

On Rate of Convergence of Jungck-T-CR Iterative Procedure

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Abstract: This paper concerns with the convergence and rate of convergence of Jungck-T-CR iterative procedure. We show that the previous iteration converges to a unique common fixed point when applied to a pair of Jungck-contraction mappings under certain condition. Also, we compare the speed of various Jungck-iterative schemes with Jungck-T-CR iterative procedure for a pair of Jungck-contraction mappings under certain condition.

Keywords: Jungck iterative procedures, convergent sequences, rate of convergent sequences, Jungck-contraction mapping

1. Introduction and Preliminaries

In 1976, Jungck [4] generalized Banach's contraction principle using the concept of commuting mappings which was given by Pfeffer [9] but Jungck has introduced it in more general context.

Proposition (1.1) [4]:

Let S be a mapping on a set X into itself. Thus S has a fixed point if and only if there is a constant mapping $T: X \rightarrow X$ which commutes with S (i.e., $T(S(x)) = S(T(x))$ for all $x \in X$).

Hence Jungck [4] has used this proposition and produced his theorem of common fixed point.

Theorem (1.2) [4]:

Let S be a continuous mapping of a complete metric space (X, d) into itself. Then S has a fixed point in X if and only if there exists $\delta \in (0, 1)$ and a mapping $T: X \rightarrow X$ which commutes with S and satisfies

$$T(X) \subset S(X) \text{ and } d(Tx, Ty) \leq \delta d(Sx, Sy) \quad (*)$$

For all $x, y \in X$. Indeed S and T have common fixed point if (*) holds.

And in 1986, Jungck [5], introduced more generalized commuting mappings, called compatible mappings which are useful for obtaining common fixed points of mappings.

Definition (1.3) [5]:

Let (X, d) be a metric space, $T, S: X \rightarrow X$ are said to be compatible if

$$\lim_{n \rightarrow \infty} d(TS(x_n), ST(x_n)) = 0$$

where $\{x_n\}_{n=0}^{\infty}$ is a sequence such that $\lim_{n \rightarrow \infty} Tx_n = \lim_{n \rightarrow \infty} Sx_n = t$ for some $t \in X$.

Thus in 1996 Jungck et. al. [6] introduced the concept of coincidence point and depending on it, in 1998, Jungck and Rhoades [7] defined the notion of weakly compatible and showed that compatible mappings are weakly compatible but the converse is not true.

Definition (1.4) [7]:

Let B be a Banach space and $T, S: B \rightarrow B$. A point $u^* \in B$ is called a coincidence point of a pair of self mappings T, S if there exists a point z (called a point of coincidence) in B such that $z = Su^* = Tu^*$. Two self mappings S and T are weakly compatible if they commute at their coincidence points, that is if $Su^* = Tu^*$ for some $u^* \in B$ then $STu^* = TSu^*$. And the point $u^* \in B$ is called common fixed point of S and T if $u^* = Su^* = Tu^*$.

$C(S, T)$ denotes the set of coincidence points of S and T .

In 2005, Singh et. al. [10] significantly improved on the result of Jungck [4] when he proved the following result which is now called Jungck-contraction principle.

Theorem (1.5) [10]:

Let (X, d) be a metric space. Let $T, S: X \rightarrow X$ satisfying $d(Tx, Ty) \leq \delta d(Sx, Sy)$, $0 \leq \delta < 1$, for all $x, y \in X$. $T(X) \subseteq S(X)$ and $S(X)$ or $T(X)$ is a complete subspace of X , then S and T have a coincidence. Indeed, for any $x_1 \in X$, there exists a sequence $\{x_n\}_{n=1}^{\infty}$ in X such that

1. $Sx_{n+1} = Tx_n$, $n = 1, 2, \dots$
2. $\{Sx_n\}_{n=1}^{\infty}$ converges to Su^* for some u^* in X , and $Su^* = Tu^*$ that is S and T have a coincidence at u^* .

Further, if S, T commute (just) at u^* then S and T have a unique common fixed point.

Remark (1.6):

If $S = id$ (identity mapping), then the Jungck-contraction mapping

$$d(Tx, Ty) \leq \delta d(Sx, Sy), 0 \leq \delta < 1 \quad (1.1)$$

is the same as the well known the contraction mapping.

Olatinwo et. al. [8] introduced Jungck-Ishikawa iterative scheme and proved its convergence of the coincidence point of a pair of certain mappings with the assumption that one of the pair of mappings is injective. Its iterative scheme is defined as follows:

Definition (1.7) [8]:

Let B be a Banach space and C be a nonempty subset of B . Let $T, S: C \rightarrow C$ be two self mappings such that $T(C) \subseteq S(C)$. For $q_1 \in C$ the Jungck-Ishikawa iterative scheme is the sequence $\{Sq_n\}_{n=1}^{\infty}$ defined by $Sq_{n+1} = (1 - \alpha_n)Sq_n + \alpha_n Tr_n$

$Sr_n = (1 - \beta_n)Sq_n + \beta_n Tq_n, n \in \mathbb{N}$ (1.2)
 where $\{\alpha_n\}_{n=1}^\infty$ and $\{\beta_n\}_{n=1}^\infty$ are real sequences in $[0,1]$ such that $\sum_{n=1}^\infty \alpha_n = \infty$.

Hussain et. al. [3] introduced the Jungck-CR iterative scheme and proved its convergence to a unique common fixed point of a pair of certain mappings without assuming the injectivity of any of the mappings but rather they proved their results for a pair of weakly compatible mappings S, T .

Definition (1.8) [3]:

Let B be a Banach space and C be a nonempty subset of B . Let $T, S: C \rightarrow C$ be two self mappings such that $T(C) \subseteq S(C)$. For $a_0 \in C$, the Jungck-CR iterative scheme is the sequence $\{Sa_n\}_{n=1}^\infty$ defined by
 $Sa_{n+1} = (1 - \alpha_n)Sb_n + \alpha_n Tb_n$
 $Sb_n = (1 - \beta_n)Ta_n + \beta_n Tc_n$
 $Sa_n = (1 - \gamma_n)Sa_n + \gamma_n Ta_n, n \in \mathbb{N}$ (1.3)
 where $\{\alpha_n\}_{n=1}^\infty, \{\beta_n\}_{n=1}^\infty$ and $\{\gamma_n\}_{n=1}^\infty$ are real sequences in $[0,1]$ such that $\sum_{n=1}^\infty \alpha_n = \infty$.
 Recently, Badri [1] defined the following Jungck-Picard-S iterative scheme.

Definition (1.9) [1]:

Let B be a Banach space and C be a nonempty subset of B . Let $T, S: C \rightarrow C$ be two self mappings such that $T(C) \subseteq S(C)$. For $x_1 \in C$, the Jungck-Picard-S iterative scheme is the sequence $\{Sx_n\}_{n=1}^\infty$ defined by
 $Sx_{n+1} = Ty_n$
 $Sy_n = (1 - \beta_n)Tx_n + \beta_n Tz_n$
 $Sz_n = (1 - \gamma_n)Sx_n + \gamma_n Tx_n, n \in \mathbb{N}$ (1.4)
 where $\{\beta_n\}_{n=1}^\infty$ and $\{\gamma_n\}_{n=1}^\infty$ are real sequences in $[0,1]$ such that $\sum_{n=1}^\infty \beta_n \gamma_n = \infty$.

In [12], we define T -CR iteration as follows:

Definition (1.10) [12]:

Let C be a nonempty closed convex subset of a Banach space X and $T: C \rightarrow C$ be a self-mapping with $\{\alpha_n\}_{n=1}^\infty, \{\beta_n\}_{n=1}^\infty$ and $\{\gamma_n\}_{n=1}^\infty$ are real sequences in $[0,1]$ such that $\sum_{n=1}^\infty \alpha_n = \infty$. The T -CR iterative scheme $\{u_n\}_{n=1}^\infty$ is defined by:

$$\begin{aligned} u_1 &\in C \\ u_{n+1} &= T[(1 - \alpha_n)v_n + \alpha_n Tv_n] \\ v_n &= T[(1 - \beta_n)Tu_n + \beta_n Tw_n] \\ w_n &= T[(1 - \gamma_n)u_n + \gamma_n Tu_n], n \in \mathbb{N} \end{aligned}$$

In this section, we define Jungck- T -CR iteration as follows:

Definition (1.11):

Let B be a Banach space and C be a nonempty subset of B . Let $T, S: C \rightarrow C$ be two self mappings such that $T(C) \subseteq S(C)$. For $u_0 \in C$ the Jungck- T -CR iterative scheme is the sequence $\{Su_n\}_{n=1}^\infty$ is defined by:
 $Su_{n+1} = T[(1 - \alpha_n)Sv_n + \alpha_n Tv_n]$
 $Sv_n = T[(1 - \beta_n)Tu_n + \beta_n Tw_n]$
 $Sw_n = T[(1 - \gamma_n)Su_n + \gamma_n Tu_n],$
 $n \in \mathbb{N}$ (1.5)

where $\{\alpha_n\}_{n=1}^\infty, \{\beta_n\}_{n=1}^\infty$ and $\{\gamma_n\}_{n=1}^\infty$ are real sequences in $[0,1]$ such that $\sum_{n=1}^\infty \alpha_n = \sum_{n=1}^\infty \beta_n = \sum_{n=1}^\infty \gamma_n = \infty$. The following definition will be needed in the sequel.

Definition (1.12), [2]:

Let X be a Banach space, C be a nonempty closed convex subset of X . A self mapping $T: C \rightarrow C$ is said to be **nonexpansive** if for all x, y in C , we have

$$\|Tx - Ty\| \leq \|x - y\|$$

Furthermore T is called **quasi-nonexpansive** if $y = u^*$ provided T has a fixed point in C and if $u^* \in C$ is a fixed point of T , then

$$\|Tx - Tu^*\| \leq \|x - u^*\| \quad (1.6)$$

is true for all $x \in C$.

2. Convergence of Jungck-T-CR Iterative Procedure

In this section, we study the convergence of Jungck- T -CR iteration (1.5) when applied to Jungck-contraction mapping (1.1) under certain condition.

Theorem (2.1):

Let C be a nonempty closed convex subset of a Banach space B , $S, T: C \rightarrow C$ be two self-mappings satisfying Jungck-contraction condition (1.1) provided that S is quasi-nonexpansive mapping (1.6) as well, assume $T(C) \subseteq S(C)$ and S, T are weakly compatible, suppose that there exists a $z \in C(S, T)$ be the coincidence points of S, T such that $Sz = Tz = u^*$. Let $\{Su_n\}_{n=1}^\infty$ be the Jungck- T -CR iterative scheme generated by (1.5), where $\{\alpha_n\}_{n=1}^\infty, \{\beta_n\}_{n=1}^\infty$ and $\{\gamma_n\}_{n=1}^\infty$ are real sequences in $[0,1]$ satisfying $\sum_{n=1}^\infty \alpha_n = \sum_{n=1}^\infty \beta_n = \sum_{n=1}^\infty \gamma_n = \infty$. Then the Jungck- T -CR iterative scheme $\{Su_n\}_{n=1}^\infty$ converges to a unique common fixed point u^* of S, T .

Proof:

It follows from (1.1) and (1.2) that:

$$\begin{aligned} \|Su_{n+1} - u^*\| &= \|T[(1 - \alpha_n)Sv_n + \alpha_n Tv_n] - u^*\| \\ &\leq \delta \|S[(1 - \alpha_n)Sv_n + \alpha_n Tv_n] - u^*\| \\ &\leq \delta \|(1 - \alpha_n)Sv_n + \alpha_n Tv_n - u^*\| \\ &\leq \delta(1 - \alpha_n)\|Sv_n - u^*\| + \alpha_n \delta \|Tv_n - u^*\| \\ &\leq \delta(1 - \alpha_n)\|Sv_n - u^*\| + \alpha_n \delta^2 \|Sv_n - u^*\| \\ &= [1 - \alpha_n(1 - \delta)]\|Sv_n - u^*\| \quad (2.1) \\ \|Sv_n - u^*\| &= \|T[(1 - \beta_n)Tu_n + \beta_n Tw_n] - u^*\| \\ &\leq \delta \|S[(1 - \beta_n)Tu_n + \beta_n Tw_n] - u^*\| \\ &\leq \delta \|(1 - \beta_n)Tu_n + \beta_n Tw_n - u^*\| \\ &\leq \delta(1 - \beta_n)\|Tu_n - u^*\| + \beta_n \delta \|Tw_n - u^*\| \\ &\leq \delta^2(1 - \beta_n)\|Su_n - u^*\| + \beta_n \delta^2 \|Sw_n - u^*\| \quad (2.2) \\ \|Sw_n - u^*\| &= \|T[(1 - \gamma_n)Su_n + \gamma_n Tu_n] - u^*\| \\ &\leq \delta \|S[(1 - \gamma_n)Su_n + \gamma_n Tu_n] - u^*\| \\ &\leq \delta \|(1 - \gamma_n)Su_n + \gamma_n Tu_n - u^*\| \\ &\leq \delta(1 - \gamma_n)\|Su_n - u^*\| + \gamma_n \delta \|Tu_n - u^*\| \\ &\leq \delta[1 - \gamma_n(1 - \delta)]\|Su_n - u^*\| \quad (2.3) \end{aligned}$$

It follows from (2.1), (2.2) and (2.3) that

$$\begin{aligned} \|Su_{n+1} - u^*\| &\leq \delta^3 [1 - \alpha_n(1 - \delta)](1 - \beta_n)\|Su_n - u^*\| \\ &\quad + \delta^4 \beta_n [1 - \alpha_n(1 - \delta)][1 - \gamma_n(1 - \delta)]\|Su_n - u^*\| \\ &\leq \delta^3 [1 - \alpha_n(1 - \delta)] \\ &\quad \cdot [1 - \beta_n(1 - \delta) - \beta_n \gamma_n \delta(1 - \delta)]\|Su_n - u^*\| \end{aligned}$$

And so on, we get:

$$\|Su_{n+1} - u^*\| \leq \delta^{3(n+1)} \|Su_1 - u^*\|$$

$$\cdot \prod_{k=1}^n [1 - \alpha_k(1 - \delta)][1 - \beta_k(1 - \delta) - \beta_k \gamma_k \delta(1 - \delta)] \quad (2.3)$$

$\leq \delta^{3(n+1)} \|Su_1 - u^*\| e^{-(1-\delta) \sum_{k=1}^{\infty} \alpha_k - (1-\delta) \sum_{k=1}^{\infty} \beta_k - (1-\delta) \sum_{k=1}^{\infty} \beta_k \gamma_k}$
 Since $0 \leq \delta < 1$, $\sum_{n=1}^{\infty} \alpha_n = \infty$, $\sum_{n=1}^{\infty} \beta_n = \infty$ and $\sum_{n=1}^{\infty} \beta_n \gamma_n = \infty$ so $\delta^{3(n+1)} e^{-(1-\delta) \sum_{k=1}^{\infty} \alpha_k - (1-\delta) \sum_{k=1}^{\infty} \beta_k - (1-\delta) \sum_{k=1}^{\infty} \beta_k \gamma_k} \rightarrow 0$ as $n \rightarrow \infty$. Which implies that $\lim_{n \rightarrow \infty} \|Su_n - u^*\| = 0$.

Therefore, $\{Su_n\}_{n=1}^{\infty}$ converges to u^* .

Now, we will prove u^* is the unique fixed point of S, T .

Suppose that there exist two points of coincidence $z_1, z_2 \in C(S, T)$ such that $Sz_1 = Tz_1 = u_1^*$ and $Sz_2 = Tz_2 = u_2^*$.

Using condition (1.1), we have

$$0 \leq \|u_1^* - u_2^*\| = \|Tz_1 - Tz_2\| \leq \delta \|Sz_1 - Sz_2\|$$

$$0 \leq \|u_1^* - u_2^*\| = \delta \|u_1^* - u_2^*\|$$

which leads $(1 - \delta) \|u_1^* - u_2^*\| \leq 0$, since $0 \leq \delta < 1$

from which it follows that $\|u_1^* - u_2^*\| = 0$, that is $u_1^* = u_2^*$.

Now, since S, T are weakly compatible and $u^* = Tz = Sz$ then $Tu^* = TTz = TSz = STz$. Hence $Tu^* = Su^*$.

Therefore, Tu^* is a point of coincidence of S, T but the coincidence point is unique, so $u^* = Tu^*$. Thus $Tu^* = Su^* = u^*$. Therefore u^* is the unique common fixed point of S, T .

3. Rate of Convergence of Jungck-T-CR Iterative Procedure

We now compare the speed of Jungck- T -CR iterative scheme (1.5) and the speed of Jungck-CR (1.3), Jungck-Ishikawa (1.2) and Jungck-Picard-S (1.4) iterative schemes by the following theorem.

Theorem (3.1):

Let C be a nonempty closed convex subset of a Banach space B , $S, T: C \rightarrow C$ be two self-mappings satisfying Jungck-contraction condition (1.1) provided that S is quasi-nonexpansive mapping (1.6) as well, assume $T(C) \subseteq S(C)$, let $\{Su_n\}_{n=1}^{\infty}$, $\{Sa_n\}_{n=1}^{\infty}$, $\{Sq_n\}_{n=1}^{\infty}$ and $\{Sx_n\}_{n=1}^{\infty}$ be the Jungck- T -CR iterative scheme (1.5) and the speed of Jungck-CR (1.3), Jungck-Ishikawa (1.2) and Jungck-Picard-S (1.4) iterative schemes respectively satisfying $\lim_{n \rightarrow \infty} \alpha_n = \lim_{n \rightarrow \infty} \beta_n = \lim_{n \rightarrow \infty} \alpha_n \beta_n = \lim_{n \rightarrow \infty} \beta_n \gamma_n = 0$. Then $\{Su_n\}_{n=0}^{\infty}$ converges to u^* faster than $\{Sa_n\}_{n=1}^{\infty}$, $\{Sq_n\}_{n=1}^{\infty}$ and $\{Sx_n\}_{n=1}^{\infty}$ do.

Proof:

From inequality (2.3), we have

$$\|Su_{n+1} - u^*\| \leq \delta^{3(n+1)} \|Su_1 - u^*\| \prod_{k=1}^n [1 - \alpha_k(1 - \delta)][1 - \beta_k(1 - \delta) - \beta_k \gamma_k \delta(1 - \delta)] \quad (3.1)$$

From Jungck-CR iteration (1.3) and Jungck-contraction condition (1.1), it is easy to get that:

$$\|Sa_{n+1} - u^*\| \leq \delta^{(n+1)} \prod_{k=1}^n [1 - \alpha_k(1 - \delta)][1 - \beta_k \gamma_k \delta(1 - \delta) - \delta Sa_1 - u^*] \quad (3.2)$$

Using (3.1) and (3.2), we obtain:

$$\frac{\|Su_{n+1} - u^*\|}{\delta^{3(n+1)} \prod_{k=1}^n [1 - \alpha_k(1 - \delta)][1 - \beta_k(1 - \delta) - \beta_k \gamma_k \delta(1 - \delta)] \|Su_1 - u^*\|} \leq \frac{\|Sa_{n+1} - u^*\|}{\delta^{(n+1)} \prod_{k=1}^n [1 - \alpha_k(1 - \delta)][1 - \beta_k \gamma_k \delta(1 - \delta)] \|Sa_1 - u^*\|}$$

$$= \delta^{2(n+1)} \prod_{k=1}^n \frac{[1 - \alpha_k(1 - \delta)][1 - \beta_k(1 - \delta) - \beta_k \gamma_k \delta(1 - \delta)] \|Su_1 - u^*\|}{[1 - \alpha_k(1 - \delta)][1 - \beta_k \gamma_k \delta(1 - \delta)] \|Sa_1 - u^*\|}$$

$$\text{Define } \theta_n = \delta^{2(n+1)} \prod_{k=1}^n \frac{[1 - \alpha_k(1 - \delta)][1 - \beta_k(1 - \delta) - \beta_k \gamma_k \delta(1 - \delta)]}{[1 - \alpha_k(1 - \delta)][1 - \beta_k \gamma_k \delta(1 - \delta)]}$$

By the assumption

$$\lim_{n \rightarrow \infty} \frac{\theta_{n+1}}{\theta_n} =$$

$$\lim_{n \rightarrow \infty} \frac{\delta^{2(n+2)} \prod_{k=1}^{n+1} [1 - \alpha_k(1 - \delta)][1 - \beta_k(1 - \delta) - \beta_k \gamma_k \delta(1 - \delta)]}{\delta^{2(n+1)} \prod_{k=1}^n [1 - \alpha_k(1 - \delta)][1 - \beta_k \gamma_k \delta(1 - \delta)]} = \lim_{n \rightarrow \infty} \frac{\delta [1 - \beta_{n+1}(1 - \delta) - \beta_{n+1} \gamma_{n+1} \delta(1 - \delta)]}{[1 - \beta_{n+1} \gamma_{n+1} \delta(1 - \delta)]} = \delta < 1$$

Thus it follows from ratio test that $\sum_{n=1}^{\infty} \theta_n < \infty$. Hence, we have $\lim_{n \rightarrow \infty} \theta_n = 0$ which implies that the iterative sequence defined by Jungck- T -CR (1.5) converges to u^* faster than the iterative sequence defined by Jungck-CR iteration method (1.3).

From Jungck-Ishikawa iterative scheme (1.2) and Jungck-contraction condition (1.1), we obtain:

$$\|Sq_{n+1} - u^*\| = \|(1 - \alpha_n)Sq_n + \alpha_n Tr_n - u^*\| \leq (1 - \alpha_n) \|Sq_n - u^*\| + \alpha_n \|Tr_n - u^*\| \leq (1 - \alpha_n) \|Sq_n - u^*\| + \alpha_n \delta \|Sr_n - u^*\| \quad (3.3)$$

$$\|Sr_n - u^*\| = \|(1 - \beta_n)Sq_n + \beta_n Tq_n - u^*\| \leq (1 - \beta_n) \|Sq_n - u^*\| + \beta_n \|Tq_n - u^*\| \leq (1 - \beta_n) \|Sq_n - u^*\| + \beta_n \delta \|Sq_n - u^*\| \leq [1 - \beta_n(1 - \delta)] \|Sq_n - u^*\| \quad (3.4)$$

Substituting (3.4) in (3.3), we have:

$$\|Sq_{n+1} - u^*\| \leq (1 - \alpha_n) \|Sq_n - u^*\| + \alpha_n \delta [1 - \beta_n(1 - \delta)] \|Sq_n - u^*\| = [1 - \alpha_n(1 - \delta) - \alpha_n \beta_n \delta(1 - \delta)] \|Sq_n - u^*\|$$

u^*

Repeating this process n times, we get

$$\|Sq_{n+1} - u^*\| \leq \prod_{k=1}^n [1 - \alpha_k(1 - \delta) - \alpha_k \beta_k \delta(1 - \delta)] \|Sq_1 - u^*\| \quad (3.5)$$

Using (3.1) and (3.5), we have

$$\frac{\|Su_{n+1} - u^*\|}{\delta^{3(n+1)} \prod_{k=1}^n [1 - \alpha_k(1 - \delta)][1 - \beta_k(1 - \delta) - \beta_k \gamma_k \delta(1 - \delta)] \|Su_1 - u^*\|} \leq \frac{\|Sq_{n+1} - u^*\|}{\delta^{3(n+1)} \prod_{k=1}^n [1 - \alpha_k(1 - \delta)][1 - \beta_k(1 - \delta) - \beta_k \gamma_k \delta(1 - \delta)] \|Su_1 - u^*\|}$$

$$\text{Define } \theta_n = \delta^{3(n+1)} \prod_{k=1}^n \frac{[1 - \alpha_k(1 - \delta)][1 - \beta_k(1 - \delta) - \beta_k \gamma_k \delta(1 - \delta)]}{[1 - \alpha_k(1 - \delta) - \alpha_k \beta_k \delta(1 - \delta)]}$$

By the assumption

$$\lim_{n \rightarrow \infty} \frac{\theta_{n+1}}{\theta_n} =$$

$$\lim_{n \rightarrow \infty} \frac{\delta^{3(n+2)} \prod_{k=1}^{n+1} [1 - \alpha_k(1 - \delta)][1 - \beta_k(1 - \delta) - \beta_k \gamma_k \delta(1 - \delta)]}{\delta^{3(n+1)} \prod_{k=1}^n [1 - \alpha_k(1 - \delta)][1 - \beta_k(1 - \delta) - \beta_k \gamma_k \delta(1 - \delta)]} = \lim_{n \rightarrow \infty} \frac{\delta^3 [1 - \alpha_{n+1}(1 - \delta)][1 - \beta_{n+1}(1 - \delta) - \beta_{n+1} \gamma_{n+1} \delta(1 - \delta)]}{[1 - \alpha_{n+1}(1 - \delta) - \alpha_{n+1} \beta_{n+1} \delta(1 - \delta)]} = \delta^3 < 1$$

Thus it follows from ratio test that $\sum_{n=1}^{\infty} \theta_n < \infty$. Hence, we have $\lim_{n \rightarrow \infty} \theta_n = 0$ which implies that the iterative sequence defined by Jungck-*T*-CR (1.5) converges to u^* faster than the iterative sequence defined by Jungck-Ishikawa iteration method (1.2).

From Jungck-Picard-S iterative scheme (1.4) and Jungck-contraction condition (1.1), we have obtained the following inequality:

$$\|Sx_{n+1} - u^*\| \leq \delta^{2(n+1)} \prod_{k=1}^n [1 - \beta_k \gamma_k (1 - \delta)] \|Sx_1 - u^*\| \quad (3.6)$$

From inequality (3.1) and (3.6), we have:

$$\frac{\|Su_{n+1} - u^*\|}{\|Sx_{n+1} - u^*\|} \leq \frac{\delta^{3(n+1)} \prod_{k=1}^n [1 - \alpha_k (1 - \delta)] [1 - \beta_k (1 - \delta) - \beta_k \gamma_k \delta (1 - \delta)] \|Su_1 - u^*\|}{\delta^{2(n+1)} \prod_{k=1}^n [1 - \beta_k \gamma_k (1 - \delta)] \|Sx_1 - u^*\|}$$

$$= \delta^{(n+1)} \prod_{k=1}^n \frac{[1 - \alpha_k (1 - \delta)] [1 - \beta_k (1 - \delta) - \beta_k \gamma_k \delta (1 - \delta)] \|Su_1 - u^*\|}{[1 - \beta_k \gamma_k (1 - \delta)] \|Sx_1 - u^*\|}$$

$$\text{Define } \theta_n = \delta^{(n+1)} \prod_{k=1}^n \frac{[1 - \alpha_k (1 - \delta)] [1 - \beta_k (1 - \delta) - \beta_k \gamma_k \delta (1 - \delta)]}{[1 - \beta_k \gamma_k (1 - \delta)]}$$

By the assumption

$$\lim_{n \rightarrow \infty} \frac{\theta_{n+1}}{\theta_n} =$$

$$\lim_{n \rightarrow \infty} \frac{\delta^{n+2} [1 - \alpha_{n+1} (1 - \delta)] [1 - \beta_{n+1} (1 - \delta) - \beta_{n+1} \gamma_{n+1} \delta (1 - \delta)]}{\delta^{n+1} [1 - \beta_{n+1} \gamma_{n+1} (1 - \delta)]} =$$

$$= \lim_{n \rightarrow \infty} \frac{\delta^3 [1 - \alpha_{n+1} (1 - \delta)] [1 - \beta_{n+1} (1 - \delta) - \beta_{n+1} \gamma_{n+1} \delta (1 - \delta)]}{[1 - \beta_{n+1} \gamma_{n+1} (1 - \delta)]}$$

$$= \delta < 1$$

Thus it follows from ratio test that $\sum_{n=1}^{\infty} \theta_n < \infty$. Hence, we have $\lim_{n \rightarrow \infty} \theta_n = 0$ which implies that the iterative sequence defined by Jungck-*T*-CR (1.5) converges to u^* faster than the iterative sequence defined by Jungck-Picard-S iteration method (1.4).

We now support the result of the above theorem by the following example using computer programming in java for comparing the speed of Jungck-*T*-CR iterative scheme (1.5) and the speed of Jungck-CR (1.3), Jungck-Ishikawa (1.2) and Jungck-Picard-S (1.4) iterative schemes.

Example (3.2): Let $B = \mathbb{R}$, $C = [0,1]$, $S, T: C \rightarrow C$ are mappings defined as $Sx = 1 - x$ and $Tx = \frac{2x+1}{4}$ for all $x \in C$. It is easily seen that the mappings S and T satisfying Jungck-contraction condition (1.1) provided that S is quasi-nonexpansive mapping (1.6) with the unique common fixed point 0.5 take $\alpha_n = \beta_n = \gamma_n = 0.1$ for all $n = 1, \dots, 99$ with initial value 0.1. The comparison of the rate of convergence of the speed of Jungck-*T*-CR iterative scheme (1.5) and the speed of Jungck-CR (1.3), Jungck-Ishikawa (1.2) and Jungck-Picard-S (1.4) iterative schemes to a common fixed point of S and T is shown in the following tables.

Jungck- <i>T</i> -CR Iterative procedure			
n	Sx_{n+1}	Tx_n	x_{n+1}
1	0.49598874208984380	0.49899718552246090	0.49598874208984367
2	0.49959831012584077	0.50005021123427000	0.50040168987415920
3	0.49995977452494555	0.49999748590780907	0.49995977452494555
4	0.49999597179578714	0.50000012588138170	0.50000402820421290
5	0.49999959661311250	0.49999999369707987	0.49999959661311255
...
99	0.50000000000000000	0.50000000000000000	0.50000000000000000

Jungck-CR			
n	Sx_{n+1}	Tx_n	x_{n+1}
1	0.57009875625000020	0.58372500000000000	0.42990124374999983
2	0.47065490816484370	0.46495062187499990	0.52934509183515630
3	0.51228458906949230	0.51467254591757820	0.48771541093050774
4	0.49485736390078383	0.49385770546525387	0.50514263609921620
5	0.50215283603703440	0.50257131804960810	0.49784716396296560
...
99	0.50000000000000000	0.50000000000000000	0.50000000000000000

Jungck-Ishikawa			
n	Sx_{n+1}	Tx_n	x_{n+1}
1	0.79412250000000010	0.32849999999999996	0.20587749999999994
2	0.75221004375000010	0.35293874999999997	0.24778995624999990
3	0.71627011251562510	0.37389497812499994	0.28372988748437490
4	0.68545162148214860	0.39186494374218744	0.31454837851785145
5	0.65902476542094250	0.40727418925892570	0.34097523457905754
...
224	0.50000000000000040	0.49999999999999970	0.49999999999999956

Jungck-Picard-S			
n	Sx_{n+1}	Tx_n	x_{n+1}
1	0.524255625000000000	0.450750000000000000	0.475744375000000000
2	0.505972947656250000	0.487872187500000000	0.494027052343750000
3	0.50147083836035160	0.497013526171875000	0.49852916163964844
4	0.50036219394623660	0.49926458081982420	0.49963780605376340
5	0.50008919025926080	0.49981890302688170	0.49991080974073920
...
99	0.500000000000000000	0.500000000000000000	0.500000000000000000

By observing the above tables, we conclude the decreasing rate of convergence of iterative schemes is as follows: Jungck-T-CR, Jungck-Picard-S, Jungck-CR and Jungck-Ishikawa iterative schemes.

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