

# Stability and Data Dependence Results for the Jungck-T-CR Iterative Scheme

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**Abstract:** In this paper, we establish improved results about stability and data dependence for the Jungck-T-CR iterative scheme.

**Keywords:** Jungck-T-CR iterative scheme, stability, data dependency

## 1. Introduction and Preliminaries

In [8], we define Jungck-T-CR iteration as follows:

### Definition (1.1)[8]:

Let  $B$  be a Banach space and  $C$  be a nonempty subset of  $B$ . Let  $T, S: C \rightarrow C$  be two self mappings such that  $T(C) \subseteq S(C)$ . For  $u_0 \in C$  the Jungck-T-CR iterative scheme is the sequence  $\{Su_n\}_{n=1}^{\infty}$  is defined by:

$$\begin{aligned} Su_{n+1} &= T[(1 - \alpha_n)Sv_n + \alpha_nTv_n] \\ Sv_n &= T[(1 - \beta_n)Tu_n + \beta_nTw_n] \\ Sw_n &= T[(1 - \gamma_n)Su_n + \gamma_nTu_n], n \in \mathbb{N} \end{aligned} \quad (1.1)$$

where  $\{\alpha_n\}_{n=1}^{\infty}$ ,  $\{\beta_n\}_{n=1}^{\infty}$  and  $\{\gamma_n\}_{n=1}^{\infty}$  are real sequences in  $[0,1]$  such that  $\sum_{n=1}^{\infty} \alpha_n = \sum_{n=1}^{\infty} \beta_n = \sum_{n=1}^{\infty} \gamma_n = \infty$ .

Thus in 1996 Jungck et. al. [6] introduced the concept of coincidence point and depending on it, in 1998, Jungck and Rhoades [7] defined the notion of weakly compatible and showed that compatible mappings are weakly compatible but the converse is not true.

### Definition (1.2) [7]:

Let  $B$  be a Banach space and  $T, S: B \rightarrow B$ . A point  $u^* \in B$  is called a coincidence point of a pair of self mappings  $T, S$  if there exists a point  $z$  (called a point of coincidence) in  $B$  such that  $z = Su^* = Tu^*$ . Two self mappings  $S$  and  $T$  are weakly compatible if they commute at their coincidence points, that is if  $Su^* = Tu^*$  for some  $u^* \in B$  then  $STu^* = TSu^*$ . And the point  $u^* \in B$  is called common fixed point of  $S$  and  $T$  if  $u^* = Su^* = Tu^*$ .

$C(S, T)$  denotes the set of coincidence points of  $S$  and  $T$ .

In 2005, Singh et. al. [5] significantly improved on the result of Jungck [2] when he proved the following result which is now called Jungck-contraction principle.

### Theorem (1.3) [5]:

Let  $(X, d)$  be a metric space. Let  $T, S: X \rightarrow X$  satisfying  $d(Tx, Ty) \leq \delta d(Sx, Sy)$ ,  $0 \leq \delta < 1$ , for all  $x, y \in X$ .  $T(X) \subseteq S(X)$  and  $S(X)$  or  $T(X)$  is a complete subspace of  $X$ , then  $S$  and  $T$  have a coincidence. Indeed, for any  $x_1 \in X$ , there exists a sequence  $\{x_n\}_{n=1}^{\infty}$  in  $X$  such that

1.  $Sx_{n+1} = Tx_n$ ,  $n = 1, 2, \dots$
2.  $\{Sx_n\}_{n=1}^{\infty}$  converges to  $Su^*$  for some  $u^*$  in  $X$ , and  $Su^* = Tu^*$  that is  $S$  and  $T$  have a coincidence at  $u^*$ .

Further, if  $S, T$  commute (just) at  $u^*$  then  $S$  and  $T$  have a unique common fixed point.

### Remark (1.4):

If  $S = id$  (identity mapping), then the Jungck-contraction mapping

$$d(Tx, Ty) \leq \delta d(Sx, Sy), 0 \leq \delta < 1 \quad (1.2)$$

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is the same as the well known the contraction mapping.

The following definition will be needed in the sequel.

**Definition (1.5), [1]:**

Let  $X$  be a Banach space,  $C$  be a nonempty closed convex subset of  $X$ . A self mapping  $T: C \rightarrow C$  is said to be **nonexpansive** if for all  $x, y$  in  $C$ , we have

$$\|Tx - Ty\| \leq \|x - y\| \tag{1.3}$$

Furthermore  $T$  is called **quasi-nonexpansive** if  $y = u^*$  provided  $T$  has a fixed point in  $C$  and if  $u^* \in C$  is a fixed point of  $T$ , then

$$\|Tx - Tu^*\| \leq \|x - u^*\| \tag{1.4}$$

is true for all  $x \in C$ .

**Lemma (1.6), [6]:**

Let  $\{a_n\}_{n=1}^\infty$  be a nonnegative sequence for which one assumes there exists  $n_1 \in \mathbb{N}$ , such that for all  $n \geq n_1$  one has satisfied the inequality

$$a_{n+1} \leq (1 - \lambda_n)a_n + \lambda_n \rho_n$$

where  $\lambda_n \in (0,1)$ , for all  $n \in \mathbb{N}$ ,  $\sum_{n=1}^\infty \lambda_n = \infty$  and  $\rho_n \geq 0$ , for all  $n \in \mathbb{N}$ . Then the following inequality holds

$$\limsup_{n \rightarrow \infty} a_n \leq \limsup_{n \rightarrow \infty} \rho_n$$

The following lemma will be needed in the next theorem.

**Lemma (1.7), [7]:**

Let  $\{\tau_n\}_{n=1}^\infty$  and  $\{\rho_n\}_{n=1}^\infty$  be nonnegative real sequences satisfying the following inequality:

$$\tau_{n+1} \leq (1 - \lambda_n)\tau_n + \rho_n,$$

where  $\lambda_n \in (0,1)$  for all  $n \geq n_1$ ,  $\sum_{n=1}^\infty \lambda_n = \infty$  and  $\frac{\rho_n}{\lambda_n} \rightarrow 0$  as  $n \rightarrow \infty$ . Then  $\lim_{n \rightarrow \infty} \tau_n = 0$ .

**2. Stability**

In the next theorem, we prove that the Jungck-  $T$ -CR (1.1) is stable with respect to  $(S, T)$ .

**Theorem (2.1):**

Let  $C$  be a nonempty closed convex subset of a Banach space  $B$ ,  $S, T: C \rightarrow C$  be two self-mappings satisfying Jungck-contraction condition (1.2) provided that  $S$  is quasi-nonexpansive mapping (1.4) as well, assume  $T(C) \subseteq S(C)$ , let  $\{Su_n\}_{n=1}^\infty$  be the Jungck-  $T$ -CR iterative scheme (1.1) converges to  $u^*$  such that  $\sum_{n=1}^\infty \alpha_n = \sum_{n=1}^\infty \beta_n = \sum_{n=1}^\infty \gamma_n = \infty$ . Then the Jungck-  $T$ -CR iterative scheme is stable with respect to  $(S, T)$ .

**Proof:**

Let  $\{Su_n\}_{n=1}^\infty$  converges to  $u^*$  and  $\{Sa_n\}_{n=1}^\infty$  be an arbitrary sequence in  $C$ .

Define  $\varepsilon_n = \|Sa_{n+1} - T[(1 - \alpha_n)Sb_n + \alpha_n Tc_n]\|$  where

$$\begin{aligned} Sb_n &= T[(1 - \beta_n)Ta_n + \beta_n Tc_n] \\ Sc_n &= T[(1 - \gamma_n)Sa_n + \gamma_n Ta_n] \end{aligned}$$

Now for some  $0 \leq \delta < 1$  such that

$\|Tx - Ty\| \leq \delta \|Sx - Sy\|$  then  
 Put  $q_n = T[(1 - \alpha_n)Sb_n + \alpha_n Tc_n]$  we get

$$\begin{aligned}
 \|Sa_{n+1} - u^*\| &= \|Sa_{n+1} - q_n + q_n - u^*\| \\
 &\leq \|Sa_{n+1} - q_n\| + \|q_n - u^*\| \\
 &= \|Sa_{n+1} - T[(1 - \alpha_n)Sb_n + \alpha_n T b_n]\| \\
 &\quad + \|T[(1 - \alpha_n)Sb_n + \alpha_n T b_n] - u^*\| \\
 &\leq \varepsilon_n + \|T[(1 - \alpha_n)Sb_n + \alpha_n T b_n] - u^*\| \\
 &\leq \varepsilon_n + \delta \|S[(1 - \alpha_n)Sb_n + \alpha_n T b_n] - u^*\| \\
 &\leq \varepsilon_n + \delta \|(1 - \alpha_n)Sb_n + \alpha_n T b_n - u^*\| \\
 &\leq \varepsilon_n + \delta(1 - \alpha_n)\|Sb_n - u^*\| + \alpha_n \delta \|T b_n - u^*\| \\
 &\leq \varepsilon_n + \delta(1 - \alpha_n)\|Sb_n - u^*\| + \alpha_n \delta^2 \|Sb_n - u^*\| \\
 &\leq \varepsilon_n + \delta[1 - \alpha_n(1 - \delta)]\|Sb_n - u^*\|
 \end{aligned} \tag{2.1}$$

On the other hand

$$\begin{aligned}
 \|Sb_n - u^*\| &= \|T[(1 - \beta_n)Ta_n + \beta_n Tc_n] - u^*\| \\
 &\leq \delta \|S[(1 - \beta_n)Ta_n + \beta_n Tc_n] - u^*\| \\
 &\leq \delta \|(1 - \beta_n)Ta_n + \beta_n Tc_n - u^*\| \\
 &\leq \delta(1 - \beta_n)\|Ta_n - u^*\| + \beta_n \delta \|Tc_n - u^*\| \\
 &\leq \delta^2(1 - \beta_n)\|Sa_n - u^*\| + \beta_n \delta^2 \|Sc_n - u^*\|
 \end{aligned} \tag{2.2}$$

Also, we get

$$\begin{aligned}
 \|Sc_n - u^*\| &= \|T[(1 - \gamma_n)Sa_n + \gamma_n Ta_n] - u^*\| \\
 &\leq \delta \|S[(1 - \gamma_n)Sa_n + \gamma_n Ta_n] - u^*\| \\
 &\leq \delta \|(1 - \gamma_n)Sa_n + \gamma_n Ta_n - u^*\| \\
 &\leq \delta(1 - \gamma_n)\|Sa_n - u^*\| + \gamma_n \delta \|Ta_n - u^*\| \\
 &\leq \delta(1 - \gamma_n)\|Sa_n - u^*\| + \gamma_n \delta^2 \|Sa_n - u^*\| \\
 &\leq \delta[1 - \gamma_n(1 - \delta)]\|Sa_n - u^*\|
 \end{aligned} \tag{2.3}$$

Substituting (2.1) and (2.2) in (2.3), we have:

$$\|Sa_{n+1} - u^*\| \leq \varepsilon_n + \delta^3[1 - \alpha_n(1 - \delta)] \left[ \frac{(1 - \beta_n)\|Sa_n - u^*\|}{+\beta_n \delta[1 - \gamma_n(1 - \delta)]\|Sa_n - u^*\|} \right]$$

$$\|Sa_{n+1} - u^*\| \leq \varepsilon_n + \delta^3[1 - \alpha_n(1 - \delta)] \cdot [1 - \beta_n(1 - \delta) - \beta_n \gamma_n \delta(1 - \delta)]\|Sa_n - u^*\|$$

By hypothesis we have  $\lim_{n \rightarrow \infty} \varepsilon_n = 0$  and  $\alpha_n, \beta_n, \gamma_n, \delta \in [0,1]$  then using Lemma (1.7) we get  $\lim_{n \rightarrow \infty} \|Sa_n - u^*\| = 0$ .

Hence, we get  $\lim_{n \rightarrow \infty} Sa_n = u^*$ .

Now suppose that  $\lim_{n \rightarrow \infty} Sa_n = u^*$  and we have to show that  $\lim_{n \rightarrow \infty} \varepsilon_n = 0$ .

We have that

$$\begin{aligned}
 \varepsilon_n &= \|Sa_{n+1} - T[(1 - \alpha_n)Sb_n + \alpha_n T b_n]\| \\
 &\leq \|Sa_{n+1} - u^*\| + \|T[(1 - \alpha_n)Sb_n + \alpha_n T b_n] - u^*\| \\
 &\leq \|Sa_{n+1} - u^*\| + \delta \|S[(1 - \alpha_n)Sb_n + \alpha_n T b_n] - u^*\| \\
 &\leq \|Sa_{n+1} - u^*\| + \delta \|(1 - \alpha_n)Sb_n + \alpha_n T b_n - u^*\| \\
 &\leq \|Sa_{n+1} - u^*\| + \delta(1 - \alpha_n)\|Sb_n - u^*\| + \alpha_n \delta \|T b_n - u^*\| \\
 &\leq \|Sa_{n+1} - u^*\| + \delta[1 - \alpha_n(1 - \delta)]\|Sb_n - u^*\| \\
 &\leq \|Sa_{n+1} - u^*\| \\
 &\quad + \delta[1 - \alpha_n(1 - \delta)]\|T[(1 - \beta_n)Ta_n + \beta_n Tc_n] - u^*\| \\
 &\leq \|Sa_{n+1} - u^*\| \\
 &\quad + \delta^2[1 - \alpha_n(1 - \delta)]\|S[(1 - \beta_n)Ta_n + \beta_n Tc_n] - u^*\| \\
 &\leq \|Sa_{n+1} - u^*\| + \delta^2[1 - \alpha_n(1 - \delta)]\|(1 - \beta_n)Ta_n + \beta_n Tc_n - u^*\| \\
 &\leq \|Sa_{n+1} - u^*\| \\
 &\quad + \delta^2[1 - \alpha_n(1 - \delta)][(1 - \beta_n)\|Ta_n - u^*\| + \beta_n \|Tc_n - u^*\|] \\
 &\leq \|Sa_{n+1} - u^*\| \\
 &\quad + \delta^3[1 - \alpha_n(1 - \delta)][(1 - \beta_n)\|Sa_n - u^*\| + \beta_n \|Sc_n - u^*\|] \\
 &\leq \|Sa_{n+1} - u^*\| \\
 &\quad + \delta^3[1 - \alpha_n(1 - \delta)] \left[ \frac{(1 - \beta_n)\|Sa_n - u^*\|}{+\beta_n \|T[(1 - \gamma_n)Sa_n + \gamma_n Ta_n] - u^*\|} \right] \\
 &\leq \|Sa_{n+1} - u^*\| \\
 &\quad + \delta^3[1 - \alpha_n(1 - \delta)] \left[ \frac{(1 - \beta_n)\|Sa_n - u^*\|}{+\beta_n \delta \|S[(1 - \gamma_n)Sa_n + \gamma_n Ta_n] - u^*\|} \right] \\
 &\leq \|Sa_{n+1} - u^*\|
 \end{aligned}$$

$$\begin{aligned}
 & +\delta^3[1 - \alpha_n(1 - \delta)] \left[ \begin{array}{l} (1 - \beta_n)\|Sa_n - u^*\| \\ +\beta_n\delta\|(1 - \gamma_n)Sa_n + \gamma_nTa_n - u^*\| \end{array} \right] \\
 \leq & \|Sa_{n+1} - u^*\| \\
 & +\delta^3[1 - \alpha_n(1 - \delta)] \left[ \begin{array}{l} (1 - \beta_n)\|Sa_n - u^*\| \\ +\beta_n(1 - \gamma_n)\delta\|Sa_n - u^*\| + \beta_n\gamma_n\delta\|Ta_n - u^*\| \end{array} \right] \\
 \leq & \|Sa_{n+1} - u^*\| \\
 & +\delta^3[1 - \alpha_n(1 - \delta)] \left[ \begin{array}{l} (1 - \beta_n)\|Sa_n - u^*\| \\ +\beta_n(1 - \gamma_n)\delta\|Sa_n - u^*\| + \beta_n\gamma_n\delta^2\|Sa_n - u^*\| \end{array} \right] \\
 \leq & \|Sa_{n+1} - u^*\| \\
 & +\delta^3[1 - \alpha_n(1 - \delta)][1 - \beta_n(1 - \delta) - \beta_n\gamma_n(1 - \delta)]\|Sa_n - u^*\|
 \end{aligned}$$

By taking  $n$  goes to infinity we get:

$\lim_{n \rightarrow \infty} \varepsilon_n = \lim_{n \rightarrow \infty} \|Sa_{n+1} - T[(1 - \alpha_n)Sb_n + \alpha_n Tb_n]\| = 0$ . Then the Jungck- $T$ -CR iterative scheme (1.1) is stable with respect to  $(S, T)$ .

### 3.Data Dependence Result

In the following theorem, we establish the data dependence result of Jungck- $T$ -CR iterative scheme (1.1).

#### **Theorem (3.1):**

Let  $X$  be a Banach space,  $C$  be a nonempty closed convex subset of  $X$  and  $(\tilde{S}, \tilde{T}): C \rightarrow C$ , be an approximate mapping of the pair  $(S, T): C \rightarrow C$  satisfying Jungck-contraction condition (1.2) provided  $S$  is a nonexpansive mapping (1.3). Suppose that  $T(C) \subseteq S(C)$  and  $\tilde{T}(C) \subseteq \tilde{S}(C)$  such that

$$\|Tx - \tilde{T}x\| \leq \varepsilon_1, \|Sx - \tilde{S}x\| \leq \varepsilon_2 \text{ for all } x \in C \quad (3.1)$$

Let  $z \in C(S, T)$  and  $\tilde{z} \in C(\tilde{S}, \tilde{T})$  be the coincidence points of  $S, T$  and  $\tilde{S}, \tilde{T}$  respectively that is  $Sz = Tz = u^*$  and  $\tilde{S}\tilde{z} = \tilde{T}\tilde{z} = \tilde{u}^*$ . Let  $\{Su_n\}_{n=1}^\infty$  be the Jungck- $T$ -CR iterative scheme generated by (1.1) with

1.  $\frac{1}{2} < \beta_n \gamma_n$  for all  $n \in \mathbb{N}$ .
2.  $\sum_{n=1}^\infty \beta_n \gamma_n = \infty$ .

Let  $\{\tilde{S}\tilde{u}_n\}_{n=1}^\infty$  be a sequence defined by

$$\begin{aligned}
 \tilde{u}_1 & \in X \\
 \tilde{S}\tilde{u}_{n+1} & = \tilde{T}[(1 - \alpha_n)\tilde{S}\tilde{v}_n + \alpha_n\tilde{T}\tilde{v}_n] \\
 \tilde{S}\tilde{v}_n & = \tilde{T}[(1 - \beta_n)\tilde{T}\tilde{u}_n + \beta_n\tilde{T}\tilde{w}_n] \\
 \tilde{S}\tilde{w}_n & = \tilde{T}[(1 - \gamma_n)\tilde{S}\tilde{u}_n + \gamma_n\tilde{T}\tilde{u}_n], n \in \mathbb{N}
 \end{aligned} \quad (3.2)$$

Assume that  $\{Su_n\}_{n=1}^\infty$  and  $\{\tilde{S}\tilde{u}_n\}_{n=1}^\infty$  converges to  $u^*$  and  $\tilde{u}^*$  respectively. Then we have

$$\|u^* - \tilde{u}^*\| \leq \frac{9\varepsilon_1 + 4\varepsilon_2}{1 - \delta}$$

**Proof:** It follows from Jungck- $T$ -CR iteration (1.1) and Jungck-contraction condition (1.2), (3.1) and (3.2) that

$$\begin{aligned}
 \|Su_{n+1} - \tilde{S}\tilde{u}_{n+1}\| & = \|T[(1 - \alpha_n)Sv_n + \alpha_nTv_n] - \tilde{T}[(1 - \alpha_n)\tilde{S}\tilde{v}_n + \alpha_n\tilde{T}\tilde{v}_n]\| \\
 \text{Put } q_n & = T[(1 - \alpha_n)Sv_n + \alpha_nTv_n] \text{ and} \\
 r_n & = T[(1 - \alpha_n)\tilde{S}\tilde{v}_n + \alpha_n\tilde{T}\tilde{v}_n] \text{ then we have:} \\
 \|Su_{n+1} - \tilde{S}\tilde{u}_{n+1}\| & = \|q_n - r_n + r_n - \tilde{T}[(1 - \alpha_n)\tilde{S}\tilde{v}_n + \alpha_n\tilde{T}\tilde{v}_n]\| \\
 & \leq \|q_n - r_n\| + \|r_n - \tilde{T}[(1 - \alpha_n)\tilde{S}\tilde{v}_n + \alpha_n\tilde{T}\tilde{v}_n]\| \\
 & = \|T[(1 - \alpha_n)Sv_n + \alpha_nTv_n] - T[(1 - \alpha_n)\tilde{S}\tilde{v}_n + \alpha_n\tilde{T}\tilde{v}_n]\| \\
 & \quad + \|T[(1 - \alpha_n)\tilde{S}\tilde{v}_n + \alpha_n\tilde{T}\tilde{v}_n] - \tilde{T}[(1 - \alpha_n)\tilde{S}\tilde{v}_n + \alpha_n\tilde{T}\tilde{v}_n]\| \\
 & \leq \delta\|S[(1 - \alpha_n)Sv_n + \alpha_nTv_n] - S[(1 - \alpha_n)\tilde{S}\tilde{v}_n + \alpha_n\tilde{T}\tilde{v}_n]\| \\
 & \quad + \varepsilon_1 \\
 & \leq \varepsilon_1 \\
 & \quad + \delta\|(1 - \alpha_n)Sv_n + \alpha_nTv_n - (1 - \alpha_n)\tilde{S}\tilde{v}_n - \alpha_n\tilde{T}\tilde{v}_n\|
 \end{aligned}$$

$$\begin{aligned}
 &\leq \varepsilon_1 + \delta(1 - \alpha_n)\|Sv_n - \tilde{S}\tilde{v}_n\| + \alpha_n\delta\|Tv_n - \tilde{T}\tilde{v}_n\| \\
 &\leq \varepsilon_1 + \delta(1 - \alpha_n)\|Sv_n - \tilde{S}\tilde{v}_n\| \\
 &\quad + \alpha_n\delta\|Tv_n - T\tilde{v}_n + T\tilde{v}_n - \tilde{T}\tilde{v}_n\| \\
 &\leq \varepsilon_1 + \delta(1 - \alpha_n)\|Sv_n - \tilde{S}\tilde{v}_n\| + \alpha_n\delta\|Tv_n - T\tilde{v}_n\| \\
 &\quad + \alpha_n\delta\|T\tilde{v}_n - \tilde{T}\tilde{v}_n\| \\
 &\leq (1 + \alpha_n\delta)\varepsilon_1 + \delta(1 - \alpha_n)\|Sv_n - \tilde{S}\tilde{v}_n\| \\
 &\quad + \alpha_n\delta^2\|S\tilde{v}_n - S\tilde{v}_n\|
 \end{aligned}$$

Therefore

$$\begin{aligned}
 \|Su_{n+1} - \tilde{S}\tilde{u}_{n+1}\| &\leq (1 + \alpha_n\delta)\varepsilon_1 + \delta(1 - \alpha_n)\|Sv_n - \tilde{S}\tilde{v}_n\| \\
 &\quad + \alpha_n\delta^2\|S\tilde{v}_n - \tilde{S}\tilde{v}_n + \tilde{S}\tilde{v}_n - S\tilde{v}_n\| \\
 &\leq (1 + \alpha_n\delta)\varepsilon_1 + \delta(1 - \alpha_n)\|Sv_n - \tilde{S}\tilde{v}_n\| \\
 &\quad + \alpha_n\delta^2\|S\tilde{v}_n - \tilde{S}\tilde{v}_n\| + \alpha_n\delta^2\|\tilde{S}\tilde{v}_n - S\tilde{v}_n\| \\
 &\leq (1 + \alpha_n\delta)\varepsilon_1 + \alpha_n\delta^2\varepsilon_2 \\
 &\quad + \delta[1 - \alpha_n(1 - \delta)]\|Sv_n - \tilde{S}\tilde{v}_n\|
 \end{aligned} \tag{3.3}$$

Also

$$\begin{aligned}
 \|Sv_n - \tilde{S}\tilde{v}_n\| &= \|T[(1 - \beta_n)Tu_n + \beta_nTw_n] - \tilde{T}[(1 - \beta_n)\tilde{T}\tilde{u}_n + \beta_n\tilde{T}\tilde{w}_n]\| \\
 \text{Put } a_n &= T[(1 - \beta_n)Tu_n + \beta_nTw_n] \text{ and} \\
 b_n &= \tilde{T}[(1 - \beta_n)\tilde{T}\tilde{u}_n + \beta_n\tilde{T}\tilde{w}_n] \text{ we get} \\
 \|Sv_n - \tilde{S}\tilde{v}_n\| &= \|a_n - b_n + b_n - \tilde{T}[(1 - \beta_n)\tilde{T}\tilde{u}_n + \beta_n\tilde{T}\tilde{w}_n]\| \\
 &\leq \|a_n - b_n\| + \|b_n - \tilde{T}[(1 - \beta_n)\tilde{T}\tilde{u}_n + \beta_n\tilde{T}\tilde{w}_n]\| \\
 &= \|T[(1 - \beta_n)Tu_n + \beta_nTw_n] - \tilde{T}[(1 - \beta_n)\tilde{T}\tilde{u}_n + \beta_n\tilde{T}\tilde{w}_n]\| \\
 &\quad + \|\tilde{T}[(1 - \beta_n)\tilde{T}\tilde{u}_n + \beta_n\tilde{T}\tilde{w}_n] - \tilde{T}[(1 - \beta_n)\tilde{T}\tilde{u}_n + \beta_n\tilde{T}\tilde{w}_n]\| \\
 &\leq \varepsilon_1 + \delta\|S[(1 - \beta_n)Tu_n + \beta_nTw_n] - S[(1 - \beta_n)\tilde{T}\tilde{u}_n + \beta_n\tilde{T}\tilde{w}_n]\| \\
 &\leq \varepsilon_1 + \delta\|(1 - \beta_n)Tu_n + \beta_nTw_n - (1 - \beta_n)\tilde{T}\tilde{u}_n - \beta_n\tilde{T}\tilde{w}_n\| \\
 &\leq \varepsilon_1 + \delta(1 - \beta_n)\|Tu_n - \tilde{T}\tilde{u}_n\| + \beta_n\delta\|Tw_n - \tilde{T}\tilde{w}_n\| \\
 &\leq \varepsilon_1 + \delta(1 - \beta_n)\|Tu_n - T\tilde{u}_n + T\tilde{u}_n - \tilde{T}\tilde{u}_n\| \\
 &\quad + \beta_n\delta\|Tw_n - \tilde{T}\tilde{w}_n\| \\
 &\leq \varepsilon_1 + \delta(1 - \beta_n)\|Tu_n - T\tilde{u}_n\| + \delta(1 - \beta_n)\|T\tilde{u}_n - \tilde{T}\tilde{u}_n\| \\
 &\quad + \beta_n\delta\|Tw_n - \tilde{T}\tilde{w}_n\| \\
 &\leq \varepsilon_1 + \delta^2(1 - \beta_n)\|Su_n - S\tilde{u}_n\| + \delta(1 - \beta_n)\varepsilon_1 \\
 &\quad + \beta_n\delta\|Tw_n - \tilde{T}\tilde{w}_n\|
 \end{aligned}$$

Therefore

$$\begin{aligned}
 \|Sv_n - \tilde{S}\tilde{v}_n\| &\leq \varepsilon_1 + \delta^2(1 - \beta_n)\|Su_n - S\tilde{u}_n\| + \delta(1 - \beta_n)\varepsilon_1 \\
 &\quad + \beta_n\delta\|Tw_n - \tilde{T}\tilde{w}_n\| \\
 &\leq \varepsilon_1 + \delta(1 - \beta_n)\varepsilon_1 + \delta^2(1 - \beta_n)\|Su_n - \tilde{S}\tilde{u}_n + \tilde{S}\tilde{u}_n - S\tilde{u}_n\| \\
 &\quad + \beta_n\delta\|Tw_n - \tilde{T}\tilde{w}_n\| \\
 &\leq \varepsilon_1 + \delta(1 - \beta_n)\varepsilon_1 + \delta^2(1 - \beta_n)\|Su_n - \tilde{S}\tilde{u}_n\| \\
 &\quad + \delta^2(1 - \beta_n)\|\tilde{S}\tilde{u}_n - S\tilde{u}_n\| + \beta_n\delta\|Tw_n - \tilde{T}\tilde{w}_n\| \\
 &\leq \varepsilon_1 + \delta(1 - \beta_n)\varepsilon_1 + \delta^2(1 - \beta_n)\varepsilon_2 \\
 &\quad + \delta^2(1 - \beta_n)\|Su_n - \tilde{S}\tilde{u}_n\| \\
 &\quad + \beta_n\delta\|Tw_n - T\tilde{w}_n + T\tilde{w}_n - \tilde{T}\tilde{w}_n\| \\
 &\leq \varepsilon_1 + \delta(1 - \beta_n)\varepsilon_1 + \delta^2(1 - \beta_n)\varepsilon_2 \\
 &\quad + \delta^2(1 - \beta_n)\|Su_n - \tilde{S}\tilde{u}_n\| \\
 &\quad + \beta_n\delta\|Tw_n - T\tilde{w}_n\| + \beta_n\delta\|T\tilde{w}_n - \tilde{T}\tilde{w}_n\| \\
 &\leq \varepsilon_1 + \delta(1 - \beta_n)\varepsilon_1 + \beta_n\delta\varepsilon_1 + \delta^2(1 - \beta_n)\varepsilon_2 \\
 &\quad + \delta^2(1 - \beta_n)\|Su_n - \tilde{S}\tilde{u}_n\| + \beta_n\delta^2\|Sw_n - S\tilde{w}_n\| \\
 &\leq (1 + \delta)\varepsilon_1 + \delta^2(1 - \beta_n)\varepsilon_2 + \delta^2(1 - \beta_n)\|Su_n - \tilde{S}\tilde{u}_n\| \\
 &\quad + \beta_n\delta^2\|Sw_n - \tilde{S}\tilde{w}_n + \tilde{S}\tilde{w}_n - S\tilde{w}_n\| \\
 &\leq (1 + \delta)\varepsilon_1 + \delta^2(1 - \beta_n)\varepsilon_2 + \delta^2(1 - \beta_n)\|Su_n - \tilde{S}\tilde{u}_n\| \\
 &\quad + \beta_n\delta^2\|Sw_n - \tilde{S}\tilde{w}_n\| + \beta_n\delta^2\|\tilde{S}\tilde{w}_n - S\tilde{w}_n\| \\
 &\leq (1 + \delta)\varepsilon_1 + \delta^2(1 - \beta_n)\varepsilon_2 + \beta_n\delta^2\varepsilon_2 \\
 &\quad + \delta^2(1 - \beta_n)\|Su_n - \tilde{S}\tilde{u}_n\| + \beta_n\delta^2\|Sw_n - \tilde{S}\tilde{w}_n\| \\
 &= (1 + \delta)\varepsilon_1 + \delta^2\varepsilon_2 + \delta^2(1 - \beta_n)\|Su_n - \tilde{S}\tilde{u}_n\| \\
 &\quad + \beta_n\delta^2\|Sw_n - \tilde{S}\tilde{w}_n\|
 \end{aligned} \tag{3.4}$$

But

$$\|Sw_n - \tilde{S}\tilde{w}_n\| = \|T[(1 - \gamma_n)Su_n + \gamma_nTu_n] - \tilde{T}[(1 - \gamma_n)\tilde{S}\tilde{u}_n + \gamma_n\tilde{T}\tilde{u}_n]\|$$

Put  $a_n = T[(1 - \gamma_n)Su_n + \gamma_nTu_n]$  and

$b_n = T[(1 - \gamma_n)\tilde{S}\tilde{u}_n + \gamma_n\tilde{T}\tilde{u}_n]$  we get

$$\begin{aligned} \|Sw_n - \tilde{S}\tilde{w}_n\| &= \|a_n - b_n + b_n - \tilde{T}[(1 - \gamma_n)\tilde{S}\tilde{u}_n + \gamma_n\tilde{T}\tilde{u}_n]\| \\ &\leq \|a_n - b_n\| + \|b_n - \tilde{T}[(1 - \gamma_n)\tilde{S}\tilde{u}_n + \gamma_n\tilde{T}\tilde{u}_n]\| \\ &= \|T[(1 - \gamma_n)Su_n + \gamma_nTu_n] - T[(1 - \gamma_n)\tilde{S}\tilde{u}_n + \gamma_n\tilde{T}\tilde{u}_n]\| \\ &\quad + \|T[(1 - \gamma_n)\tilde{S}\tilde{u}_n + \gamma_n\tilde{T}\tilde{u}_n] - \tilde{T}[(1 - \gamma_n)\tilde{S}\tilde{u}_n + \gamma_n\tilde{T}\tilde{u}_n]\| \\ &\leq \varepsilon_1 + \delta \|S[(1 - \gamma_n)Su_n + \gamma_nTu_n] - S[(1 - \gamma_n)\tilde{S}\tilde{u}_n + \gamma_n\tilde{T}\tilde{u}_n]\| \\ &\leq \varepsilon_1 + \delta \|(1 - \gamma_n)Su_n + \gamma_nTu_n - (1 - \gamma_n)\tilde{S}\tilde{u}_n - \gamma_n\tilde{T}\tilde{u}_n\| \\ &\leq \varepsilon_1 + \delta(1 - \gamma_n)\|Su_n - \tilde{S}\tilde{u}_n\| + \delta\gamma_n\|Tu_n - \tilde{T}\tilde{u}_n\| \\ &\leq \varepsilon_1 + \delta(1 - \gamma_n)\|Su_n - \tilde{S}\tilde{u}_n\| \\ &\quad + \delta\gamma_n\|Tu_n - \tilde{T}\tilde{u}_n + T\tilde{u}_n - \tilde{T}\tilde{u}_n\| \\ &\leq \varepsilon_1 + \delta(1 - \gamma_n)\|Su_n - \tilde{S}\tilde{u}_n\| + \delta\gamma_n\|Tu_n - T\tilde{u}_n\| \\ &\quad + \delta\gamma_n\|T\tilde{u}_n - \tilde{T}\tilde{u}_n\| \\ &\leq \varepsilon_1 + \gamma_n\delta\varepsilon_1 + \delta(1 - \gamma_n)\|Su_n - \tilde{S}\tilde{u}_n\| \\ &\quad + \delta^2\gamma_n\|Su_n - S\tilde{u}_n\| \\ &= (1 + \gamma_n\delta)\varepsilon_1 + \delta(1 - \gamma_n)\|Su_n - \tilde{S}\tilde{u}_n\| \\ &\quad + \delta^2\gamma_n\|Su_n - \tilde{S}\tilde{u}_n + \tilde{S}\tilde{u}_n - S\tilde{u}_n\| \\ &\leq (1 + \gamma_n\delta)\varepsilon_1 + \delta(1 - \gamma_n)\|Su_n - \tilde{S}\tilde{u}_n\| \\ &\quad + \delta^2\gamma_n\|Su_n - \tilde{S}\tilde{u}_n\| + \delta^2\gamma_n\|\tilde{S}\tilde{u}_n - S\tilde{u}_n\| \\ &\leq (1 + \gamma_n\delta)\varepsilon_1 + \delta^2\gamma_n\varepsilon_2 \\ &\quad + \delta[1 - \gamma_n(1 - \delta)]\|Su_n - \tilde{S}\tilde{u}_n\| \end{aligned} \tag{3.5}$$

Substituting (3.3) and (3.4) in (3.5), we get:

$$\begin{aligned} \|Su_{n+1} - \tilde{S}\tilde{u}_{n+1}\| &\leq (1 + \alpha_n\delta)\varepsilon_1 + \alpha_n\delta^2\varepsilon_2 + \delta[1 - \alpha_n(1 - \delta)] \\ &\quad \cdot \left[ \begin{aligned} &(1 + \delta)\varepsilon_1 + \delta^2\varepsilon_2 + \delta^2(1 - \beta_n)\|Su_n - \tilde{S}\tilde{u}_n\| \\ &+ \beta_n\delta^2(1 + \gamma_n\delta)\varepsilon_1 + \delta^2\gamma_n\varepsilon_2 \\ &+ \delta[1 - \gamma_n(1 - \delta)]\|Su_n - \tilde{S}\tilde{u}_n\| \end{aligned} \right] \\ &\leq \varepsilon_1[1 + \alpha_n\delta + 1 + \delta + \beta_n\delta^2(1 + \gamma_n\delta)] \\ &\quad + \varepsilon_2[\delta^2\alpha_n + \delta^2 + \beta_n\gamma_n\delta^4] \\ &\quad + [\delta^2(1 - \beta_n) + \delta^3\beta_n[1 - \gamma_n(1 - \delta)]]\|Su_n - \tilde{S}\tilde{u}_n\| \\ &= \varepsilon_1[2 + (\alpha_n + 1)\delta + \beta_n\delta^2(1 + \gamma_n\delta)] \\ &\quad + \varepsilon_2[(\alpha_n + 1)\delta^2 + \beta_n\gamma_n\delta^4] \\ &\quad + \delta^2[1 - \beta_n(1 - \delta) - \beta_n\gamma_n\delta(1 - \delta)]\|Su_n - \tilde{S}\tilde{u}_n\| \end{aligned} \tag{3.6}$$

Since  $\alpha_n, \beta_n, \gamma_n \in [0, 1]$  and  $\frac{1}{2} < \beta_n\gamma_n$  for all  $n \in \mathbb{N}$  and using of the fact  $\delta \in (0, 1)$  in (3.6)

Yields:

$$\|Su_{n+1} - \tilde{S}\tilde{u}_{n+1}\| \leq [1 - \beta_n\gamma_n(1 - \delta)]\|Su_n - \tilde{S}\tilde{u}_n\| + \beta_n\gamma_n(1 - \delta) \frac{9\varepsilon_1 + 4\varepsilon_2}{1 - \delta} \tag{3.7}$$

Put

$$\begin{aligned} a_n &= \|Su_n - \tilde{S}\tilde{u}_n\| \\ \lambda_n &= \beta_n\gamma_n(1 - \delta) \in (0, 1) \\ \rho_n &= \frac{9\varepsilon_1 + 4\varepsilon_2}{1 - \delta} \end{aligned}$$

Hence, the inequality (3.7) perform all assumptions in lemma (1.6) and thus an application of lemma (1.6) to (3.7) yields

$$\begin{aligned} 0 &\leq \lim_{n \rightarrow \infty} \sup \|Su_n - \tilde{S}\tilde{u}_n\| \\ &\leq \lim_{n \rightarrow \infty} \sup \frac{9\varepsilon_1 + 4\varepsilon_2}{1 - \delta} \end{aligned}$$

Since  $\lim_{n \rightarrow \infty} Su_n = u^*$  and  $\lim_{n \rightarrow \infty} \tilde{S}\tilde{u}_n = \tilde{u}^*$ , then

$$\|u^* - \tilde{u}^*\| \leq \frac{9\varepsilon_1 + 4\varepsilon_2}{1 - \delta}$$



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