Totaly *-paranormal and the Operators Equations $SRS = S^2$ and $RSR = R^2$

Buthainah A.A. Ahmed[1], Hassan N. Almrayatee[2]

[1, 2]Department of Mathematics -College of Science -University of Baghdad

Abstract: ***In this paper we study the property of the operator equations $SRS = S^2$ and $RSR = R^2$ where S is a *-paranormal operator we show that if S or S^* is a polynomial root of *-paranormal operator then $f(A) \in gW$ for all $f \in H(\sigma(A))$, where $A \in \{SR, RS, R\}$ and we show that For the operator equation $SRS = S^2$ and $RSR = R^2$ we have $\sigma_\beta(S) = \sigma_\beta(SR) = \sigma_\beta(RS) = \sigma_\beta(R)$***

1. Introduction

Let H be an infinite dimensional separable Hilbert space and let $B(H), B_0(H)$ denote the algebra of bounded linear operators and the ideal of compact operator acting on H. If $T \in B(H)$ we shall write $Im(T)$ and $ker(T)$ for the range and null space of T. Let $\alpha(T) := dimker(T)$, $\beta(T) := dimker(T^*)$, and let $\sigma_a(T), \sigma_s(T), \sigma_p(T), p_0(T)$, and $\pi_0(T)$ denote the spectrum, approximate point spectrum, surjective spectrum, point spectrum of T, the set of the resolvent of T and the set of all eigenvalues of T which are isolated in $\sigma(T)$.

Recall that $T \in B(H)$ is called *-paranormal operator if $\| T^* x \|^2 \le \| T^2 x \| \| x \|$ and T is called isoloid if every isolated point of $\sigma(T)$ is an eigenvalue of T. If $T \in B(H)$ we write $r(T)$ for the spectral radius of T where $r(T) \le \| T \|$. An operator $T \in B(H)$ is called normaliod if $r(T) = \| T \|$. An operator $T \in B(H)$ is said to be nilpotent if $T^n = 0$ for a natural number n and called quasinilpotent if $r(T) = 0$ [11, 12]

The operator $E := \frac{1}{2\pi i} \int_{\partial D} (\lambda - T)^{-1}$ is called Riesz idempotent with respect to λ where D is a closed disk centered at λ and $D \cap \sigma(T) = \{\lambda\}$ where $\lambda \in \sigma(T)$ be an isolated point of $\sigma(T)$ see[11]

Recall that if $T \in B(H)$, the asent $a(T)$ and the descent $d(T)$ given by

$$a(T) = inf\{n \ge 0 : ker(T^n) = ker(T^{n+1})\}$$

and

$$d(T) = inf\{n \ge 0 : Im(T^n) = Im(T^{n+1})\}$$

An operator $T \in B(H)$ is called Fredholm if it has closed range , finite dimensional null space and its range has finite co-dimensional.the index of a Fredholm operator

$$i(T) = \alpha(T) - \beta(T)$$

T is called Weyle if it is Fredholm of index zero , and Browder if it is Fredholm of finite ascent and descent.The essential spectrum $\sigma_e(T)$, the Wely spectrum $\sigma_w(T)$ and the Browder spectrum $\sigma_b(T)$ define as [7, 10]

$$\sigma_e(T) := \{\lambda \in C : T - \lambda \quad is \quad not \quad Fredholm\}$$

$$\sigma_w(T) := \{\lambda \in C : T - \lambda \quad is \quad not \quad Weyl\}$$

$$\sigma_b(T) := \{\lambda \in C : T - \lambda \quad is \quad not \quad Browder\}$$

$$\sigma_e(T) \subseteq W(T) \subseteq \sigma_b(T) := \sigma_e(T) \cup_{acc} \sigma(T)$$

we write $accK$ for the accumulation point of $K \subset C$ if we write $isoK = K \setminus accK$ then we let

$$\pi_{00}(T) := \{\lambda \in iso\sigma(T) : 0 \le \alpha(T - \lambda) \le \infty\}$$

$$P_{00}(T) := \sigma(T) \setminus \sigma_b(T)$$

we say that Weyl's theorem hold for T if

$$\sigma(T) \setminus W(T) = \pi_{00}(T)$$

and Browder's theorem hold for T if

$$\sigma(T) \setminus W(T) = P_{00}(T)$$

An operator $T \in B(H)$ is called B-Freadholm if there exists a natural number n for the induced operator $T_n : Im(T) \to Im(T^n)$ is Freadholm in the usual sense and B-Weyl's if in addition T_n has zero index .

the B-Fredholm spectrum $\sigma_{BF}(T)$ and B-Weyl spectrum $\sigma_{BW}(T)$ are define by

$$\sigma_{BF}(T) := \{\lambda \in C : T - \lambda \quad is \quad not \quad B - Freadholm\}$$

$$\sigma_{BW}(T) := \{\lambda \in C : T - \lambda \quad is \quad not \quad B - Weyl\}$$

An element x of A is Drazin invertible if there is an element b of A and non-negative integer k such that[17]

$$x^k bx = x^k \quad , bxb = b, \quad , \quad xb = bx$$

the Drazin spectrum of $a \in A$ is define by [8]

$$\sigma_D(a) := \{\lambda \in C : a - \lambda \quad is \quad not \quad Drazin\,invertible\}$$

If $T \in B(H)$ it is well known that T is Drazin invertible if and only if it has finite ascent and descent and that is also equivalent to the fact that T decomposed as $T_1 \oplus T_2$ where T_1 is invertible and T_2 is nilpotent[15] and

$$\sigma_{BW}(T) = \cap\{\sigma_D(T + F) : F \in B_0(H)\}$$

$$\sigma_{BB}(T) = \cap\{\sigma_D(T + F) : F \in B_0(H) \quad and \quad TF = FT\}$$

2. Main Results

Lemma 2.1 *[19] every *-paranormal operator is normoliod.*

Lemma 2.2 *[19] If $T \in B(H)$ is *-paranormal then $ker(T-\lambda I) \subseteq ker(T^*-\lambda I)$ for each $\lambda \in C$ thus $T-\lambda I$ is reduced by its eigenspace for every $\lambda \in C$*

Theorem 2.3 *[19] If H is finite dimension .every *-paranormal operator T is normal*

Theorem 2.4 *let S be a *-paranormal operator on finite dimensional Hilbert space H and $ker(S) = ker(SR)$ then we have*
(1) SR is normal
(2)If $N(S-\lambda) = ker(R-\lambda)$ for each $\lambda \in C$ then for all of S, RS, SR, R are normal.

Proof. Since S is *-paranormal operator and $dimH < \infty$ use theorem(2.3) that is S is normal operator hence S is paranormal operator then by [3] that is SR is normal (2)since $ker(S) = ker(RS)$ then by [3, Theorem3.1] it is obtain

Lemma 2.5 *Let A be a *-paranormal operator then we have $A = \lambda$ if $\sigma(A) = \{\lambda\}$ for $\lambda \in C$*
Proof. case(1) if $(\lambda = 0)$ since A is *-paranormal then by lemma(2.1) T is normoliod therefore $T = 0$

Case(2) if $(\lambda \neq 0)$ that is A is invertible ,since A is *-paranormal then A^{-1} is also *-paranormal then A^{-1} is normoliod hence $\sigma(A^{-1}) = \{\frac{1}{\lambda}\}$ so $\| A \| \| A^{-1} \| = | \lambda | \frac{1}{\lambda} = 1$ that is convexoid so $w(A) = \{\lambda\}$ then $A = \lambda$

Lemma 2.6 *Let S be a *-paranormal operator and $\sigma(S) = \{\lambda\}$ then we have*
(1) If $\lambda = 0,$ then $R^2 = 0$
(2) If $\lambda \neq 0$ then $\lambda = 1$ *and* $R = S = I$
Proof. case(1) is $\lambda = 0$ then by lemma(2.5) $R^2 = 0$
case(2) if $\lambda \neq 0$ and S is *-paranormal , $S = \lambda I$ and since $SRS = S^2$ that is $\lambda^2(R-I) = 0$ so that $R = I$ and also that $RSR = R^2$ then $(\lambda - I)R^2 = 0$ and $\lambda = 1$ that is $\sigma(S) = \sigma(R) = \{1\}$ which is $R = S = I$

Remark 2.7 *Let S be a*-paranormal operator then we have*
(1) If S is quasinilpotent , then SR, RS, R are nilpotent
(2) IF $S-I$ is quasinilpotent , then $R = I$ therefore $RS-\lambda, SR-\lambda$ and $R-\lambda$ are invertible for each $\lambda \in C \setminus \{1\}$

Corollary 2.8 *If S is a*-paranormal operator, then $iso(A) \subseteq \{0,1\}$ where $A \in \{S, SR, RS, R\}$*
Proof. Let λ_0 be a nonzero isolated point of $\sigma(S)$ by Riesz decomposition $E_{\lambda_0}(A)$ with respect to λ_0 we can act A as the direct sum

$$S = S_1 \oplus S_2, \quad where \sigma(S_1) = \{\lambda_0\} \quad and \quad \sigma(S_2) = \sigma(S) \setminus \{\lambda_0\}$$

since S_1 is *-paranormal then by lemma (2.6) that is $\lambda_0 = 1$ that is $iso\sigma(A) \subseteq \{0,1\}$

Lemma 2.9 *If S is *-paranormal and λ_0 is nonzero isolated point of $\sigma(SR)$, then for the Riesz idempotent $E_{\lambda_0}(S)$ with respect to λ_0, we have*

$$Im(E_{\lambda_0}(A)) = ker(SR-\lambda_0) = ker(S^*R^*-\overline{\lambda_0})$$

Proof. since S is *-paranormal and $\lambda \in iso\sigma(S) \setminus \{0\}$,then by [20, Theorem 3] $Im(E_{\lambda_0}(S)) = ker(RS-\lambda_0) = ker(S^*-\overline{\lambda_0})$ for the Riesz idempotent $E_{\lambda_0}(S)$ with respect to λ_0 but a pair (S, R) is solution of the operator equation $SRS = S^2$ and $RSR = R^2$ then by [18, corollary 2.2]

$$ker(S-\lambda_0) = ker(SR-\lambda_0) \quad and \quad ker(S^*-\overline{\lambda_0}) = ker(S^*R^*-\overline{\lambda_0}),$$

for $\lambda_0 \neq 0$

Definition 2.10 *we called the set δ be the collection of every pair (S, R) of the operators as following*
$\delta := \{(S, R) : S$ and R are the solution of the operator equation $SRS = S^2$ and $RSR = R^2$ with $ker(S-\lambda) = ker(R-\lambda)$ for $\lambda \neq \{0\}$

Lemma 2.11 *Suppose that $(S, R) \in \delta$ and S is *-paranormal if $\lambda_0 \in iso\sigma(RS) \setminus \{0\}$ then for the Riesz idempotent $E_{\lambda_0}(A)$ with respect to λ_0 we have that*

$$Im(E_{\lambda_0}(S)) = ker(RS-\lambda_0) = ker(S^*R^*-\overline{\lambda_0})$$

Proof. since $(S, R) \in \delta$ and S is *-paranormal then by [18, Corollary 2.2] and lemma (2.9) that is $ker(RS-\lambda_0) = ker(SR-\lambda_0) = ker(S^*R^*-\overline{\lambda_0})$ for $\lambda_0 \in iso(RS) \setminus \{0\}$

Proposition 2.12 *Let $(S, R) \in \delta$ and S be a *-paranormal operator*
(1) If λ_0 is a nonzero isolated point of $\sigma(RS)$ then the range of $RS-\lambda_0$ is closed
(2) If R^* is injective and $\lambda_0 \in iso\sigma(A) \setminus \{0\}$ then, $ker(A-\lambda_0)$ reduces A where $\in \{SR, R\}$
Proof. (1) Let λ_0 is a nonzero isolated point of $\sigma(RS)$ then by corollary (2.8) that is $iso\sigma(RS) \subseteq \{1\}$. If $iso\sigma(RS) = \phi$ then the prove done. If $iso\sigma(RS) = \{1\}$, since $SRS = S^2$

and $RSR = R^2$, by [18] 1 is an isolated point of $\sigma(S)$ by using the Riesz idempotent $E_1(S)$ with respect to 1 we can act S as the direct sum

$$S = S_1 \oplus S_2 \quad where \quad \sigma(S_1) = \{1\} \quad and \quad \sigma(S_2) = \sigma(S) \setminus \{1\}$$

since (S,R) is δ and S is *-paranormal then by lemma (2.9)

$$H = Im(E) \oplus Im(E)^{\perp} = ker(RS - I) \oplus ker(RS - I)^{\perp}$$

which implies that

$$RS = C_1 \oplus C_2, \quad where \quad \sigma(C_1) = \{1\} \quad and \quad \sigma(C_2) = \sigma(RS) \setminus \{1\}$$

since S_1 and C_1 are the restriction of S and RS to $Im(E_1(S))$ therefore we not that if $R_1 := R \mid Im(E_1(S))$ then $S_1R_1S_1 = S_1^2$ and $R_1S_1R_1 = R_1^2$ since S_1 is *-paranormal then be lemma (2.6) that is $C_1 = I$ thus

$$RS - I = 0 \oplus (C_2 - I)$$

so that

$$Im(RS - I) = (RS - I)(H) = 0 \oplus (C_2 - I)(N(RS - I)^{\perp})$$

since $(C_2 - I)$ is invertible, that is $RS - I$ has closed range

(2) since a pair (S^*, S^*) is a solution of the operator equation $S^*R^*S^* = S^{*2}$ and $R^*S^*R^* = R^{*2}$ and R^* is injective $S^*R^* = R^*$ but $(S,R) \in \delta$ then by lemma (2.9) and lemma(2.11) that for the Riesz idempotent $E_{\lambda_0}(A)$

$$Im(E_{\lambda_0}(S)) = ker(A - \lambda_0) = ker(A^* - \overline{\lambda_0})$$

where $A \in \{SR, R\}$

Lemma 2.13 *We have the following properties*

(1) $\pi_0(S) = \pi_0(SR) = \pi_0(RS) = \pi_0(R)$

(2) S is isolated if and only if A is is isolated where $A \in \{SR, RS, R\}$

Proof. Since $SRS = S^2$ and $RSR = R^2$ then by [18] and [9, Lemma 2.3], it is known that $\sigma(S) = \sigma(SR) = \sigma(RS) = \sigma(R)$ and $\sigma_p(S) = \sigma_p(SR) = \sigma_p(RS) = \sigma_p(R)$ that is (2) is satisfied . Also for every $\lambda \in C$

$$\alpha(S - \lambda) > 0 \Leftarrow \alpha(SR_\lambda) > 0 \Leftarrow \alpha(RS - \lambda) > 0 \Leftarrow \alpha(R - \lambda) > 0$$

that is (1) satisfied

3. Generalized Weyl's theorem for algebraically totally *-paranormal

Definition 3.1 *[14] An operator $A \in B(H)$ is said to be totally *-paranormal if $T - \lambda$ is *-paranormal for all $\lambda \in C$*

Definition 3.2 *Let $A \in B(H)$, we called A is an algebraically totally *-paranormal if there exists a non-constant complex polynomial P such that $P(A)$ is totally *-paranormal*

normal operator $\Rightarrow$ totally *-paranormal $\Rightarrow$ algebraically totally *-paranormal

Theorem 3.3 *Suppose that S or S^* is a polynomial root of *-paranormal operator then $f(A) \in gW$ for all $f \in H(\sigma(A))$, where $A \in \{SR, RS, R\}$*

Proof. suppose that $\lambda \in \sigma(A) \setminus \sigma_{BW}(A)$. Then $A - \lambda$ is B-Weyl but not invertible then by [5, Lemma 4.1] we can act $A - \lambda$ as the direct sum

$$A - \lambda = A_1 \oplus A_2 \quad where \quad A_1 \quad is \quad Weyl \quad and \quad A_2 \quad is \quad nilpotent$$

Since S is polynomial root of *-paranormal then by[13] S has $(SVEP)$ therefore by [16][Theorem 3.3.9] and [9, Theorem 2.1], A has $(SVEP)$. This is implies that A_1 has $(SVEP)$ at 0. Therefore A_1 is Wyle, so that A_1 has finite ascent and descent that is $A - \lambda$ has finite ascent and descent so that $\lambda \in \pi_0(A)$

Conversely, suppose that $\lambda \in \pi_0(A)$. Then by lemma(2.13) $\lambda \in \pi_0(S)$. But S is polynomial root of *-paranormal operator hence by [2] $S \in gB$ therefore λ is a pole of the resolvent of S so that $A - \lambda$ is Drazin invertible by [9, Theorem 2.1] we can act $A - \lambda$ as the direct sum

$$A - \lambda = A_1 \oplus A_2 \quad where \quad A_1 \quad is \quad invertible \quad and \quad A_2 \quad is \quad nilpote$$

Therefore $A - \lambda$ is B-Weyl, and so $\lambda \in \sigma_{BW}(A)$ .thus

$$\sigma(A) \setminus \sigma_{BW}(A) = \pi_0(A)$$

therefore $A \in gW$

We claim that $\sigma_{BW}(f(A)) = f(\sigma_{BW}(A))$ for all $f \in H(\sigma(A))$. Since $A \in gW$, $A \in gB$ then by [6, Theorem 2.1] that is $\sigma_{BW}(A) = \sigma_D(A)$. Since S is polynomial root of *-paranormal operator, A has $(SVEP)$ so that $f(A)$ has $(SVEP)$ for all $f \in H(\sigma(A))$ therefore $f(A) \in gB$ by [6, Theorem 2.9] hence we have

$$\sigma_{BW}(f(A)) = \sigma_D(f(A)) = f(\sigma_D(A)) = f(\sigma_{BW}(A))$$

Since S is a polynomial root of *-paranormal operator then by [2] that S is isoliod therefore by lemma(2.13) S is isoloid for all $f \in H(\sigma(A))$,

$$\sigma(f(A)) \setminus \pi_0(f(A)) = f(\sigma(A) \setminus \pi_0(A))$$

since $A \in gW$, we have

$$\sigma(f(A)) \setminus \pi_0(f(A)) = f(\sigma(A) \setminus \pi_0(A)) = f(\sigma_{BW}(A)) = \sigma_{BW}(f(A))$$

that is $f(A) \in gW$.

Now suppose that S^* is polynomial root of *-paranormal operator. Let $\lambda \in \sigma(A) \setminus \sigma_{BW}(A)$ observe that

$\sigma(A^*) = \overline{\sigma(A)}$ and $\sigma_{BW}(A^*) = \overline{\sigma_{BW}(A)}$. so $\bar{\lambda} \in \sigma(A^*) \setminus \sigma_{BW}(A^*)$. But, $S^*R^*S^* = S^{*^2}$ and $R^*S^*R^* = R^{*^2}$ hence $A^* \in gW$ so $\bar{\lambda} \in P_0(A^*)$, therefore $\bar{\lambda} \in P_0(S^*)$. Since S^* is polynomial root of *-paranormal operator and $\bar{\lambda}$ is pole if the resolvent of S^* that is λ is a pole of the resolvent of T so $\lambda \in \pi_0(T)$

Conversely let $\lambda \in \pi_0(A)$ then $\lambda \in \pi_0(S)$. Since $\lambda \in iso\sigma(S^*)$ and S^* is a polynomial root of *-paranormal operators and λ is a pole of the resolvent of S, so that $T-\lambda$ is Drazin invertible. hence $\lambda \in \sigma(A) \setminus \sigma_{BW}(A)$ so that $\sigma(A) \setminus \sigma_{BW}(A) = \pi_0(A)$. Hence $A \in gW$. If S^* is polynomial root of *-paranormal operators then by lemma(2.13) A is isoloid. Hence $f(A) \in gW$.

Corollary 3.4 *Suppose that* $(S,R) \in \delta$ *and* A *is a compact operator. Then we have*

$$RA = I \oplus Q \quad on \quad ker(RS-I) \oplus ker(RS-I)^{\perp}$$

where $\sigma(Q) = 0$

Proof. Let S be a compact operator and *-paranormal. Then by theorem (3.3) RS satisfies generalized Weyl's theorem and by corollary (2.8) that is $iso\sigma(RS) \subseteq \{0,1\}$ hence

$\sigma(RS) \setminus \sigma_{BW}(RS) \subseteq \{0,1\}$

Assume that $\sigma_{BW}(RS)$ is not finite. Then $\sigma(RS)$ is finite. Since S is compact, $\sigma(RS)$ is countable set $\sigma(RS) := \{0, \lambda_1, \lambda_2, ...\}$, where $\lambda_j \neq 0$ for $j = 1,2,..., \lambda_i \neq \lambda_j$ for all $i \neq j$ and $\lambda_j \to 0$ as $j \to \infty$ then by corollary (2.8) $\{\lambda_1, \lambda_2, ...\} \subseteq iso\sigma(RS) \setminus \{0\} \subseteq \{1\}$. But this is a contradiction therefore $\sigma_{BW}(RS)$ is finite. That is means for all point in $\sigma_{BW}(RS)$ is isolated. So $\sigma(RS) \subseteq \{0,1\}$. If $1 \notin \sigma(RS)$, then $\sigma(RS) = 0$. Since S is *-paranormal then by lemma (2.5) $S = 0$ hence $RS = 0$. If $1 \in \sigma(RS)$, then by proposition (2.12)

$$RS = I \oplus Q \quad on \quad H = ker(RS-I) \oplus ker(RS-I)^{\perp}, \quad where \quad \sigma(Q) = \{0\}.$$

Theorem 3.5 *Let* S *is a polynomial root of *-paranormal operators then generalized a-Browder's theorem holds for* A *where* $A \in \{SR, RS, R\}$

Proof. First we must show that $\sigma_{ea}(f(A)) = f(\sigma_{ea}(A))$ for all $f \in H(\sigma(A))$

Let $f \in H(\sigma(A))$. since the inclusion $\sigma_{ea}(f(A)) \subseteq f(\sigma_{ea}(A))$ hold for each operator. Suppose that $\lambda \notin \sigma_{ea}(f(A))$ then $f(A) - \lambda$ is upper semi-Fredholm and $i(f(A)-\lambda) \leq 0$ let

$$f(A) - \lambda = c(A-\mu_1)(A-\mu_2)...(A-\mu_n)g(A)$$

where $c, \mu_1, \mu_2, ..., \mu_n \in C$ and $g(A)$ is invertible. Since S is polynomial root of *-paranormal operators then by [13] and [1,

Theorem 2.40] that is S has $(SVEP)$ therefore A has $(SVEP)$ by [9, Theorem 2.1]. Since $A-\mu_i$ is upper semi-Fredholm, then by [15, Proposition 2.2] that is $i(A-\mu_i) \leq 0$ for all $i = 1,2,...,n$ hence $\lambda \notin f(\sigma_{ea}(A))$.

Suppose that S or S^* is a polynomial root of *-paranormal operators. Since $S^*R^*S^* = S^{*^2}$ and $R^*S^*R^* = R^{*^2}$ A^* has also $(SVEP)$. So $i(A-\mu_i) \geq 0$ for all $i = 1,2,...,n$. From the classical index product theorem , $A-\mu_i$ is Weyl for all $i - 1,2,...,n$ therefore $\lambda \notin f(\sigma_{ea}(A))$ so that $\sigma_{ea}(f(A)) = f(\sigma_{ea}(A))$. Since S or S^* is a root of *-paranormal operators then A or A^* has $(SVEP)$ therefore a-Browder's theorem holds for A. Hence

$$\sigma_{ab}(f(A)) = f(\sigma_{ab}(A)) = f(\sigma_{ea}(A)) = \sigma_{ea}(f(A))$$

for all $f \in H(\sigma(A))$

Definition 3.6 *[16] An operator* $A \in B(H)$ *on* C *is said has a Bishop's property* (β) *if for every open subset* U *of* C *and every sequence of analytic functions* $f_n : U \to x$ *with the property that* $(A - \lambda I) f_n(\lambda) \to 0$ *as* $n \to \infty$

Let

$$\sigma_\beta(A) = \{\lambda \in C : A \quad dose \quad not \quad have \quad a\, Bishop's \quad property\}$$

Theorem 3.7 *For the operator equation* $SRS = S^2$ *and* $RSR = R^2$ *we have* $\sigma_\beta(S) = \sigma_\beta(SR) = \sigma_\beta(RS) = \sigma_\beta(R)$

Proof. The equivalence SR has a property (β) at a point $\mu \Leftarrow RS$ has a property (β) at μ holds for all $S, R \in B(H)$ [4]. We prove that S has a property (β) at $\mu \Leftarrow SR$ has a property (β) at μ the equivalence R has a property (β) at $\mu \Leftarrow RS$ has a property (β) at μ is similarly proved. Let U be an open neighborhood of μ and $(f_n) : U \to X$ be a sequence of analytic functions in a neighborhood of λ such that

$$\lim_{n \to +\infty} (S-\lambda) f_n(\lambda) = 0 \quad in \quad U. \quad Then$$

$$\lim_{n \to +\infty} (S^2 - \lambda S) f_n(\lambda) = 0 \Leftarrow \lim_{n \to +\infty} S^2 f_n(\lambda) = \lambda \lim_{n \to +\infty} S f_n(\lambda) = \lambda^2 \lim_{n \to +\infty} f_n(\lambda) \quad in \quad U$$

and so

$$\lim_{n \to +\infty} SR(S-\lambda) f_n(\lambda) = 0 \Rightarrow \lim_{n \to +\infty} (S^2 - \lambda SR) f_n(\lambda) = 0 \Rightarrow \lim_{n \to +\infty} (SR-\lambda)(-\lambda f_n(\lambda)) = 0 \quad in \quad U$$

Thus if SR has a property (β) at μ then

$$\lambda \lim_{n\to+\infty} f_n(\lambda)=0 \Rightarrow \lim_{n\to+\infty} f_n(\lambda)=0 \quad for \quad all \quad \lambda \quad in \quad U$$

implies S has a property β at μ. Conversely assume that S has a property β at μ and let $g_n U \to x$ be an analytic sequence such that

$$\lim_{n\to+\infty} (SR-\lambda)g_n(\lambda)=0 \quad in \quad U.$$

Then

$$\lim_{n\to\infty} SR(SR-\lambda)=0 \Leftarrow \lim_{n\to\infty} (S^2R-\lambda SR)g_n(\lambda)=\lim_{n\to\infty} (S-\lambda)SRg_n(\lambda)=0 \quad \in \quad U$$

$$\Rightarrow \lim_{n\to+\infty} SRg_n(\lambda)=0 \Rightarrow \lambda \lim_{n\to+\infty} g_n(\lambda)=0 \quad in \quad U \Rightarrow \lim_{n\to+\infty} g_n(\lambda)=0 \quad in \quad U$$

this implies that S has a property β

References

[1] P Aiena, *Fredholm and local spectral theory*, UIT Applications to Multipliers (2004).
[2] Hiba F. Al-Janaby, *Generalized weyl's theorem for spectral classes of operators*, Journal of college of education **3** (2010), no. 6, 297–314.
[3] Il Ju An and Eungil Ko, *Paranormal operators and some operator equations*, Filomat **29** (2015), no. 6, 1195–1207.
[4] C Benhida and EH Zerouali, *Local spectral theory of linear operators rs and sr*, Integral Equations and Operator Theory **54** (2006), no. 1, 1–8.
[5] M Berkani, *Index of b-fredholm operators and generalization of a weyl theorem*, Proceedings of the American Mathematical Society **130** (2002), no. 6, 1717–1723.
[6] Raúl E Curto and Young Min Han, *Generalized browder's and weyl's theorems for banach space operators*, Journal of Mathematical Analysis and Applications **336** (2007), no. 2, 1424–1442.
[7] HR Dowson, *R. harte, invertibility and singularity for bounded linear operators (marcel dekker inc., new york and basel, 1987) xii+ 590 pp. 0 8247 7754 9, 119.50.*, Proceedings of the Edinburgh Mathematical Society (Series 2) **32** (1989), no. 02, 334–335.
[8] MP Drazin, *Pseudo-inverses in associative rings and semigroups*, The American Mathematical Monthly **65** (1958), no. 7, 506–514.
[9] BP Duggal, *Operator equations aba= a2 and bab= b2*, Funct. Anal. Approx. Comput **3** (2011), no. 1, 9–18.
[10] Robin Harte, *Fredholm, weyl and browder theory*, Proceedings of the Royal Irish Academy. Section A: Mathematical and Physical Sciences, JSTOR, 1985, pp. 151–176.
[11] G.H. Heuser, *Functional analysis*, Jonhn Wiley and Sons Lid, 1982.
[12] Istratusco, *Introduction to linear operator theory*, Mareel Dekker, INC.New York and Basel, 1981.
[13] K Laursen, *Essential spectra through local spectral theory*, Proceedings of the American Mathematical Society **125** (1997), no. 5, 1425–1434.
[14] Sang Hun Lee and Cheon Seoung Ryoo, *Some properties of certain nonhyponormal operators*, Bull. Korean Math. Soc **31** (1994), no. 1, 133–141.
[15] Mourad Oudghiri, *Weyl's and browder's theorems for operators satisfying the svep*, Studia Mathematica **163** (2004), 85–101.
[16] BEBE PRUNARU, Kjeld B Laursen, and Michael M Neumann, *An introduction to local spectral theory: London mathematical society monographs (new series)*, 2002.
[17] Steffen Roch and Bernd Silbermann, *Continuity of generalized inverses in banach algebras*, Studia Mathematica **136** (1999), no. 3, 197–227.
[18] Christoph Schmoeger et al., *Common spectral properties of linear operators a and b such that aba= a2 and bab= b2*, Publications de lâ€™Institut Math.(NS) **79** (2006), no. 93, 109–114.
[19] Cheon seoung Ryoo, *Some class of operators*, Math.J.Toyamn Univ **21** (1998), no. 1, 147–152.
[20] Kotoro Tanahashi and Atsushi Uchiyama, *A note onâˆ—-paranormal operators and related classes of operators*, Bulletin of the Korean Mathematical Society **51** (2014), no. 2, 357–371.