\le \varepsilon_n+\delta\|(1-\alpha_n)Sb_n+\alpha_n Tb_n-u^*\| \\
&\le \varepsilon_n+\delta(1-\alpha_n)\|Sb_n-u^*\|+\alpha_n\delta\|Tb_n-u^*\| \\
&\le \varepsilon_n+\delta(1-\alpha_n)\|Sb_n-u^*\|+\alpha_n\delta^2\|Sb_n-u^*\| \\
&\le \varepsilon_n+\delta[1-\alpha_n(1-\delta)]\|Sb_n-u^*\| \qquad (2.1)
\end{aligned}
$$

On the other hand

$$
\begin{aligned}
\|Sb_n-u^*\| &= \|T[(1-\beta_n)Ta_n+\beta_n Tc_n]-u^*\| \\
&\le \delta\|S[(1-\beta_n)Ta_n+\beta_n Tc_n]-u^*\| \\
&\le \delta\|(1-\beta_n)Ta_n+\beta_n Tc_n-u^*\| \\
&\le \delta(1-\beta_n)\|Ta_n-u^*\|+\beta_n\delta\|Tc_n-u^*\| \\
&\le \delta^2(1-\beta_n)\|Sa_n-u^*\|+\beta_n\delta^2\|Sc_n-u^*\| \qquad (2.2)
\end{aligned}
$$

Also, we get

$$
\begin{aligned}
\|Sc_n-u^*\| &= \|T[(1-\gamma_n)Sa_n+\gamma_n Ta_n]-u^*\| \\
&\le \delta\|S[(1-\gamma_n)Sa_n+\gamma_n Ta_n]-u^*\| \\
&\le \delta\|(1-\gamma_n)Sa_n+\gamma_n Ta_n-u^*\| \\
&\le \delta(1-\gamma_n)\|Sa_n-u^*\|+\gamma_n\delta\|Ta_n-u^*\| \\
&\le \delta(1-\gamma_n)\|Sa_n-u^*\|+\gamma_n\delta^2\|Sa_n-u^*\| \\
&\le \delta[1-\gamma_n(1-\delta)]\|Sa_n-u^*\| \qquad (2.3)
\end{aligned}
$$

Substituting (2.1) and (2.2) in (2.3), we have:

$$
\|Sa_{n+1}-u^*\| \le \varepsilon_n +\delta^3[1-\alpha_n(1-\delta)]\begin{bmatrix}(1-\beta_n)\|Sa_n-u^*\| \\ +\beta_n\delta[1-\gamma_n(1-\delta)]\|Sa_n-u^*\|\end{bmatrix}
$$

$$
\|Sa_{n+1}-u^*\| = \varepsilon_n+\delta^3[1-\alpha_n(1-\delta)] .[1-\beta_n(1-\delta)-\beta_n\gamma_n\delta(1-\delta)]\|Sa_n-u^*\|
$$

By hypothesis we have $\lim_{n\to\infty}\varepsilon_n=0$ and $\alpha_n,\beta_n,\gamma_n,\delta\in[0,1)$ then using Lemma (1.7) we get $\lim_{n\to\infty}\|Sa_n-u^*\|=0$.

Hence, we get $\lim_{n\to\infty}Sa_n=u^*$.

Now suppose that $\lim_{n\to\infty}Sa_n=u^*$ and we have to show that $\lim_{n\to\infty}\varepsilon_n=0$.

We have that

$$
\begin{aligned}
\varepsilon_n &= \|Sa_{n+1}-T[(1-\alpha_n)Sb_n+\alpha_n Tb_n]\| \\
&\le \|Sa_{n+1}-u^*\|+\|T[(1-\alpha_n)Sb_n+\alpha_n Tb_n]-u^*\| \\
&\le \|Sa_{n+1}-u^*\|+\delta\|S[(1-\alpha_n)Sb_n+\alpha_n Tb_n]-u^*\| \\
&\le \|Sa_{n+1}-u^*\|+\delta\|(1-\alpha_n)Sb_n+\alpha_n Tb_n-u^*\| \\
&\le \|Sa_{n+1}-u^*\|+\delta(1-\alpha_n)\|Sb_n-u^*\|+\alpha_n\delta\|Tb_n-u^*\| \\
&\le \|Sa_{n+1}-u^*\|+\delta[1-\alpha_n(1-\delta)]\|Sb_n-u^*\| \\
&\le \|Sa_{n+1}-u^*\| \\
&\quad +\delta[1-\alpha_n(1-\delta)]\|T[(1-\beta_n)Ta_n+\beta_n Tc_n]-u^*\| \\
&\le \|Sa_{n+1}-u^*\| \\
&\quad +\delta^2[1-\alpha_n(1-\delta)]\|S[(1-\beta_n)Ta_n+\beta_n Tc_n]-u^*\| \\
&\le \|Sa_{n+1}-u^*\|+\delta^2[1-\alpha_n(1-\delta)]\|(1-\beta_n)Ta_n+\beta_n Tc_n-u^*\| \\
&\le \|Sa_{n+1}-u^*\| \\
&\quad +\delta^2[1-\alpha_n(1-\delta)][(1-\beta_n)\|Ta_n-u^*\|+\beta_n\|Tc_n-u^*\|] \\
&\le \|Sa_{n+1}-u^*\| \\
&\quad +\delta^3[1-\alpha_n(1-\delta)][(1-\beta_n)\|Sa_n-u^*\|+\beta_n\|Sc_n-u^*\|] \\
&\le \|Sa_{n+1}-u^*\| \\
&\quad +\delta^3[1-\alpha_n(1-\delta)]\begin{bmatrix}(1-\beta_n)\|Sa_n-u^*\| \\ +\beta_n\|T[(1-\gamma_n)Sa_n+\gamma_n Ta_n]-u^*\|\end{bmatrix} \\
&\le \|Sa_{n+1}-u^*\| \\
&\quad +\delta^3[1-\alpha_n(1-\delta)]\begin{bmatrix}(1-\beta_n)\|Sa_n-u^*\| \\ +\beta_n\delta\|S[(1-\gamma_n)Sa_n+\gamma_n Ta_n]-u^*\|\end{bmatrix} \\
&\le \|Sa_{n+1}-u^*\|
\end{aligned}
$$

$$+\delta^3[1-\alpha_n(1-\delta)]\begin{bmatrix}(1-\beta_n)\|Sa_n-u^*\| \\ +\beta_n\delta\|(1-\gamma_n)Sa_n+\gamma_nTa_n-u^*\|\end{bmatrix}$$
$$\leq \|Sa_{n+1}-u^*\|$$
$$+\delta^3[1-\alpha_n(1-\delta)]\begin{bmatrix}(1-\beta_n)\|Sa_n-u^*\| \\ +\beta_n(1-\gamma_n)\delta\|Sa_n-u^*\|+\beta_n\gamma_n\delta\|Ta_n-u^*\|\end{bmatrix}$$
$$\leq \|Sa_{n+1}-u^*\|$$
$$+\delta^3[1-\alpha_n(1-\delta)]\begin{bmatrix}(1-\beta_n)\|Sa_n-u^*\| \\ +\beta_n(1-\gamma_n)\delta\|Sa_n-u^*\|+\beta_n\gamma_n\delta^2\|Sa_n-u^*\|\end{bmatrix}$$
$$\leq \|Sa_{n+1}-u^*\|$$
$$+\delta^3[1-\alpha_n(1-\delta)][1-\beta_n(1-\delta)-\beta_n\gamma_n(1-\delta)]\|Sa_n-u^*\|$$

By taking n goes to infinity we get:

$\lim_{n\to\infty}\varepsilon_n = \lim_{n\to\infty}\|Sa_{n+1}-T[(1-\alpha_n)Sb_n+\alpha_nTb_n]\| = 0$. Then the Jungck-$T$-CR iterative scheme (1.1) is stable with respect to (S,T).

3.Data Dependence Result

In the following theorem, we establish the data dependence result of Jungck-T-CR iterative scheme (1.1).

Theorem (3.1):

Let X be a Banach space, C be a nonempty closed convex subset of X and $(\tilde{S},\tilde{T}): C\to C$, be an approximate mapping of the pair $(S,T): C\to C$ satisfying Jungck-contraction condition (1.2) provided S is a nonexpansive mapping (1.3). Suppose that $T(C)\subseteq S(C)$and $\tilde{T}(C)\subseteq \tilde{S}(C)$ such that

$$\|Tx-\tilde{T}x\|\leq\varepsilon_1,\ \|Sx-\tilde{S}x\|\leq\varepsilon_2 \text{ for all } x\in C \tag{3.1}$$

Let $z\in C(S,T)$ and $\tilde{z}\in C(\tilde{S},\tilde{T})$ be the coincidence points of S,T and $\tilde{S},\tilde{T}$ respectively that is $Sz=Tz=u^*$ and $\tilde{S}\tilde{z}=\tilde{T}\tilde{z}=\tilde{u}^*$. Let $\{Su_n\}_{n=1}^{\infty}$ be the Jungck-T-CR iterative scheme generated by (1.1) with

1. $\frac{1}{2}<\beta_n\gamma_n$ for all $n\in\mathbb{N}$.
2. $\sum_{n=1}^{\infty}\beta_n\gamma_n=\infty$.

Let $\{\tilde{S}\tilde{u}_n\}_{n=1}^{\infty}$ be a sequence defined by

$$\tilde{u}_1\in X$$
$$\tilde{S}\tilde{u}_{n+1}=\tilde{T}[(1-\alpha_n)\tilde{S}\tilde{v}_n+\alpha_n\tilde{T}\tilde{v}_n]$$
$$\tilde{S}\tilde{v}_n=\tilde{T}[(1-\beta_n)\tilde{T}\tilde{u}_n+\beta_n\tilde{T}\tilde{w}_n]$$
$$\tilde{S}\tilde{w}_n=\tilde{T}[(1-\gamma_n)\tilde{S}\tilde{u}_n+\gamma_n\tilde{T}\tilde{u}_n],\ n\in\mathbb{N} \tag{3.2}$$

Assume that $\{Su_n\}_{n=1}^{\infty}$ and $\{\tilde{S}\tilde{u}_n\}_{n=1}^{\infty}$ converges to u^* and $\tilde{u}^*$ respectively. Then we have

$$\|u^*-\tilde{u}^*\|\leq\frac{9\varepsilon_1+4\varepsilon_2}{1-\delta}$$

Proof: It follows from Jungck-T-CR iteration (1.1) and Jungck-contraction condition (1.2), (3.1) and (3.2) that

$$\|Su_{n+1}-\tilde{S}\tilde{u}_{n+1}\|=\|T[(1-\alpha_n)Sv_n+\alpha_nTv_n]-\tilde{T}[(1-\alpha_n)\tilde{S}\tilde{v}_n+\alpha_n\tilde{T}\tilde{v}_n]\|$$

Put $q_n=T[(1-\alpha_n)Sv_n+\alpha_nTv_n]$ and

$r_n=T[(1-\alpha_n)\tilde{S}\tilde{v}_n+\alpha_n\tilde{T}\tilde{v}_n]$ then we have:

$$\|Su_{n+1}-\tilde{S}\tilde{u}_{n+1}\|=\|q_n-r_n+r_n-\tilde{T}[(1-\alpha_n)\tilde{S}\tilde{v}_n+\alpha_n\tilde{T}\tilde{v}_n]\|$$
$$\leq\|q_n-r_n\|+\|r_n-\tilde{T}[(1-\alpha_n)\tilde{S}\tilde{v}_n+\alpha_n\tilde{T}\tilde{v}_n]\|$$
$$=\|T[(1-\alpha_n)Sv_n+\alpha_nTv_n]-T[(1-\alpha_n)\tilde{S}\tilde{v}_n+\alpha_n\tilde{T}\tilde{v}_n]\|$$
$$+\|T[(1-\alpha_n)\tilde{S}\tilde{v}_n+\alpha_n\tilde{T}\tilde{v}_n]-\tilde{T}[(1-\alpha_n)\tilde{S}\tilde{v}_n+\alpha_n\tilde{T}\tilde{v}_n]\|$$
$$\leq\delta\|S[(1-\alpha_n)Sv_n+\alpha_nTv_n]-S[(1-\alpha_n)\tilde{S}\tilde{v}_n+\alpha_n\tilde{T}\tilde{v}_n]\|$$
$$+\varepsilon_1$$
$$\leq\varepsilon_1$$
$$+\delta\|(1-\alpha_n)Sv_n+\alpha_nTv_n-(1-\alpha_n)\tilde{S}\tilde{v}_n-\alpha_n\tilde{T}\tilde{v}_n\|$$

$$\leq \varepsilon_1 + \delta(1-\alpha_n)\|Sv_n - \tilde{S}\tilde{v}_n\| + \alpha_n\delta\|Tv_n - \tilde{T}\tilde{v}_n\|$$
$$\leq \varepsilon_1 + \delta(1-\alpha_n)\|Sv_n - \tilde{S}\tilde{v}_n\|$$
$$+\alpha_n\delta\|Tv_n - T\tilde{v}_n + T\tilde{v}_n - \tilde{T}\tilde{v}_n\|$$
$$\leq \varepsilon_1 + \delta(1-\alpha_n)\|Sv_n - \tilde{S}\tilde{v}_n\| + \alpha_n\delta\|Tv_n - T\tilde{v}_n\|$$
$$+\alpha_n\delta\|T\tilde{v}_n - \tilde{T}\tilde{v}_n\|$$
$$\leq (1+\alpha_n\delta)\varepsilon_1 + \delta(1-\alpha_n)\|Sv_n - \tilde{S}\tilde{v}_n\|$$
$$+\alpha_n\delta^2\|S\tilde{v}_n - S\tilde{v}_n\|$$

Therefore

$$\|Su_{n+1} - \tilde{S}\tilde{u}_{n+1}\| \leq (1+\alpha_n\delta)\varepsilon_1 + \delta(1-\alpha_n)\|Sv_n - \tilde{S}\tilde{v}_n\|$$
$$+\alpha_n\delta^2\|S\tilde{v}_n - \tilde{S}\tilde{v}_n + \tilde{S}\tilde{v}_n - S\tilde{v}_n\|$$
$$\leq (1+\alpha_n\delta)\varepsilon_1 + \delta(1-\alpha_n)\|Sv_n - \tilde{S}\tilde{v}_n\|$$
$$+\alpha_n\delta^2\|S\tilde{v}_n - \tilde{S}\tilde{v}_n\| + \alpha_n\delta^2\|\tilde{S}\tilde{v}_n - S\tilde{v}_n\|$$
$$\leq (1+\alpha_n\delta)\varepsilon_1 + \alpha_n\delta^2\varepsilon_2$$
$$+\delta[1-\alpha_n(1-\delta)]\|Sv_n - \tilde{S}\tilde{v}_n\| \qquad (3.3)$$

Also

$$\|Sv_n - \tilde{S}\tilde{v}_n\| = \|T[(1-\beta_n)Tu_n + \beta_n Tw_n] - \tilde{T}[(1-\beta_n)\tilde{T}\tilde{u}_n + \beta_n\tilde{T}\tilde{w}_n]\|$$

Put $a_n = T[(1-\beta_n)Tu_n + \beta_n Tw_n]$ and

$b_n = T[(1-\beta_n)\tilde{T}\tilde{u}_n + \beta_n\tilde{T}\tilde{w}_n]$ we get

$$\|Sv_n - \tilde{S}\tilde{v}_n\| = \|a_n - b_n + b_n - \tilde{T}[(1-\beta_n)\tilde{T}\tilde{u}_n + \beta_n\tilde{T}\tilde{w}_n]\|$$
$$\leq \|a_n - b_n\| + \|b_n - \tilde{T}[(1-\beta_n)\tilde{T}\tilde{u}_n + \beta_n\tilde{T}\tilde{w}_n]\|$$
$$= \|T[(1-\beta_n)Tu_n + \beta_n Tw_n] - T[(1-\beta_n)\tilde{T}\tilde{u}_n + \beta_n\tilde{T}\tilde{w}_n]\|$$
$$+\|T[(1-\beta_n)\tilde{T}\tilde{u}_n + \beta_n\tilde{T}\tilde{w}_n] - \tilde{T}[(1-\beta_n)\tilde{T}\tilde{u}_n + \beta_n\tilde{T}\tilde{w}_n]\|$$
$$\leq \varepsilon_1 + \delta\|S[(1-\beta_n)Tu_n + \beta_n Tw_n] - S[(1-\beta_n)\tilde{T}\tilde{u}_n + \beta_n\tilde{T}\tilde{w}_n]\|$$
$$\leq \varepsilon_1 + \delta\|(1-\beta_n)Tu_n + \beta_n Tw_n - (1-\beta_n)\tilde{T}\tilde{u}_n - \beta_n\tilde{T}\tilde{w}_n\|$$
$$\leq \varepsilon_1 + \delta(1-\beta_n)\|Tu_n - \tilde{T}\tilde{u}_n\| + \beta_n\delta\|Tw_n - \tilde{T}\tilde{w}_n\|$$
$$\leq \varepsilon_1 + \delta(1-\beta_n)\|Tu_n - T\tilde{u}_n + T\tilde{u}_n - \tilde{T}\tilde{u}_n\|$$
$$+\beta_n\delta\|Tw_n - \tilde{T}\tilde{w}_n\|$$
$$\leq \varepsilon_1 + \delta(1-\beta_n)\|Tu_n - T\tilde{u}_n\| + \delta(1-\beta_n)\|T\tilde{u}_n - \tilde{T}\tilde{u}_n\|$$
$$+\beta_n\delta\|Tw_n - \tilde{T}\tilde{w}_n\|$$
$$\leq \varepsilon_1 + \delta^2(1-\beta_n)\|Su_n - S\tilde{u}_n\| + \delta(1-\beta_n)\varepsilon_1$$
$$+\beta_n\delta\|Tw_n - \tilde{T}\tilde{w}_n\|$$

Therefore

$$\|Sv_n - \tilde{S}\tilde{v}_n\| \leq \varepsilon_1 + \delta^2(1-\beta_n)\|Su_n - S\tilde{u}_n\| + \delta(1-\beta_n)\varepsilon_1$$
$$+\beta_n\delta\|Tw_n - \tilde{T}\tilde{w}_n\|$$
$$\leq \varepsilon_1 + \delta(1-\beta_n)\varepsilon_1 + \delta^2(1-\beta_n)\|Su_n - \tilde{S}\tilde{u}_n + \tilde{S}\tilde{u}_n - S\tilde{u}_n\|$$
$$+\beta_n\delta\|Tw_n - \tilde{T}\tilde{w}_n\|$$
$$\leq \varepsilon_1 + \delta(1-\beta_n)\varepsilon_1 + \delta^2(1-\beta_n)\|Su_n - \tilde{S}\tilde{u}_n\|$$
$$+\delta^2(1-\beta_n)\|\tilde{S}\tilde{u}_n - S\tilde{u}_n\| + \beta_n\delta\|Tw_n - \tilde{T}\tilde{w}_n\|$$
$$\leq \varepsilon_1 + \delta(1-\beta_n)\varepsilon_1 + \delta^2(1-\beta_n)\varepsilon_2$$
$$+\delta^2(1-\beta_n)\|Su_n - \tilde{S}\tilde{u}_n\|$$
$$+\beta_n\delta\|Tw_n - T\tilde{w}_n + T\tilde{w}_n - \tilde{T}\tilde{w}_n\|$$
$$\leq \varepsilon_1 + \delta(1-\beta_n)\varepsilon_1 + \delta^2(1-\beta_n)\varepsilon_2$$
$$+\delta^2(1-\beta_n)\|Su_n - \tilde{S}\tilde{u}_n\|$$
$$+\beta_n\delta\|Tw_n - T\tilde{w}_n\| + \beta_n\delta\|T\tilde{w}_n - \tilde{T}\tilde{w}_n\|$$
$$\leq \varepsilon_1 + \delta(1-\beta_n)\varepsilon_1 + \beta_n\delta\varepsilon_1 + \delta^2(1-\beta_n)\varepsilon_2$$
$$+\delta^2(1-\beta_n)\|Su_n - \tilde{S}\tilde{u}_n\| + \beta_n\delta^2\|Sw_n - S\tilde{w}_n\|$$
$$\leq (1+\delta)\varepsilon_1 + \delta^2(1-\beta_n)\varepsilon_2 + \delta^2(1-\beta_n)\|Su_n - \tilde{S}\tilde{u}_n\|$$
$$+\beta_n\delta^2\|Sw_n - \tilde{S}\tilde{w}_n + \tilde{S}\tilde{w}_n - S\tilde{w}_n\|$$
$$\leq (1+\delta)\varepsilon_1 + \delta^2(1-\beta_n)\varepsilon_2 + \delta^2(1-\beta_n)\|Su_n - \tilde{S}\tilde{u}_n\|$$
$$+\beta_n\delta^2\|Sw_n - \tilde{S}\tilde{w}_n\| + \beta_n\delta^2\|\tilde{S}\tilde{w}_n - S\tilde{w}_n\|$$
$$\leq (1+\delta)\varepsilon_1 + \delta^2(1-\beta_n)\varepsilon_2 + \beta_n\delta^2\varepsilon_2$$
$$+\delta^2(1-\beta_n)\|Su_n - \tilde{S}\tilde{u}_n\| + \beta_n\delta^2\|Sw_n - \tilde{S}\tilde{w}_n\|$$
$$= (1+\delta)\varepsilon_1 + \delta^2\varepsilon_2 + \delta^2(1-\beta_n)\|Su_n - \tilde{S}\tilde{u}_n\|$$
$$+\beta_n\delta^2\|Sw_n - \tilde{S}\tilde{w}_n\| \qquad (3.4)$$

But

$\|Sw_n - \tilde{S}\tilde{w}_n\| = \|T[(1-\gamma_n)Su_n + \gamma_n Tu_n] - \tilde{T}[(1-\gamma_n)\tilde{S}\tilde{u}_n + \gamma_n \tilde{T}\tilde{u}_n]\|$

Put $a_n = T[(1-\gamma_n)Su_n + \gamma_n Tu_n]$ and

$b_n = T[(1-\gamma_n)\tilde{S}\tilde{u}_n + \gamma_n \tilde{T}\tilde{u}_n]$ we get

$$\begin{aligned}
\|Sw_n - \tilde{S}\tilde{w}_n\| &= \|a_n - b_n + b_n - \tilde{T}[(1-\gamma_n)\tilde{S}\tilde{u}_n + \gamma_n \tilde{T}\tilde{u}_n]\| \\
&\le \|a_n - b_n\| + \|b_n - \tilde{T}[(1-\gamma_n)\tilde{S}\tilde{u}_n + \gamma_n \tilde{T}\tilde{u}_n]\| \\
&= \|T[(1-\gamma_n)Su_n + \gamma_n Tu_n] - T[(1-\gamma_n)\tilde{S}\tilde{u}_n + \gamma_n \tilde{T}\tilde{u}_n]\| \\
&\quad + \|T[(1-\gamma_n)\tilde{S}\tilde{u}_n + \gamma_n \tilde{T}\tilde{u}_n] - \tilde{T}[(1-\gamma_n)\tilde{S}\tilde{u}_n + \gamma_n \tilde{T}\tilde{u}_n]\| \\
&\le \varepsilon_1 + \delta\|S[(1-\gamma_n)Su_n + \gamma_n Tu_n] - S[(1-\gamma_n)\tilde{S}\tilde{u}_n + \gamma_n \tilde{T}\tilde{u}_n]\| \\
&\le \varepsilon_1 + \delta\|(1-\gamma_n)Su_n + \gamma_n Tu_n - (1-\gamma_n)\tilde{S}\tilde{u}_n - \gamma_n \tilde{T}\tilde{u}_n\| \\
&\le \varepsilon_1 + \delta(1-\gamma_n)\|Su_n - \tilde{S}\tilde{u}_n\| + \delta\gamma_n\|Tu_n - \tilde{T}\tilde{u}_n\| \\
&\le \varepsilon_1 + \delta(1-\gamma_n)\|Su_n - \tilde{S}\tilde{u}_n\| \\
&\quad + \delta\gamma_n\|Tu_n - T\tilde{u}_n + T\tilde{u}_n - \tilde{T}\tilde{u}_n\| \\
&\le \varepsilon_1 + \delta(1-\gamma_n)\|Su_n - \tilde{S}\tilde{u}_n\| + \delta\gamma_n\|Tu_n - T\tilde{u}_n\| \\
&\quad + \delta\gamma_n\|T\tilde{u}_n - \tilde{T}\tilde{u}_n\| \\
&\le \varepsilon_1 + \gamma_n\,\delta\varepsilon_1 + \delta(1-\gamma_n)\|Su_n - \tilde{S}\tilde{u}_n\| \\
&\quad + \delta^2\gamma_n\|Su_n - S\tilde{u}_n\| \\
&= (1+\gamma_n\delta)\varepsilon_1 + \delta(1-\gamma_n)\|Su_n - \tilde{S}\tilde{u}_n\| \\
&\quad + \delta^2\gamma_n\|Su_n - \tilde{S}\tilde{u}_n + \tilde{S}\tilde{u}_n - S\tilde{u}_n\| \\
&\le (1+\gamma_n\delta)\varepsilon_1 + \delta(1-\gamma_n)\|Su_n - \tilde{S}\tilde{u}_n\| \\
&\quad + \delta^2\gamma_n\|Su_n - \tilde{S}\tilde{u}_n\| + \delta^2\gamma_n\|\tilde{S}\tilde{u}_n - S\tilde{u}_n\| \\
&\le (1+\gamma_n\delta)\varepsilon_1 + \delta^2\gamma_n\varepsilon_2 \\
&\quad + \delta[1-\gamma_n(1-\delta)]\|Su_n - \tilde{S}\tilde{u}_n\| \qquad (3.5)
\end{aligned}$$

Substituting (3.3) and (3.4) in (3.5), we get:

$$\begin{aligned}
\|Su_{n+1} - \tilde{S}\tilde{u}_{n+1}\| &\le (1+\alpha_n\delta)\varepsilon_1 + \alpha_n\delta^2\varepsilon_2 + \delta[1-\alpha_n(1-\delta)] \\
&\quad \cdot\begin{bmatrix} (1+\delta)\varepsilon_1 + \delta^2\varepsilon_2 + \delta^2(1-\beta_n)\|Su_n - \tilde{S}\tilde{u}_n\| \\ +\beta_n\delta^2(1+\gamma_n\delta)\varepsilon_1 + \delta^2\gamma_n\varepsilon_2 \\ +\delta[1-\gamma_n(1-\delta)]\|Su_n - \tilde{S}\tilde{u}_n\| \end{bmatrix} \\
&\le \varepsilon_1[1 + \alpha_n\delta + 1 + \delta + \beta_n\delta^2(1+\gamma_n\delta)] \\
&\quad + \varepsilon_2[\delta^2\alpha_n + \delta^2 + \beta_n\gamma_n\delta^4] \\
&\quad + [\delta^2(1-\beta_n) + \delta^3\beta_n[1-\gamma_n(1-\delta)]]\|Su_n - \tilde{S}\tilde{u}_n\| \\
&= \varepsilon_1[2 + (\alpha_n+1)\delta + \beta_n\delta^2(1+\gamma_n\delta)] \\
&\quad + \varepsilon_2[(\alpha_n+1)\delta^2 + \beta_n\gamma_n\delta^4] \\
&\quad + \delta^2[1-\beta_n(1-\delta) - \beta_n\gamma_n\delta(1-\delta)]\|Su_n - \tilde{S}\tilde{u}_n\| \qquad (3.6)
\end{aligned}$$

Since $\alpha_n, \beta_n, \gamma_n \in [0,1]$ and $\frac{1}{2} < \beta_n\gamma_n$ for all $n \in \mathbb{N}$ and using the fact $\delta \in (0,1)$in (3.6)

Yields:

$$\begin{aligned}
\|Su_{n+1} - \tilde{S}\tilde{u}_{n+1}\| &\le [1-\beta_n\gamma_n(1-\delta)]\|Su_n - \tilde{S}\tilde{u}_n\| \\
&\quad + \beta_n\gamma_n(1-\delta)\left[\frac{9\varepsilon_1 + 4\varepsilon_2}{1-\delta}\right] \qquad (3.7)
\end{aligned}$$

Put

$a_n = \|Su_n - \tilde{S}\tilde{u}_n\|$

$\lambda_n = \beta_n\gamma_n(1-\delta) \in (0,1)$

$\rho_n = \dfrac{9\varepsilon_1 + 4\varepsilon_2}{1-\delta}$

Hence, the inequality (3.7) perform all assumptions in lemma (1.6) and thus an application of lemma (1.6) to (3.7) yields

$$\begin{aligned}
0 &\le \lim_{n\to\infty} sup\|Su_n - \tilde{S}\tilde{u}_n\| \\
&\le \lim_{n\to\infty} sup\frac{9\varepsilon_1 + 4\varepsilon_2}{1-\delta}
\end{aligned}$$

Since $\lim_{n\to\infty} S\,u_n = u^*$ and $\lim_{n\to\infty} \tilde{S}\tilde{u}_n = \tilde{u}^*$, then

$$\|u^* - \tilde{u}^*\| \le \frac{9\varepsilon_1 + 4\varepsilon_2}{1-\delta}$$

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