Stability and Data Dependence Results for the Jungck-T-CR Iterative Scheme

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Abstract: In this paper, we establish improved results about stability and data dependence for the Jungck-T-CR iterative scheme.

Keywords: Jungck-T-CR iterative scheme, stability, data dependency

1. Introduction and Preliminaries

In [8], we define Jungck-T-CR iteration as follows:

**Definition (1, 1)[8]:**

Let $B$ be a Banach space and $C$ be a nonempty subset of $B$. Let $T, S : C \to C$ be two self mappings such that $T(C) \subseteq S(C)$. For $u_0 \in C$ the Jungck-T-CR iterative scheme is the sequence $\{S_{n+1}\}_{n=1}^{\infty}$ is defined by:

\[
\begin{align*}
S_{n+1} &= T[(1 - \alpha_n)Su_n + \alpha_nTv_n] \\
S_n &= T[(1 - \beta_n)Tu_n + \beta_nTw_n] \\
Sw_n &= T[(1 - \gamma_n)Su_n + \gamma_nTu_n], n \in \mathbb{N}
\end{align*}
\] (1.1)

where $\{\alpha_n\}_{n=1}^{\infty}$, $\{\beta_n\}_{n=1}^{\infty}$ and $\{\gamma_n\}_{n=1}^{\infty}$ are real sequences in $[0, 1]$ such that $\sum_{n=1}^{\infty} \alpha_n = \sum_{n=1}^{\infty} \beta_n = \sum_{n=1}^{\infty} \gamma_n y_n = \infty$.

Thus in 1996 Jungck et. al. [6] introduced the concept of coincidence point and depending on it, in 1998, Jungck and Rhoades [7] defined the notion of weakly compatible and showed that compatible mappings are weakly compatible but the converse is not true.

**Definition (1.2) [7]:**

Let $B$ be a Banach space and , $T, S : B \to B$. A point $u^* \in B$ is called a coincidence point of a pair of self mappings $T, S$ if there exists a point $x$ (called a point of coincidence) in $B$ such that $x = Su^* = Tu^*$. Two self mappings $S$ and $T$ are weakly compatible if they commute at their coincidence points, that is if $Su^* = Tu^*$ for some $u^* \in B$ then $STu^* = TSu^*$. And the point $u^* \in B$ is called common fixed point of $S$ and $T$ if $u^* = Su^* = Tu^*$.

$C(S, T)$ denotes the set of coincidence points of $S$ and $T$.

In 2005, Singh et. al. [5] significantly improved on the result of Jungck [2] when he proved the following result which is now called Jungck-contraction principle.

**Theorem (1.3) [5]:**

Let $(X, d)$ be a metric space. Let $T, S : X \to X$ satisfying $d(Tx, Ty) \leq \delta d(Sx, Sy), 0 \leq \delta < 1$, for all $x, y \in X$. $T(X) \subseteq S(X)$ and $S(X)$ or $T(X)$ is a complete subspace of $X$, then $S$ and $T$ have a coincidence. Indeed, for any $x_i \in X$, there exists a sequence $\{x_n\}_{n=1}^{\infty}$ in $X$ such that

1. $Sx_{n+1} = Tx_n, n = 1, 2, ...$
2. $\{Sx_n\}_{n=1}^{\infty}$ converges to $Su^*$, for some $u^* \in X$, and $Su^* = Tu^*$ that is $S$ and $T$ have a coincidence at $u^*$.

Further, if $S, T$ commute (just) at $u^*$ then $S$ and $T$ have a unique common fixed point.

**Remark (1.4):**

If $S = id$ (identity mapping), then the Jungck-contraction mapping

\[
d(Tx, Ty) \leq \delta d(Sx, Sy), 0 \leq \delta < 1\] (1.2)

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is the same as the well known the contraction mapping.

The following definition will be needed in the sequel.

**Definition (1.5), [1]:**

Let $X$ be a Banach space, $C$ be a nonempty closed convex subset of $X$. A self mapping $T : C \to C$ is said to be **nonexpansive** if for all $x, y$ in $C$, we have

$$
\|Tx - Ty\| \leq \|x - y\|
$$

(1.3)

Furthermore $T$ is called **quasi-nonexpansive** if $y = u^*$ provided $T$ has a fixed point in $C$ and if $u^* \in C$ is a fixed point of $T$, then

$$
\|Tx - Tu^*\| \leq \|x - u^*\|
$$

(1.4)

is true for all $x \in C$.

**Lemma (1.6), [6]:**

Let $\{a_n\}_{n=1}^\infty$ be a nonnegative sequence for which one assumes there exists $n_1 \in \mathbb{N}$, such that for all $n \geq n_1$ one has satisfied the inequality

$$
a_{n+1} \leq (1 - \lambda_n)a_n + \lambda_n\rho_n
$$

where $\lambda_n \in (0,1)$, for all $n \in \mathbb{N}$, $\sum_{n=1}^\infty \lambda_n = \infty$ and $\rho_n \geq 0$, for all $n \in \mathbb{N}$. Then the following inequality holds

$$
\lim \sup a_n \leq \lim \sup \rho_n
$$

The following lemma will be needed in the next theorem.

**Lemma (1.7), [7]:**

Let $\{\tau_n\}_{n=1}^\infty$ and $\{\rho_n\}_{n=1}^\infty$ be nonnegative real sequences satisfying the following inequality:

$$
\tau_{n+1} \leq (1 - \lambda_n)\tau_n + \rho_n,
$$

where $\lambda_n \in (0,1)$ for all $n \geq n_1$, $\sum_{n=1}^\infty \lambda_n = \infty$ and $\rho_n \to 0$ as $n \to \infty$. Then $\lim_{n \to \infty} \tau_n = 0$.

2. **Stability**

In the next theorem, we prove that the Jungck-$T$-CR (1.1) is stable with respect to $(S,T)$.

**Theorem (2.1):**

Let $C$ be a nonempty closed convex subset of a Banach space $B$, $S, T : C \to C$ be two self-mappings satisfying Jungck-contraction condition (1.2) provided that $S$ is quasi-nonexpansive mapping (1.4) as well, assume $T(C) \subseteq S(C)$, let $\{Su_n\}_{n=1}^\infty$ be the Jungck-$T$-CR iterative scheme (1.1) converges to $u^*$ such that $\sum_{n=1}^\infty \alpha_n = \sum_{n=1}^\infty \beta_n = \sum_{n=1}^\infty \gamma_n = \infty$. Then the Jungck-$T$-CR iterative scheme is stable with respect to $(S,T)$.

**Proof:**

Let $\{Su_n\}_{n=1}^\infty$ converges to $u^*$ and $\{Sa_n\}_{n=1}^\infty$ be an arbitrary sequence in $C$.

Define $\varepsilon_n = \|Sa_{n+1} - T[(1 - \alpha_n)Sb_n + \alpha_n Tb_n]\|$ where

$$
Sb_n = T[(1 - \beta_n)Ta_n + \beta_n Tc_n]
$$

$$
Sc_n = T[(1 - \gamma_n)Sa_n + \gamma_n Ta_n]
$$

Now for some $0 \leq \delta < 1$ such that

$$
\|Tx - Ty\| \leq \delta\|x - y\|
$$

then

Put $q_n = T[(1 - \alpha_n)Sb_n + \alpha_n Tb_n]$ we get
We have that

Now suppose that By hypothesis we have

Substituting

(2.1)

On the other hand

Also, we get

(2.2)

Substituting (2.1) and (2.2) in (2.3), we have:

(2.3)

By hypothesis we have \( \lim_{n \to x} \epsilon_n = 0 \) and \( \alpha_n, \beta_n, \gamma_n, \delta \in [0,1) \) then using Lemma (1.7) we get \( \lim_{n \to x} \| S_{a_n} - u^* \| = 0 \).

Hence, we get \( \lim_{n \to x} S_{a_n} = u^* \).

Now suppose that \( \lim_{n \to x} S_{a_n} = u^* \) and we have to show that \( \lim_{n \to x} \epsilon_n = 0 \).

We have that

\[
\epsilon_n = \| S_{a_{n+1}} - u^* \| = \| S_{a_{n+1}} - q_n + q_n - u^* \| \\
\leq \| S_{a_{n+1}} - q_n \| + \| q_n - u^* \| \\
= \| S_{a_{n+1}} - T[(1 - \alpha_n)Sb_n + \alpha_n Tb_n] \| \\
+ \| T[(1 - \alpha_n)Sb_n + \alpha_n Tb_n] - u^* \| \\
\leq \epsilon_n + \| T[(1 - \alpha_n)Sb_n + \alpha_n Tb_n] - u^* \| \\
\leq \epsilon_n + \delta \| S[(1 - \alpha_n)Sb_n + \alpha_n Tb_n] - u^* \| \\
\leq \epsilon_n + \delta \| (1 - \alpha_n)Sb_n + \alpha_n Tb_n - u^* \| \\
\leq \epsilon_n + \delta (1 - \alpha_n)\| Sb_n - u^* \| + \alpha_n \delta \| Tb_n - u^* \| \\
\leq \epsilon_n + \delta (1 - \alpha_n)\| Sb_n - u^* \| + \alpha_n \delta \| Tb_n - u^* \| \\
\leq \epsilon_n + \delta (1 - \alpha_n(1 - \delta))\| Sb_n - u^* \| \\
\text{(2.1)}
\]

\[
\| S_{b_n} - u^* \| = \| T[(1 - \beta_n)Ta_n + \beta_n Tc_n] - u^* \| \\
\leq \| T[(1 - \beta_n)Ta_n + \beta_n Tc_n] - u^* \| \\
\leq \| (1 - \beta_n)Ta_n + \beta_n Tc_n - u^* \| \\
\leq \| (1 - \beta_n)Ta_n - u^* \| + \beta_n \delta \| Tc_n - u^* \| \\
\leq \| (1 - \beta_n)Ta_n - u^* \| + \beta_n \delta \| Tc_n - u^* \| \\
\text{(2.2)}
\]

\[
\| S_{c_n} - u^* \| = \| T[(1 - \gamma_n)Sa_n + \gamma_n Ta_n] - u^* \| \\
\leq \| T[(1 - \gamma_n)Sa_n + \gamma_n Ta_n] - u^* \| \\
\leq \| (1 - \gamma_n)Sa_n + \gamma_n Ta_n - u^* \| \\
\leq \| (1 - \gamma_n)Sa_n - u^* \| + \gamma_n \delta \| Ta_n - u^* \| \\
\leq \| (1 - \gamma_n)Sa_n - u^* \| + \gamma_n \delta \| Ta_n - u^* \| \\
\leq \| (1 - \gamma_n(1 - \delta))\| Sa_n - u^* \| \\
\text{(2.3)}
\]

By hypothesis we have \( \lim_{n \to x} \epsilon_n = 0 \) and \( \alpha_n, \beta_n, \gamma_n, \delta \in [0,1) \) then using Lemma (1.7) we get \( \lim_{n \to x} \| S_{a_n} - u^* \| = 0 \).

Hence, we get \( \lim_{n \to x} S_{a_n} = u^* \).

Now suppose that \( \lim_{n \to x} S_{a_n} = u^* \) and we have to show that \( \lim_{n \to x} \epsilon_n = 0 \).

We have that
Proof: Let $\epsilon_n = \lim_{n \to \infty} \epsilon_n = \lim_{n \to \infty} \|S_{a_{n+1}} - T[(1 - \alpha_n)Sb_n + \alpha_n Tb_n]\| = 0$. Then the Jungck-T-CR iterative scheme (1.1) is stable with respect to $(S, T)$.

3. Data Dependence Result

In the following theorem, we establish the data dependence result of Jungck-T-CR iterative scheme (1.1).

Theorem (3.1):

Let $X$ be a Banach space, $C$ be a nonempty closed convex subset of $X$ and $(S, T): C \to C$, be an approximate mapping of the pair $(S, T): C \to C$ satisfying Jungck-contraction condition (1.2) provided $S$ is a nonexpansive mapping (1.3). Suppose that $T(C) \subseteq S(C)$ and $T(C) \subseteq S(C)$ such that

$$\|Tx - \tilde{T}x\| \leq \epsilon_1, \|Sx - \tilde{S}x\| \leq \epsilon_2 \text{ for all } x \in C \quad (3.1)$$

Let $z \in C(S, T)$ and $\tilde{x} \in C(S, \tilde{T})$ be the coincidence points of $S, T$ and $\tilde{S}, \tilde{T}$ respectively that is $Sz = Tx = u^*$ and $\tilde{S}z = \tilde{T}z = \tilde{u}^*$. Let $(S_{u_n})_{n=1}^\infty$ be the Jungck-T-CR iterative scheme generated by (1.1) with

1. $\frac{1}{2} < \beta_n \gamma_n \text{ for all } n \in N$.
2. $\sum_{n=1}^\infty \beta_n \gamma_n = \infty$.

Let $(\tilde{S}_{\tilde{u}})_{n=1}^\infty$ be a sequence defined by

$$\tilde{u}_n = X \quad \tilde{S}_{\tilde{u}_{n+1}} = \tilde{T}[(1 - \alpha_n)\tilde{S}_{\tilde{u}_n} + \alpha_n \tilde{T}_{\tilde{u}_n}]$$
$$\tilde{S}_{\tilde{u}_n} = \tilde{T}[(1 - \beta_n)\tilde{T}_{\tilde{u}_n} + \beta_n \tilde{T}_{\tilde{u}_n}]$$
$$\tilde{S}_{\tilde{u}_n} = \tilde{T}[(1 - \gamma_n)\tilde{S}_{\tilde{u}_n} + \gamma_n \tilde{T}_{\tilde{u}_n}], \quad n \in N \quad (3.2)$$

Assume that $(S_{u_n})_{n=1}^\infty$ and $(\tilde{S}_{\tilde{u}_n})_{n=1}^\infty$ converges to $u^*$ and $\tilde{u}^*$ respectively. Then we have

$$\|u^* - \tilde{u}^*\| \leq \frac{9\epsilon_1 + 4\epsilon_2}{1 - \delta}$$

Proof: It follows from Jungck-T-CR iteration (1.1) and Jungck-contraction condition (1.2), (3.1) and (3.2) that

$$\|S_{u_{n+1}} - \tilde{S}_{\tilde{u}_{n+1}}\| = \|T[(1 - \alpha_n)Sv_n + \alpha_n Tv_n] - \tilde{T}[(1 - \alpha_n)S\tilde{v}_n + \alpha_n \tilde{T}\tilde{v}_n]\|$$

Put $q_n = T[(1 - \alpha_n)Sv_n + \alpha_n Tv_n]$ and

$$r_n = \tilde{T}[(1 - \alpha_n)S\tilde{v}_n + \alpha_n \tilde{T}\tilde{v}_n]$$

then we have:

$$\|S_{u_{n+1}} - \tilde{S}_{\tilde{u}_{n+1}}\| = \|q_n - r_n + r_n - \tilde{T}[(1 - \alpha_n)S\tilde{v}_n + \alpha_n \tilde{T}\tilde{v}_n]\|$$

$$\leq \|q_n - r_n\| + \|r_n - \tilde{T}[(1 - \alpha_n)S\tilde{v}_n + \alpha_n \tilde{T}\tilde{v}_n]\|$$

$$+ \|\tilde{T}[(1 - \alpha_n)S\tilde{v}_n + \alpha_n \tilde{T}\tilde{v}_n] - \tilde{T}[(1 - \alpha_n)S\tilde{v}_n + \alpha_n \tilde{T}\tilde{v}_n]\|$$

$$\leq \delta \|S[(1 - \alpha_n)Sv_n + \alpha_n Tv_n] - S[(1 - \alpha_n)S\tilde{v}_n + \alpha_n \tilde{T}\tilde{v}_n]\| + \epsilon_1$$

$$\leq \epsilon_1$$

$$+ \delta \|S[(1 - \alpha_n)Sv_n + \alpha_n Tv_n] - S[(1 - \alpha_n)S\tilde{v}_n + \alpha_n \tilde{T}\tilde{v}_n]\|$$

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\[
\begin{align*}
&\leq \epsilon_1 + \delta(1 - \alpha_n) \| S v_n - S \tilde{v}_n \| + \alpha_n \delta \| T v_n - T \tilde{v}_n \| \\
&+ \alpha_n \delta \| T v_n - T \tilde{v}_n + T \tilde{v}_n - T \tilde{v}_n \|
\end{align*}
\]
Therefore
\[
\| S u_{n+1} - S \tilde{u}_{n+1} \| \leq (1 + \alpha_n \delta) \epsilon_1 + \delta(1 - \alpha_n) \| S v_n - S \tilde{v}_n \|
\]
Also
\[
\| S v_n - S \tilde{v}_n \| = \| T[(1 - \beta_n) T u_n + \beta_n T w_n] - T[(1 - \beta_n) T \tilde{u}_n + \beta_n T \tilde{w}_n] \|
\]
Put \( a_n = T[(1 - \beta_n) T u_n + \beta_n T w_n] \) and
\[
\| S v_n - S \tilde{v}_n \| \leq \| a_n - b_n + b_n - T[(1 - \beta_n) T \tilde{u}_n + \beta_n T \tilde{w}_n] \|
\]
Therefore
\[
\| S v_n - S \tilde{v}_n \| \leq \epsilon_1 + \delta^2(1 - \beta_n) \| S u_n - S \tilde{u}_n \| + \delta(1 - \beta_n) \epsilon_1
\]

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But
\[ \|S\bar{u}_n - S\tilde{u}_n\| = \|T[(1 - \gamma_n)Su_n + \gamma_n Tu_n] - T[(1 - \gamma_n)\bar{S}u_n + \gamma_n \bar{T}u_n]\| \]
Put \( a_n = T[(1 - \gamma_n)Su_n + \gamma_n Tu_n] \) and
\[ b_n = T[(1 - \gamma_n)S\bar{u}_n + \gamma_n \bar{T}u_n] \] we get
\[ \|S\bar{u}_n - S\tilde{u}_n\| = \|a_n - b_n + b_n - T[(1 - \gamma_n)\bar{S}u_n + \gamma_n \bar{T}u_n]\| \]
\[ + \|T[(1 - \gamma_n)S\bar{u}_n + \gamma_n \bar{T}u_n] - T[(1 - \gamma_n)S\bar{u}_n + \gamma_n \bar{T}u_n]\| \]
\[ \leq \varepsilon_1 + \delta\|S[(1 - \gamma_n)Su_n + \gamma_n Tu_n] - S[(1 - \gamma_n)\bar{S}u_n + \gamma_n \bar{T}u_n]\| \]
\[ \leq \varepsilon_1 + \delta\|(1 - \gamma_n)Su_n + \gamma_n Tu_n - (1 - \gamma_n)S\bar{u}_n - \gamma_n \bar{T}u_n\| \]
\[ \leq \varepsilon_1 + \delta(1 - \gamma_n)\|Su_n - S\bar{u}_n\| + \delta\gamma_n\|Tu_n - \bar{T}u_n\| \]
\[ \leq \varepsilon_1 + \delta(1 - \gamma_n)\|Su_n - S\bar{u}_n\| + \delta\gamma_n\|Tu_n - \bar{T}u_n\| \]
(3.5)
Substituting (3.3) and (3.4) in (3.5), we get:
\[ \|S\bar{u}_{n+1} - S\tilde{u}_{n+1}\| \leq \left[ (1 + \alpha_n)\varepsilon_1 + \alpha_n\varepsilon_2 + \delta[1 - \alpha_n(1 - \delta)] \right] \]
\[ \left[ + (1 + \delta)\varepsilon_1 + \delta^2\varepsilon_2 + \delta^2(1 - \beta_n)\|Su_n - S\bar{u}_n\| \right] \]
\[ + \varepsilon_1[1 + \alpha_n\delta + 1 + \delta + \beta_n\delta^2(1 + \gamma_n\delta)] \]
\[ + \varepsilon_1[2 + (\alpha_n + 1)\delta + \beta_n\delta^2(1 + \gamma_n\delta)] \]
\[ + \varepsilon_1(\alpha_n + 1)\delta^2 + \beta_n\gamma_n\delta^4 \]
\[ + \varepsilon_1[1 - \beta_n(1 - \delta) - \beta_n\gamma_n\delta(1 - \delta)]\|Su_n - S\bar{u}_n\| \]
(3.6)
Since \( \alpha_n, \beta_n, \gamma_n \in [0,1] \) and \( \frac{1}{2} < \beta_n \gamma_n \) for all \( n \in \mathbb{N} \) and using of the fact \( \delta \in (0,1) \) in (3.6)
Yields:
\[ \|S\bar{u}_{n+1} - S\tilde{u}_{n+1}\| \leq \|1 - \beta_n\gamma_n(1 - \delta)\|\|Su_n - S\bar{u}_n\| \]
(3.7)
Put
\[ a_n = \|Su_n - S\bar{u}_n\| \]
\[ \lambda_n = \beta_n\gamma_n(1 - \delta) \in (0,1) \]
\[ \rho_n = \frac{9\varepsilon_1 + 4\varepsilon_2}{1 - \delta} \]
Hence, the inequality (3.7) perform all assumptions in lemma (1.6) and thus an application of lemma (1.6) to (3.7) yields
\[ 0 \leq \lim_{n \to \infty} \sup \|S\bar{u}_n - S\tilde{u}_n\| \]
\[ \leq \lim_{n \to \infty} \sup \frac{9\varepsilon_1 + 4\varepsilon_2}{1 - \delta} \]
Since \( \lim_{n \to \infty} S\bar{u}_n = u^* \) and \( \lim_{n \to \infty} S\tilde{u}_n = \bar{u}^* \), then
\[ \|u^* - \bar{u}^*\| \leq \frac{9\varepsilon_1 + 4\varepsilon_2}{1 - \delta} \]
References


