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On Rate of Convergence of Jungck-T-CR Iterative Procedure

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Abstract: ***This paper concerns with the convergence and rate of convergence of Jungck-T-CR iterative procedure. We show that the previous iteration converges to a unique common fixed point when applied to a pair of Jungck-contraction mappings under certain condition. Also, we compare the speed of various Jungck-iterative schemes with Jungck-T-CR iterative procedure for a pair of Jungck-contraction mappings under certain condition.***

Keywords: Jungck iterative procedures, convergent sequences, rate of convergent sequences, Jungck-contraction mapping

1. Introduction and Preliminaries

In 1976, Jungck [4] generalized Banach's contraction principle using the concept of commuting mappings which was given by Pfeffer [9] but Jungck has introduced it in more general context.

Proposition (1.1) [4]:
Let S be a mapping on a set X into itself. Thus S has a fixed point if and only if there is a constant mapping $T: X \to X$ which commutes with $\left(i.e., T\big(S(x)\big) = S\big(T(x)\big)\ for\ all\ x \in X\right)$.

Hence Jungck [4] has used this proposition and produced his theorem of common fixed point.

Theorem (1.2) [4]:
Let S be a continuous mapping of a complete metric space (X, d) into itself. Then S has a fixed point in X if and only if there exists $\delta \in (0,1)$ and a mapping $T: X \to X$ which commutes with S and satisfies
$T(X) \subset S(X)$ and $d(Tx, Ty) \le \delta d(Sx, Sy)$ (*)

For all $x, y \in X$. Indeed S and T have common fixed point if (*) holds.

And in 1986, Jungck [5], introduced more generalized commuting mappings, called compatible mappings which are useful for obtaining common fixed points of mappings.

Definition (1.3) [5]:
Let (X, d) be a metric space, $T, S: X \to X$ are said to be compatible if
$\lim_{n\to\infty} d\big(TS(x_n), ST(x_n)\big) = 0$
where $\{x_n\}_{n=0}^{\infty}$ is a sequence such that $\lim_{n\to\infty} Tx_n = \lim_{n\to\infty} Sx_n = t$ for some $t \in X$.

Thus in 1996 Jungck et. al. [6] introduced the concept of coincidence point and depending on it , in 1998, Jungck and Rhoades [7] defined the notion of weakly compatible and showed that compatible mappings are weakly compatible but the converse is not true.

Definition (1.4) [7]:
Let B be a Banach space and , $T, S: B \to B$. A point $u^* \in B$ is called a coincidence point of a pair of self mappings T, S if there exists a point z (called a point of coincidence) in B such that $z = Su^* = Tu^*$. Two self mappings S and T are weakly compatible if they commute at there coincidence points, that is if $Su^* = Tu^*$ for some $u^* \in B$ then $STu^* = TSu^*$. And the point $u^* \in B$ is called common fixed point of S and T if $u^* = Su^* = Tu^*$.
$C(S, T)$ denotes the set of coincidence points of S and T.

In 2005, Singh et. al. [10] significantly improved on the result of Jungck [4] when he proved the following result which is now called Jungck-contraction principle.

Theorem (1.5) [10]:
Let (X, d) be a metric space. Let $T, S: X \to X$ satisfying $d(Tx, Ty) \le \delta d(Sx, Sy)$, $0 \le \delta < 1$, for all $x, y \in X$. $T(X) \subseteq S(X)$ and $S(X)$ or $T(X)$ is a complete subspace of X, then S and T have a coincidence. Indeed, for any $x_1 \in X$, there exists a sequence $\{x_n\}_{n=1}^{\infty}$ in X such that

1. $Sx_{n+1} = Tx_n, n = 1,2,\cdots$
2. $\{Sx_n\}_{n=1}^{\infty}$ converges to Su^* for some u^* in X, and $Su^* = Tu^*$ that is S and T have a coincidence at u^*.

Further, if S, T commute (just) at u^* then S and T have a unique common fixed point.

Remark (1.6):
If $S = id$ (identity mapping), then the Jungck-contraction mapping

$$d(Tx, Ty) \le \delta d(Sx, Sy), 0 \le \delta < 1 \qquad (1.1)$$

is the same as the well known the contraction mapping.

Olatinwo et. al. [8] introduced Jungck-Ishikawa iterative scheme and proved its convergence of the coincidence point of a pair of certain mappings with the assumption that one of the pair of mappings is injective. Its iterative scheme is defined as follows:

Definition (1.7) [8]:
Let B be a Banach space and C be a nonempty subset of B. Let $T, S: C \to C$ be two self mappings such that $T(C) \subseteq S(C)$. For $q_1 \in C$ the Jungck-Ishikawa iterative scheme is the sequence $\{Sq_n\}_{n=1}^{\infty}$ defined by
$Sq_{n+1} = (1 - \alpha_n)Sq_n + \alpha_n Tr_n$

$$Sr_n = (1-\beta_n)Sq_n + \beta_n Tq_n, n \in \mathbb{N} \quad (1.2)$$

where $\{\alpha_n\}_{n=1}^{\infty}$ and $\{\beta_n\}_{n=1}^{\infty}$ are real sequences in [0,1) such that $\sum_{n=1}^{\infty} \alpha_n = \infty$.

Hussain et. al. [3] introduced the Jungck-CR iterative scheme and proved its convergence to a unique common fixed point of a pair of certain mappings without assuming the injectivity of any of the mappings but rather they proved their results for a pair of weakly compatible mappings S,T.

Definition (1.8) [3]:

Let B be a Banach space and C be a nonempty subset of B. Let $T,S: C \to C$ be two self mappings such that $T(C) \subseteq S(C)$. For $a_0 \in C$, the Jungck-CR iterative scheme is the sequence $\{Sa_n\}_{n=1}^{\infty}$ defined by

$$Sa_{n+1} = (1-\alpha_n)Sb_n + \alpha_n Tb_n$$
$$Sb_n = (1-\beta_n)Ta_n + \beta_n Tc_n$$
$$Sc_n = (1-\gamma_n)Sa_n + \gamma_n Ta_n \quad n \in \mathbb{N} \quad (1.3)$$

where $\{\alpha_n\}_{n=1}^{\infty}$, $\{\beta_n\}_{n=1}^{\infty}$ and $\{\gamma_n\}_{n=1}^{\infty}$ are real sequences in [0,1) such that $\sum_{n=1}^{\infty} \alpha_n = \infty$.

Recently, Badri [1] defined the following Jungck-Picard-S iterative scheme.

Definition (1.9) [1]:

Let B be a Banach space and C be a nonempty subset of B. Let $T,S: C \to C$ be two self mappings such that $T(C) \subseteq S(C)$. For $x_1 \in C$, the Jungck-Picard-S iterative scheme is the sequence $\{Sx_n\}_{n=1}^{\infty}$ defined by

$$Sx_{n+1} = Ty_n$$
$$Sy_n = (1-\beta_n)Tx_n + \beta_n Tz_n$$
$$Sz_n = (1-\gamma_n)Sx_n + \gamma_n Tx_n \quad n \in \mathbb{N} \quad (1.4)$$

where $\{\beta_n\}_{n=1}^{\infty}$ and $\{\gamma_n\}_{n=1}^{\infty}$ are real sequences in [0,1) such that $\sum_{n=1}^{\infty} \beta_n \gamma_n = \infty$.

In [12], we define T-CR iteration as follows:

Definition (1.10) [12]:

Let C be a nonempty closed convex subset of a Banach space X and T:C→C be a self-mapping with $\{\alpha_n\}_{n=1}^{\infty}$, $\{\beta_n\}_{n=1}^{\infty}$ and $\{\gamma_n\}_{n=1}^{\infty}$ are real sequences in [0,1] such that $\sum_{n=1}^{\infty} \alpha_n = \infty$. The T-CR iterative scheme $\{u_n\}_{n=1}^{\infty}$ is defined by:

$$u_1 \in C$$
$$u_{n+1} = T[(1-\alpha_n)v_n + \alpha_n Tv_n]$$
$$v_n = T[(1-\beta_n)Tu_n + \beta_n Tw_n]$$
$$w_n = T[(1-\gamma_n)u_n + \gamma_n Tu_n], n \in \mathbb{N}$$

In this section, we define Jungck-T-CR iteration as follows:

Definition (1.11):

Let B be a Banach space and C be a nonempty subset of B. Let $T,S: C \to C$ be two self mappings such that $T(C) \subseteq S(C)$. For $u_0 \in C$ the Jungck- T-CR iterative scheme is the sequence $\{Su_n\}_{n=1}^{\infty}$ is defined by:

$$Su_{n+1} = T[(1-\alpha_n)Sv_n + \alpha_n Tv_n]$$
$$Sv_n = T[(1-\beta_n)Tu_n + \beta_n Tw_n]$$
$$Sw_n = T[(1-\gamma_n)Su_n + \gamma_n Tu_n],$$
$$n \in \mathbb{N} \quad (1.5)$$

where $\{\alpha_n\}_{n=1}^{\infty}$, $\{\beta_n\}_{n=1}^{\infty}$ and $\{\gamma_n\}_{n=1}^{\infty}$ are real sequences in [0,1] such that $\sum_{n=1}^{\infty} \alpha_n = \sum_{n=1}^{\infty} \beta_n = \sum_{n=1}^{\infty} \beta_n \gamma_n = \infty$.

The following definition will be needed in the sequel.

Definition (1.12), [2]:

Let X be a Banach space, C be a nonempty closed convex subset of X. A self mapping $T: C \to C$ is said to be **nonexpansive** if for all x, y in C, we have

$$\|Tx - Ty\| \le \|x - y\|$$

Furthermore T is called **quasi-nonexpansive** if $y = u^*$ provided T has a fixed point in C and if $u^* \in C$ is a fixed point of T, then

$$\|Tx - Tu^*\| \le \|x - u^*\| \quad (1.6)$$

is true for all $x \in C$.

2. Convergence of Jungck-T-CR Iterative Procedure

In this section, we study the convergence of Jungck-T-CR iteration (1.5) when applied to Jungck-contraction mapping (1.1) under certain condition.

Theorem (2.1):

Let C be a nonempty closed convex subset of a Banach space B, $S,T: C \to C$ be two self-mappings satisfying Jungck-contraction condition (1.1) provided that S is quasi-nonexpansive mapping (1.6) as well, assume $T(C) \subseteq S(C)$ and S, T are weakly compatible, suppose that there exists a $z \in C(S,T)$ be the coincidence points of S,T such that $Sz = Tz = u^*$. Let $\{Su_n\}_{n=1}^{\infty}$ be the Jungck- T-CR iterative scheme generated by (1.5), where $\{\alpha_n\}_{n=1}^{\infty}$, $\{\beta_n\}_{n=1}^{\infty}$ and $\{\gamma_n\}_{n=1}^{\infty}$ are real sequences in [0,1] satisfying $\sum_{n=1}^{\infty} \alpha_n =$ *n=1∞βn=n=1∞βnγn=∞*.Then the Jungck- *T*-CR iterative scheme $\{Su_n\}_{n=1}^{\infty}$ converges to a unique common fixed point u^* of S,T.

Proof:

It follows from (1.1) and (1.2) that:

$$\|Su_{n+1} - u^*\| = \|T[(1-\alpha_n)Sv_n + \alpha_n Tv_n] - u^*\|$$
$$\le \delta\|S[(1-\alpha_n)Sv_n + \alpha_n Tv_n] - u^*\|$$
$$\le \delta\|(1-\alpha_n)Sv_n + \alpha_n Tv_n - u^*\|$$
$$\le \delta(1-\alpha_n)\|Sv_n - u^*\| + \alpha_n\delta\|Tv_n - u^*\|$$
$$\le \delta(1-\alpha_n)\|Sv_n - u^*\| + \alpha_n\delta^2\|Sv_n - u^*\|$$
$$= [1-\alpha_n(1-\delta)]\|Sv_n - u^*\| \quad (2.1)$$
$$\|Sv_n - u^*\| = \|T[(1-\beta_n)Tu_n + \beta_n Tw_n] - u^*\|$$
$$\le \delta\|S[(1-\beta_n)Tu_n + \beta_n Tw_n] - u^*\|$$
$$\le \delta\|(1-\beta_n)Tu_n + \beta_n Tw_n - u^*\|$$
$$\le \delta(1-\beta_n)\|Tu_n - u^*\| + \beta_n\delta\|Tw_n - u^*\|$$
$$\le \delta^2(1-\beta_n)\|Su_n - u^*\| + \beta_n\delta^2\|Sw_n - u^*\| \quad (2.2)$$
$$\|Sw_n - u^*\| = \|T[(1-\gamma_n)Su_n + \gamma_n Tu_n] - u^*\|$$
$$\le \delta\|S[(1-\gamma_n)Su_n + \gamma_n Tu_n] - u^*\|$$
$$\le \delta\|(1-\gamma_n)Su_n + \gamma_n Tu_n - u^*\|$$
$$\le \delta(1-\gamma_n)\|Su_n - u^*\| + \gamma_n\delta\|Tu_n - u^*\|$$
$$\le \delta[1-\gamma_n(1-\delta)]\|Su_n - u^*\| \quad (2.3)$$

It follows from (2.1), (2.2) and (2.3) that

$$\|Su_{n+1} - u^*\| \le \delta^3[1-\alpha_n(1-\delta)](1-\beta_n)\|Su_n - u^*\|$$
$$+\delta^4\beta_n[1-\alpha_n(1-\delta)][1-\gamma_n(1-\delta)]\|Su_n - u^*\|$$
$$\le \delta^3[1-\alpha_n(1-\delta)]$$
$$.[1-\beta_n(1-\delta) - \beta_n\gamma_n\delta(1-\delta)]\|Su_n - u^*\|$$

And so on, we get:

$$\|Su_{n+1} - u^*\| \le \delta^{3(n+1)}\|Su_1 - u^*\|$$

$$\cdot \prod_{k=1}^{n} [1-\alpha_k(1-\delta)][1-\beta_k(1-\delta)-\beta_k\gamma_k\delta(1-\delta)]$$

(2.3)

$\leq \delta^{3(n+1)} \|Su_1 - u^*\| e^{-(1-\delta)\sum_{k=1}^{\infty}\alpha_k - (1-\delta)\sum_{k=1}^{\infty}\beta_k - (1-\delta)\sum_{k=1}^{\infty}\beta_k\gamma_k}$

Since $0 \leq \delta < 1, \sum_{n=1}^{\infty}\alpha_n = \infty, \sum_{n=1}^{\infty}\beta_n = \infty$ and $\sum_{n=1}^{\infty}\beta_n\gamma_n = \infty$ so $\delta^{3(n+1)} e^{-(1-\delta)\sum_{k=1}^{\infty}\alpha_k - (1-\delta)\sum_{k=1}^{\infty}\beta_k - (1-\delta)\sum_{k=1}^{\infty}\beta_k\gamma_k} \to 0$ as $n \to \infty$. Which implies that $\lim_{n\to\infty}\|Su_n - u^*\| = 0$.

Therefore, $\{Su_n\}_{n=1}^{\infty}$ converges to u^*.
Now, we will prove u^* is the unique fixed point of S, T.

Suppose that there exist two points of coincidence z_1, $z_2 \in C(S,T)$ such that $Sz_1 = Tz_1 = u_1^*$ and $Sz_2 = Tz_2 = u_2^*$.

Using condition (1.1), we have

$0 \leq \|u_1^* - u_2^*\| = \|Tz_1 - Tz_2\|$

$\leq \delta\|Sz_1 - Sz_2\|$

$0 \leq \|u_1^* - u_2^*\| = \delta\|u_1^* - u_2^*\|$

which leads $(1-\delta)\|u_1^* - u_2^*\| \leq 0$, since $0 \leq \delta < 1$ from which it follows that $\|u_1^* - u_2^*\| = 0$, that is $u_1^* = u_2^*$.
Now, since S,T are weakly compatible and $u^* = Tz = Sz$ then $Tu^* = TTz = TSz = STz$. Hence $Tu^* = Su^*$.
Therefore, Tu^* is a point of coincidence of S,T but the coincidence point is unique, so $u^* = Tu^*$. Thus $Tu^* = Su^* = u^*$. Therefore u^* is the unique common fixed point of S,T.

3. Rate of Convergence of Jungck-T-CR Iterative Procedure

We now compare the speed of Jungck- T-CR iterative scheme (1.5) and the speed of Jungck-CR (1.3), Jungck-Ishikawa (1.2) and Jungck-Picard-S (1.4) iterative schemes by the following theorem.

Theorem (3.1):
Let C be a nonempty closed convex subset of a Banach space B, $S,T: C \to C$ be two self-mappings satisfying Jungck-contraction condition (1.1) provided that S is quasi-nonexpansive mapping (1.6) as well, assume $T(C) \subseteq S(C)$, let $\{Su_n\}_{n=1}^{\infty}$, $\{Sa_n\}_{n=1}^{\infty}$, $\{Sq_n\}_{n=1}^{\infty}$ and $\{Sx_n\}_{n=1}^{\infty}$ be the Jungck- T-CR iterative scheme (1.5) and the speed of Jungck-CR (1.3), Jungck-Ishikawa (1.2) and Jungck-Picard-S (1.4) iterative schemes respectively satisfying $\lim_{n\to\infty}\alpha_n = \lim_{n\to\infty}\beta_n = \lim_{n\to\infty}\alpha_n\beta_n = \lim_{n\to\infty}\beta_n\gamma_n = 0$. Then $\{Su_n\}_{n=0}^{\infty}$ converges to u^* faster than $\{Sa_n\}_{n=1}^{\infty}$, $\{Sq_n\}_{n=1}^{\infty}$ and $\{Sx_n\}_{n=1}^{\infty}$ do.

Proof:
From inequality (2.3), we have

$\|Su_{n+1} - u^*\| \leq \delta^{3(n+1)}\|Su_1 - u^*\| \prod_{k=1}^{n}[1-\alpha_k(1-\delta)][1-$
βk1−δ−βkγkδ1−δ (3.1)

From Jungck-CR iteration (1.3) and Jungck-contraction condition (1.1), it is easy to get that:

$\|Sa_{n+1} - u^*\| \leq \delta^{(n+1)} \prod_{k=1}^{n}[1-\alpha_k(1-\delta)][1-$
*βkγk1−δSa1−u** (3.2)

Using (3.1) and (3.2), we obtain:

$$\frac{\|Su_{n+1}-u^*\|}{\|Sa_{n+1}-u^*\|} \leq \frac{\delta^{3(n+1)}\prod_{k=1}^{n}[1-\alpha_k(1-\delta)][1-\beta_k(1-\delta)-\beta_k\gamma_k\delta(1-\delta)]\|Su_1-u^*\|}{\delta^{(n+1)}\prod_{k=1}^{n}[1-\alpha_k(1-\delta)][1-\beta_k\gamma_k(1-\delta)]\|Sa_1-u^*\|}$$

$$= \delta^{2(n+1)}\prod_{k=1}^{n}\frac{[1-\alpha_k(1-\delta)][1-\beta_k(1-\delta)-\beta_k\gamma_k\delta(1-\delta)]\|Su_1-u^*\|}{[1-\alpha_k(1-\delta)][1-\beta_k\gamma_k(1-\delta)]\|Sa_1-u^*\|}$$

Define $\theta_n = \delta^{2(n+1)}\prod_{k=1}^{n}\frac{[1-\alpha_k(1-\delta)][1-\beta_k(1-\delta)-\beta_k\gamma_k\delta(1-\delta)]}{[1-\alpha_k(1-\delta)][1-\beta_k\gamma_k(1-\delta)]}$

By the assumption

$\lim_{n\to\infty}\frac{\theta_{n+1}}{\theta_n} =$

limn→∞δ2(n+2)k=1n+11−αk1−δ1−βk1−δ−βkγkδ1−δ1−αk1−δ1−βkγk1−δδ2(n+1)k=1n1−αk1−δ1−βk1−δ−βkγkδ1−δ1−αk1−δ1−βkγk1−δ

$$= \lim_{n\to\infty}\frac{\delta[1-\beta_{n+1}(1-\delta)-\delta\beta_{n+1}\gamma_{n+1}(1-\delta)]}{[1-\beta_{n+1}\gamma_{n+1}(1-\delta)]}$$

$= \delta < 1$

Thus it follows from ratio test that $\sum_{n=1}^{\infty}\theta_n < \infty$. Hence, we have $\lim_{n\to\infty}\theta_n = 0$ which implies that the iterative sequence defined by Jungck- T-CR (1.5) converges to u^* faster than the iterative sequence defined by Jungck-CR iteration method (1.3).

From Jungck-Ishikawa iterative scheme (1.2) and Jungck-contraction condition (1.1), we obtain:

$\|Sq_{n+1} - u^*\| = \|(1-\alpha_n)Sq_n + \alpha_n Tr_n - u^*\|$

$\leq (1-\alpha_n)\|Sq_n - u^*\| + \alpha_n\|Tr_n - u^*\|$

$\leq (1-\alpha_n)\|Sq_n - u^*\| + \alpha_n\delta\|Sr_n - u^*\|$

(3.3)

$\|Sr_n - u^*\| = \|(1-\beta_n)Sq_n + \beta_n Tq_n - u^*\|$

$\leq (1-\beta_n)\|Sq_n - u^*\| + \beta_n\|Tq_n - u^*\|$

$\leq (1-\beta_n)\|Sq_n - u^*\| + \beta_n\delta\|Sq_n - u^*\|$

$\leq [1-\beta_n(1-\delta)]\|Sq_n - u^*\|$

(3.4)

Substituting (3.4) in (3.3), we have:

$\|Sq_{n+1} - u^*\| \leq (1-\alpha_n)\|Sq_n - u^*\|$

$+\alpha_n\delta[1-\beta_n(1-\delta)]\|Sq_n - u^*\|$

$= [1-\alpha_n(1-\delta) - \alpha_n\beta_n\delta(1-\delta)]\|Sq_n -$
*u**

Repeating this process n times, we get

$\|Sq_{n+1} - u^*\| \leq \prod_{k=1}^{n}[1-\alpha_k(1-\delta) - \alpha_k\beta_k\delta(1-$
*δSq1−u** (3.5)

Using (3.1) and (3.5), we have

$$\frac{\|Su_{n+1}-u^*\|}{\|Sq_{n+1}-u^*\|} \leq \frac{\delta^{3(n+1)}\prod_{k=1}^{n}[1-\alpha_k(1-\delta)][1-\beta_k(1-\delta)-\beta_k\gamma_k\delta(1-\delta)]\|Su_1-u^*\|}{\prod_{k=1}^{n}[1-\alpha_k(1-\delta)-\alpha_k\beta_k\delta(1-\delta)]\|Sq_1-u^*\|}$$

Define $\theta_n = \delta^{3(n+1)}\prod_{k=1}^{n}\frac{[1-\alpha_k(1-\delta)][1-\beta_k(1-\delta)-\beta_k\gamma_k\delta(1-\delta)]}{[1-\alpha_k(1-\delta)-\alpha_k\beta_k\delta(1-\delta)]}$

By the assumption

$\lim_{n\to\infty}\frac{\theta_{n+1}}{\theta_n} =$

limn→∞δ3n+2k=1n+11−αk1−δ1−βk1−δ−βkγkδ1−δ1−αk1−δ−αkβkδ1−δδ3n+1k=1n1−αk1−δ1−βk1−δ−βkγkδ1−δ1−αk1−δ−αkβkδ1−δ

$$= \lim_{n\to\infty}\frac{\delta^3[1-\alpha_{n+1}(1-\delta)][1-\beta_{n+1}(1-\delta)-\beta_{n+1}\gamma_{n+1}\delta(1-\delta)]}{[1-\alpha_{n+1}(1-\delta)-\alpha_{n+1}\beta_{n+1}\delta(1-\delta)]}$$

$= \delta^3 < 1$

Thus it follows from ratio test that $\sum_{n=1}^{\infty} \theta_n < \infty$. Hence, we have $\lim_{n\to\infty} \theta_n = 0$ which implies that the iterative sequence defined by Jungck- T-CR (1.5) converges to u^* faster than the iterative sequence defined by Jungck-Ishikawa iteration method (1.2).

From Jungck-Picard-S iterative scheme (1.4) and Jungck-contraction condition (1.1), we have obtained the following inequality:

$\|Sx_{n+1} - u^*\| \leq \delta^{2(n+1)} \prod_{k=1}^{n}[1 - \beta_k\gamma_k(1-\delta)]\|Sx_1 - u^*\|$ (3.6)

From inequality (3.1) and (3.6), we have:

$$\frac{\|Su_{n+1}-u^*\|}{\|Sx_{n+1}-u^*\|} \leq \frac{\delta^{3(n+1)} \prod_{k=1}^{n}[1-\alpha_k(1-\delta)][1-\beta_k(1-\delta)-\beta_k\gamma_k\delta(1-\delta)]\|Su_1-u^*\|}{\delta^{2(n+1)} \prod_{k=1}^{n}[1-\beta_k\gamma_k(1-\delta)]\|Sx_1-u^*\|}$$

$$= \delta^{(n+1)} \prod_{k=1}^{n} \frac{[1-\alpha_k(1-\delta)][1-\beta_k(1-\delta)-\beta_k\gamma_k\delta(1-\delta)]\|Su_1-u^*\|}{[1-\beta_k\gamma_k(1-\delta)]\|Sx_1-u^*\|}$$

Define $\theta_n = \delta^{(n+1)} \prod_{k=1}^{n} \frac{[1-\alpha_k(1-\delta)][1-\beta_k(1-\delta)-\beta_k\gamma_k\delta(1-\delta)]}{[1-\beta_k\gamma_k(1-\delta)]}$

By the assumption

$\lim_{n\to\infty} \frac{\theta_{n+1}}{\theta_n} =$

$limn\to\infty\delta n+2k=1n+11-\alpha k1-\delta1-\beta k1-\delta-\beta k\gamma k\delta1-\delta1$
$-\beta k\gamma k1-\delta\delta n+1k=1n1-\alpha k1-\delta1-\beta k1-\delta-\beta k\gamma k\delta1-\delta1$
$-\beta k\gamma k1-\delta$

$$= \lim_{n\to\infty} \frac{\delta^3[1-\alpha_{n+1}(1-\delta)][1-\beta_{n+1}(1-\delta)-\beta_{n+1}\gamma_{n+1}\delta(1-\delta)]}{[1-\beta_{n+1}\gamma_{n+1}(1-\delta)]}$$

$$= \delta < 1$$

Thus it follows from ratio test that $\sum_{n=1}^{\infty} \theta_n < \infty$. Hence, we have $\lim_{n\to\infty} \theta_n = 0$ which implies that the iterative sequence defined by Jungck- T-CR (1.5) converges to u^* faster than the iterative sequence defined by Jungck-Picard-S iteration method (1.4).

We now support the result of the above theorem by the following example using computer programming in java for comparing the speed of Jungck- T-CR iterative scheme (1.5) and the speed of Jungck-CR (1.3), Jungck-Ishikawa (1.2) and Jungck-Picard-S (1.4) iterative schemes.

Example (3.2): Let $B = \mathbb{R}$, $C = [0,1]$, $S,T: C \to C$ are mappings defined as $Sx = 1 - x$ and $Tx = \frac{2x+1}{4}$ for all $x \in C$. It is easily seen that the mappings S and T satisfying Jungck-contraction condition (1.1) provided that S is quasi-nonexpansive mapping (1.6) with the unique common fixed point 0.5 take $\alpha_n = \beta_n = \gamma_n = 0.1$or all $n = 1, \cdots, 99$ with initial value 0.1 . The comparison of the rate of convergence of the speed of Jungck- T-CR iterative scheme (1.5) and the speed of Jungck-CR (1.3), Jungck-Ishikawa (1.2) and Jungck-Picard-S (1.4) iterative schemes to a common fixed point of S and T is shown in the following tables.

	Jungck-T-CR Iterative procedure		
n	Sx_{n+1}	Tx_n	x_{n+1}
1	0.49598874208984380	0.49899718552246090	0.49598874208984367
2	0.49959831012584077	0.50005021123427000	0.50040168987415920
3	0.49995977452494555	0.49999748590780907	0.49995977452494555
4	0.49999597179578714	0.50000012588138170	0.50000402820421290
5	0.49999959661311250	0.49999999369707987	0.49999959661311255
…	…	…	…
99	0.50000000000000000	0.50000000000000000	0.50000000000000000

	Jungck-CR		
n	Sx_{n+1}	Tx_n	x_{n+1}
1	0.57009875625000020	0.58372500000000000	0.42990124374999983
2	0.47065490816484370	0.46495062187499990	0.52934509183515630
3	0.51228458906949230	0.51467254591757820	0.48771541093050774
4	0.49485736390078383	0.49385770546525387	0.50514263609921620
5	0.50215283603703440	0.50257131804960810	0.49784716396296560
…	…	…	…
99	0.50000000000000000	0.50000000000000000	0.50000000000000000

	Jungck-Ishikawa		
n	Sx_{n+1}	Tx_n	x_{n+1}
1	0.79412250000000010	0.32849999999999996	0.20587749999999994
2	0.75221004375000010	0.35293874999999997	0.24778995624999990
3	0.71627011251562510	0.37389497812499994	0.28372988748437490
4	0.68545162148214860	0.39186494374218744	0.31454837851785145
5	0.65902476542094250	0.40727418925892570	0.34097523457905754
…	…	…	…
224	0.50000000000000040	0.49999999999999970	0.49999999999999956

	Jungck-Picard-S		
n	Sx_{n+1}	Tx_n	x_{n+1}
1	0.52425562500000000	0.45075000000000000	0.47574437500000000
2	0.50597294765625000	0.48787218750000000	0.49402705234375000
3	0.50147083836035160	0.49701352617187500	0.49852916163964844
4	0.50036219394623660	0.49926458081982420	0.49963780605376340
5	0.50008919025926080	0.49981890302688170	0.49991080974073920
…	…	…	…
99	0.50000000000000000	0.50000000000000000	0.50000000000000000

By observing the above tables, we conclude the decreasing rate of convergence of iterative schemes is as follows: Jungck-T-CR, Jungck-Picard-S, Jungck-CR and Jungck-Ishikawa iterative schemes.

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