Totaly *-paranormal and the Operators Equations

\[ SRS = S^2 \text{ and } RSR = R^2 \]

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Abstract: In this paper we study the property of the operator equations \( SRS = S^2 \) and \( RSR = R^2 \) where \( S \) is a *-paranormal operator we show that if \( S \) or \( S^\ast \) is a polynomial root of *-paranormal operator then \( f(A) \in \mathbb{G} \) for all \( f \in (H(\sigma(A))) \), where \( A \in \{SR, RS, R\} \) and we show that For the operator equation \( SRS = S^2 \) and \( RSR = R^2 \) we have \( \sigma_\beta(S) = \sigma_\beta(SR) = \sigma_\beta(RS) = \sigma_\beta(R) \)

1. Introduction

Let \( H \) be an infinite dimensional separable Hilbert space and let \( B(H), B_0(H) \) denote the algebra of bounded linear operators and the ideal of compact operator acting on \( H \). If \( T \in B(H) \) we shall write \( \text{Im}(T) \) and \( \text{ker}(T) \) for the range and null space of \( T \). Let \( \alpha(T) := \text{dimker}(T) \), \( \beta(T) := \text{dimker}(T^\ast) \), and let \( \sigma_\alpha(T), \sigma_\beta(T), \sigma_\rho(T), p_0(T), \) and \( \pi_0(T) \) denote the spectrum, approximate point spectrum, surjective spectrum, point spectrum of \( T \), the set of the resolvent of \( T \) and the set of all eigenvalues of \( T \) which are isolated in \( \sigma(T) \).

Recall that \( T \in B(H) \) is called *-paranormal operator if \( \|T^*x\| \leq \|T^2x\| \|x\| \) and \( T \) is called isosolid if every isolated point of \( \sigma(T) \) is an eigenvalue of \( T \). If \( T \in B(H) \) we write \( r(T) \) for the spectral radius of \( T \) where \( r(T) = \|T\| \). An operator \( T \in B(H) \) is called normal if \( \|T^*\| = \|T\| \). An operator \( T \in B(H) \) is said to be nilpotent if \( T^n = 0 \) for a natural number \( n \) and called quasinilpotent if \( r(T) = 0 \) [11, 12]

The operator \( E := \frac{1}{2\pi i} \int_{\mathbb{C}} (\lambda - T)^{-1} \) is called Riesz idempotent with respect to \( \lambda \) where \( D \) is a closed disk centered at \( \lambda \) and \( D \cap \sigma(T) = \{\lambda\} \) where \( \lambda \in \sigma(T) \) be an isolated point of \( \sigma(T) \) see[11]

Recall that if \( T \in B(H) \), the asent \( \alpha(T) \) and the descent \( d(T) \) given by

\[ \alpha(T) := \inf \{ n \geq 0 : \text{ker}(T^n) = \text{ker}(T^{n+1}) \} \]
\[ d(T) := \inf \{ n \geq 0 : \text{Im}(T^n) = \text{Im}(T^{n+1}) \} \]

An operator \( T \in B(H) \) is called Fredholm if it has closed range , finite dimensional null space and its range has finite co-dimensional:the index of a Fredholm operator \( i(T) = \alpha(T) - \beta(T) \)

\( T \) is called Weyle if it is Fredholm of index zero , and Browder if it is Fredholm of finite ascent and descent.The essential spectrum \( \sigma_e(T) \), the Weyl spectrum \( \sigma_w(T) \) and the Browder spectrum \( \sigma_B(T) \) define as [7, 10]

\[ \sigma_e(T) := \{ \lambda \in C : T - \lambda \text{ is not Fredholm} \} \]
\[ \sigma_w(T) := \{ \lambda \in C : T - \lambda \text{ is not Weyl} \} \]
\[ \sigma_B(T) := \{ \lambda \in C : T - \lambda \text{ is not Browder} \} \]

\[ \sigma_T(W) \subseteq \sigma_T(T) \cup \text{acc} \sigma(T) \]

we write \( \text{acc} K \) for the accumulation point of \( K \subset C \) if we write \( \text{iso} K = K \backslash \text{acc} K \) then let \( \pi_0(T) := \{ \lambda \in \text{iso} \sigma(T) : 0 \leq \alpha(T - \lambda) \leq \infty \} \)

\[ P_0(T) := \sigma(T) \backslash \sigma_0(T) \]

we say that Weyl’s theorem hold for \( T \) if \( \sigma(T) \backslash \sigma_0(T) \)

and Browder’s theorem hold for \( T \) if \( \sigma(T) \backslash \sigma_0(T) = P_0(T) \)

An operator \( T \in B(H) \) is called B-Fredholm if there exists a natural number \( n \) for the induced operator \( T_n : \text{Im}(T) \to \text{Im}(T^n) \) is Fredholm in the usual sense and B-Weyl’s if in addition \( T_n \) has zero index .

the B-Fredholm spectrum \( \sigma_{BF}(T) \) and B-Weyl spectrum \( \sigma_{BW}(T) \) are define by

\[ \sigma_{BF}(T) := \{ \lambda \in C : T - \lambda \text{ is not } B - \text{Fredholm} \} \]
\[ \sigma_{BW}(T) := \{ \lambda \in C : T - \lambda \text{ is not } B - \text{Weyl} \} \]

An element \( x \) of \( A \) is Drazin invertible if there is an element \( b \) of \( A \) and non-negative integer \( k \) such that [17]

\[ x^k b x = x^k, b x b = b, \quad x b = b x \]

the Drazin spectrum of \( a \in A \) is define by [8]

\[ \sigma_D(a) := \{ \lambda \in C : a - \lambda \text{ is not Drazin invertible} \} \]

If \( T \in B(H) \) it is well known that \( T \) is Drazin invertible if and only if it has finite ascent and descent and that is also equivalent to the fact that \( T \) decomposed as \( T_1 \oplus T_2 \) where \( T_1 \) is invertible and \( T_2 \) is nilpotent[15] and

\[ \sigma_{BW}(T) = \sigma(\sigma_{D}(T+F)) : F \in B_0(H) \]
\[ \sigma_{BB}(T) = \sigma(\sigma_{D}(T+F)) : F \in B_0(H) \text{ and } TF = FT \]
2. Main Results

Lemma 2.1 [19] every *-paranormal operator is normaloid.

Lemma 2.2 [19] If \( T \in B(H) \) is *-paranormal then \( \ker(T-\lambda I) \subseteq \ker(T^*-\lambda I) \) for each \( \lambda \in \mathbb{C} \) thus \( T-\lambda I \) is reduced by its eigenspace for every \( \lambda \in \mathbb{C} \)

Theorem 2.3 [19] If \( H \) is finite dimension .every *-paranormal operator \( T \) is normal.

Theorem 2.4 Let \( S \) be a *-paranormal operator on finite dimensional Hilbert space \( H \) and \( \ker(S)=\ker(S^*) \) then we have

(1) \( SR \) is normal.

(2) If \( N(S-\lambda I) \subseteq \ker(R-\lambda I) \) for each \( \lambda \in \mathbb{C} \) then for all of \( S, RS, SR, R \) are normal.

**Proof.** Since \( S \) is *-paranormal operator and \( \dim H < \infty \) use theorem (2.3) that \( S \) is normal operator hence \( S \) is paranormal operator then by [3] that \( SR \) is normal (2) since \( \ker(S) = \ker(RS) \) then by [3, Theorem3.1] it is obtain

Lemma 2.5 Let \( A \) be a *-paranormal operator then we have \( A=\lambda \) if \( \sigma(A) = \{\lambda\} \) for \( \lambda \in \mathbb{C} \)

**Proof.** case (1) if \( \lambda = 0 \) since \( A \) is *-paranormal then by lemma (2.1) \( T \) is normaloid therefore \( T = 0 \)

Case (2) if \( \lambda \neq 0 \) that is \( A \) is invertible since \( A \) is *-paranormal then \( A^{-1} \) is also *-paranormal then \( A^{-1} \) is normaloid hence \( \sigma(A^{-1}) = \{\frac{1}{\lambda}\} \) so \( \|A\| \left\|A^{-1}\right\| = \frac{1}{|\lambda|} = 1 \)

that is convexoid so \( \omega(A) = \{\lambda\} \) then \( A = \lambda \)

Lemma 2.6 Let \( S \) be a *-paranormal operator and \( \sigma(S) = \{\lambda\} \) then we have

(1) If \( \lambda = 0 \), then \( R^2 = 0 \)

(2) If \( \lambda \neq 0 \) then \( \lambda = 1 \) and \( R = S = I \)

**Proof.** case (1) is \( \lambda = 0 \) then by lemma (2.5) \( R^2 = 0 \) case (2) if \( \lambda \neq 0 \) and \( S \) is *-paranormal \( S = \lambda I \) and since \( SRS = S^2 \) that is \( SRS = (R-I) = 0 \) so that \( R = I \) and also that \( RSR = R^2 \) then \( (\lambda - I)R^2 = 0 \) and \( \lambda = 1 \) that is \( \sigma(S) = \sigma(R) = \{1\} \) which is \( R = S = I \)

Remark 2.7 Let \( S \) be a *-paranormal operator then we have

(1) If \( S \) is quasinilpotent, then \( SR, RS, R \) are nilpotent

(2) If \( S - I \) is quasinilpotent, then \( R = I \) therefore \( RS - \lambda, SR - \lambda, R - \lambda \) are invertible for each \( \lambda \in \mathbb{C} \setminus \{1\} \)

Corollary 2.8 If \( S \) is a *-paranormal operator, then \( iso(A) \subseteq \{0,1\} \) where \( A \in \{S, SR, RS, R\} \)

**Proof.** Let \( \lambda_0 \) be a nonzero isolated point of \( \sigma(S) \) by Riesz decomposition \( E^\bullet_\lambda(A) \) with respect to \( \lambda_0 \) we can act \( A \) as the direct sum

\[
S = S_1 \oplus S_2 , \quad \text{where} \sigma(S_1) = \{\lambda_0\} \quad \text{and} \quad \sigma(S_2) = \sigma(S) \setminus \{\lambda_0\}
\]

since \( S_1 \) is *-paranormal then by lemma (2.6) that is \( \lambda_0 = 1 \) that is \( iso(A) \subseteq \{0,1\} \)

**Lemma 2.9** If \( S \) is *-paranormal and \( \lambda_0 \) is nonzero isolated point of \( \sigma(S) \), then for the Riesz idempotent \( E^\bullet_\lambda(S) \) with respect to \( \lambda_0 \), we have

\[
Im(E^\bullet_\lambda(A)) = ker(SR - \lambda_0) = ker(S^*R^* - \lambda_0)
\]

**Proof.** since \( S \) is *-paranormal and \( \lambda \in iso(S) \setminus \{0\} \) then by [20, Theorem 3]

\[
Im(E^\bullet_\lambda(S)) = ker(RS - \lambda_0) = ker(S^* - \lambda_0)
\]

for the Riesz idempotent \( E^\bullet_\lambda(S) \) with respect to \( \lambda_0 \) but a pair \( (S, R) \) is solution of the operator equation \( SRS = S^2 \) and \( RSR = R^2 \) then by [18, corollary 2.2]

\[
ker(S - \lambda_0) = ker(SR - \lambda_0) \quad \text{and} \quad ker(S^* - \lambda_0) = ker(S^*R^* - \lambda_0)
\]

**Definition 2.10** we called the set \( \delta \) the collection of every pair \( (S, R) \) the operators as following

\( \delta := \{(S, R) : S \quad \text{and} \quad R \quad \text{are the solution of the operator equation} \quad SRS = S^2 \quad \text{and} \quad RSR = R^2 \quad \text{with} \quad ker(S - \lambda_0) = ker(R - \lambda) \quad \text{for} \quad \lambda \neq 0 \} \)

**Lemma 2.11** Suppose that \( (S, R) \in \delta \) and \( S \) is *-paranormal then by [18, Corollary 2.2] and lemma (2.9) that \( ker(RS - \lambda_0) = ker(S^*R^* - \lambda_0) \) for \( \lambda_0 \in iso(RS) \setminus \{0\} \)

**Proposition 2.12** Let \( (S, R) \in \delta \) and \( S \) be a *-paranormal operator

(1) If \( \lambda_0 \) is a nonzero isolated point of \( \sigma(S) \) then the range of \( RS - \lambda_0 \) is closed

(2) If \( R^* \) is injective and \( \lambda_0 \in iso(A) \setminus \{0\} \) then, \( ker(A - \lambda_0) \) reduces A where \( \in \{SR, R\} \)

**Proof.** (1) Let \( \lambda_0 \) is a nonzero isolated point of \( \sigma(S) \) then by corollary (2.8) that is \( iso(S) \subseteq \{1\} \). If \( iso(RS) = \phi \) then the prove done. If \( iso(RS) = \{1\} \), since \( SRS = S^2 \).
and \( RSR = R^2 \), by [18] 1 is an isolated point of \( \sigma(S) \) by
using the Riesz idempotent \( E_1(S) \) with respect to 1 we can act \( S \) as the direct sum
\[ S = S_1 \oplus S_2 \] where \( \sigma(S_1) = \{1\} \) and \( \sigma(S_2) = \sigma(S) \setminus \{1\} \)

since \((S, R)\) is \( \delta \) and \( S \) is \(*\)-paranormal then by lemma (2.9)
\[ H = \text{Im}(E) \oplus \text{Im}(E)^\perp = \ker(RS - I) \oplus \ker(RS - I)^\perp \]
which implies that
\[ RS = C_1 \oplus C_2 \text{ where } \sigma(C_1) = \{1\} \text{ and } \sigma(C_2) = \sigma(RS) \setminus \{1\} \]

since \( S_1 \) and \( C_1 \) are the restriction of \( S \) and \( RS \) to
\( \text{Im}(E_1(S)) \) therefore we not that if \( R := R \mid \text{Im}(E_1(S)) \) then
\( S_1 R_1 S_1 = S_1^2 \) and \( R S_1 R_1 = R_1^2 \) since \( S_1 \) is \(*\)-paranormal
then be lemma (2.6) that \( C_1 = I \) thus
\[ RS - I = 0 \oplus (C_2 - I) \]
so that
\[ \text{Im}(RS - I) = (RS - I)(H) = 0 \oplus (C_2 - I)(N(RS - I)^\perp) \]

since \( C_2 - I \) is invertible, that is \( RS - I \) has closed range
(2) since a pair \((S^*, R^*)\) is a solution of the operator
equation \( S^* R^* \) is \( S^2 \) and \( R^* R^* = R^2 \) and \( R^* \) is
injective \( S^* R^* = R^* \) but \((S, R) \in \delta \) then by lemma (2.9)
and lemma(2.11) that for the Riesz idempotent \( E_{\lambda_0}(A) \)
\[ \text{Im}(E_{\lambda_0}(S)) = \ker(A - \lambda_0) = \ker(A^* - \lambda_0) \]
where \( A \in \{SR, RS\} \)

Lemma 2.13 We have the following properties
(1) \( \pi_0(S) = \pi_0(SR) = \pi_0(RS) = \pi_0(R) \)
(2) \( S \) is isolated if and only if \( A \) is is isolated where \( A \in \{SR, RS, R\} \)

Proof. Since \( SRS = S^2 \) and \( RSR = R^2 \) then by [18] and [9, Lemma 2.3], it is known that
\[ \sigma(S) = \sigma(SR) = \sigma(RS) = \sigma(R) \]
and
\[ \sigma_p(S) = \sigma_p(SR) = \sigma_p(RS) = \sigma_p(R) \]
that is (2) is satisfied .
Also for every \( \lambda \in C \)
\[ \alpha(S - \lambda) > 0 \iff \alpha(SR_1) > 0 \iff \alpha(RS - \lambda) > 0 \iff \alpha(R - \lambda) > 0 \]
that is (1) satisfied

3. Generalized Weyl’s theorem for
algebraically totally \(*\)-paranormal

Definition 3.1 [14] An operator \( A \in B(H) \) is said to be
totally \(*\)-paranormal if \( T - \lambda \) is \(*\)-paranormal for all \( \lambda \in C \)

Definition 3.2 Let \( A \in B(H) \), we called \( A \) is an
algebraically totally \(*\)-paranormal if there exists a
non-constant complex polynomial \( P \) such that \( P(A) \) is
totally \(*\)-paranormal

normal operator ⇒ totally \(*\)-paranormal ⇒ algebraically
totally \(*\)-paranormal

Theorem 3.3 Suppose that \( S \) or \( S^* \) is a polynomial root of
\(*\)-paranormal operator then \( f(A) \in gW \) for all
\( f \in H(\sigma(A)) \), where \( A \in \{SR, RS, R\} \)

Proof. suppose that \( \lambda \in \sigma(A) \setminus \sigma_{BW}(A) \). Then \( A - \lambda \) is
B-Weyl but not invertible then by [5, Lemma 4.1] we can act \( A - \lambda \) as the direct sum
\( A - \lambda = A_1 \oplus A_2 \) where \( A_1 \) is Weyl and \( A_2 \) is nilpotent

Since \( S \) is polynomial root of \(*\)-paranormal then by[13] \( S \)
has (SVEP) therefore by [16][Theorem 3.3.9] and [9, Theorem 2.1], \( A \) has (SVEP).
This is implies that \( A_1 \) has (SVEP) at 0. Therefore \( A_1 \) is Wyle, so that \( A_1 \) has finite ascent and descent that is \( A - \lambda \) has finite ascent and
descent so that \( \lambda \in \pi_0(A) \)

Conversely, suppose that \( \lambda \in \pi_0(A) \). Then by lemma(2.13)
\( \lambda \in \pi_0(S) \). But \( S \) is polynomial root of \(*\)-paranormal
operator hence by [2] \( S \in gB \) therefore \( \lambda \) is a pole of
the resolvent of \( S \) so that \( A - \lambda \) is Drazin invertible by [9, Theorem 2.1] we can act \( A - \lambda \) as the direct sum
\( A - \lambda = A_1 \oplus A_2 \) where \( A_1 \) is invertible and \( A_2 \) is nilpotent.

Therefore \( A - \lambda \) is B-Weyl, and so \( \lambda \in \sigma_{BW}(A) \).
thus \( \sigma(A) \setminus \sigma_{BW}(A) = \pi_0(A) \)
therefore \( A \in gW \)

We claim that \( \sigma_{BW}(f(A)) = f(\sigma_{BW}(A)) \) for all
\( f \in H(\sigma(A)) \). Since \( A \in gW \), \( A \in gB \) then by [6, Theorem 2.1]
that \( \sigma_{BW}(A) = \sigma_{D}(A) \). Since \( S \) is polynomial root of \(*\)-paranormal operator, \( A \) has (SVEP)
so that \( f(A) \) has (SVEP) for all \( f \in H(\sigma(A)) \) therefore
\( f(A) \in gB \) by [6, Theorem 2.9] hence we have
\( \sigma_{BW}(f(A)) = \sigma_{D}(f(A)) = f(\sigma_{D}(A)) = f(\sigma_{BW}(A)) \)

Since \( S \) is a polynomial root of \(*\)-paranormal operator then
by [2] that \( S \) is isolated therefore by lemma(2.13) \( S \) is
isolated for all \( f \in H(\sigma(A)) \),
\( \sigma(f(A)) \setminus \pi_0(f(A)) = f(\sigma(A)) \setminus \pi_0(A) \)
since \( A \in gW \), we have
\( \sigma(f(A)) \setminus \pi_0(f(A)) = f(\sigma(A)) \setminus \pi_0(A) = f(\sigma_{BW}(A)) = \sigma_{BW}(f(A)) \)
that is \( f(A) \in gW \).

Now suppose that \( S^* \) is polynomial root of \(*\)-paranormal
operator. Let \( \lambda \in \sigma(A) \setminus \sigma_{BW}(A) \) observe that
Corollary 3.4 Suppose that $(S, R) \in \delta$ and $A$ is a compact operator. Then we have 
$$RA = I \oplus Q \text{ on } ker(RS-I) \oplus ker(RS-I)$$
where $\sigma(Q) = 0$

Proof. Let $S$ be a compact operator and $*$-paranormal. Then by theorem (3.3) $RS$ satisfies generalized Weyl’s theorem and by corollary (2.8) that is $isoaRS(0) \subseteq \{0\}$ hence 
$$\sigma(RS) \setminus \sigma_{BW}(RS) \subseteq \{0\}$$

Assume that $\sigma_{BW}(RS)$ is not finite. Then $\sigma(RS)$ is finite. Since $S$ is compact, $\sigma(RS)$ is countable set $\sigma(RS) = \{0, \lambda_1, \lambda_2, \ldots\}$, where $\lambda_i \neq 0$ for $j = 1, 2, \ldots$ and $\lambda_i \neq \lambda_j$ for all $i \neq j$ and $\lambda_j \to 0$ as $j \to \infty$ then by corollary (2.8) $\{\lambda_1, \lambda_2, \ldots\} \subseteq isoaRS(0) \subseteq \{0\}$. But this is a contradiction therefore $\sigma_{BW}(RS)$ is finite. That is means for all point in $\sigma_{BW}(RS)$ is isolated. So $\sigma(RS) \subseteq \{0\}$ if $1 \notin \sigma(RS)$, then $\sigma(RS) = 0$. If $S$ is $*$-paranormal then by lemma (2.5) $S = 0$ hence $RS = 0$. If $1 \in \sigma(RS)$, then by proposition (2.12)

$$RS = I \oplus Q \text{ on } H = ker(RS-I) \oplus ker(RS-I)$$

Theorem 3.5 Let $S$ is a polynomial root of $*$-paranormal operators then generalized $a$-Browder’s theorem holds for $A$ where $A \in \{SR, RS, R\}$

Proof. First we must show that $\sigma_{ea}(f(A)) = f(\sigma_{ea}(A))$ for all $f \in H(\sigma(A))$

Let $f \in H(\sigma(A))$. Since the inclusion $\sigma_{ea}(f(A)) \subseteq f(\sigma_{ea}(A))$ hold for each operator. Suppose that $\lambda \notin \sigma_{ea}(f(A))$ then $f(A) - \lambda$ is upper semi-Fredholm and $i(f(A) - \lambda) \leq 0$

$$f(A) - \lambda = c(A - \mu_1)(A - \mu_2)\ldots(A - \mu_n)_g(A)$$

where $c, \mu_1, \mu_2, \ldots, \mu_n \in C$ and $g(A)$ is invertible. Since $S$ is polynomial root of $*$-paranormal operators then by [13] and [1],

Theorem 2.40] that is $S$ has (SVEP) therefore $A$ has (SVEP) by [9, Theorem 2.1]. Since $A - \mu_i$ is upper semi-Fredholm, then by [15, Proposition 2.5] that is $i(A - \mu_i) \leq 0$ for all $i = 1, 2, \ldots, n$ hence $\lambda \notin f(\sigma_{ea}(A))$.

Suppose that $S$ or $S^*$ is a polynomial root of $*$-paranormal operators. Then $S^*R^*S^* = S^2$ and $R^*R^* = R^2$ $A^*$ has also (SVEP). So $i(A - \mu_i) \leq 0$ for all $i = 1, 2, \ldots, n$. From the classical index product theorem $A - \mu_i$ is Weyl for all $i = 1, 2, \ldots, n$ therefore $\lambda \notin f(\sigma_{ea}(A))$. Since $S$ or $S^*$ is a root of $*$-paranormal operators then $A$ or $A^*$ has (SVEP) therefore $a$-Browder’s theorem holds for $A$. Hence $\sigma_{ab}(f(A)) = f(\sigma_{ab}(A)) = f(\sigma_{ea}(A)) = f(\sigma_{ea}(A))$ for all $f \in H(\sigma(A))$

Definition 3.6 [16] An operator $A \in B(H)$ on $C$ is said has a Bishop’s property ($\beta$) if for every open subset $U$ of $C$ and every sequence of analytic functions $f_n : U \to x$ with the property that $(A - \lambda I)f_n(\lambda) \to 0$ as $n \to \infty$

Let $\sigma_{\beta}(A) = \{\lambda \in C : A \text{ does not have a Bishop’s property}\}$

Theorem 3.7 For the operator equation $SRS = S^2$ and $RSR = R^2$ we have $\sigma_{\beta}(S) = \sigma_{\beta}(SR) = \sigma_{\beta}(RS) = \sigma_{\beta}(R)$

Proof. The equivalence $SR$ has a property ($\beta$) at a point $\mu \in RS$ has a property ($\beta$) at $\mu$ holds for all $H^* \subseteq B(H)$ [4]. We prove that $S$ has a property ($\beta$) at $\mu$ the equivalence $R$ has a property ($\beta$) at $\mu$ is similarly proved. Let $U$ be an open neighborhood of $\mu$ and $(f_n) : U \to x$ be a sequence of analytic functions in a neighborhood of $\lambda$ such that

$$\lim_{n \to \infty} (S - \lambda)f_n(\lambda) = 0 \text{ in } U.$$ Then

$$\lim_{n \to \infty} (S^2 - \lambda S)f_n(\lambda) = 0 \Leftrightarrow \lim_{n \to \infty} S^2f_n(\lambda) = \lambda \text{ limit } Sf_n(\lambda) = \lambda^2 \text{ limit } f_n(\lambda) \text{ in } U$$

and so

$$\lim_{n \to \infty} SR(S - \lambda)f_n(\lambda) = 0 \Rightarrow \lim_{n \to \infty} (S^2 - \lambda SR)f_n(\lambda) = 0 \Rightarrow \lim_{n \to \infty} (SR - \lambda)(-\lambda f_n(\lambda)) = 0$$

in $U$.
Thus if SR has a property (β) at μ then
\[ \lambda \lim_{n \to \infty} f_{g_n}(\lambda) = 0 \Rightarrow \lim_{n \to \infty} g_n(\lambda) = 0 \quad \text{for all} \; \lambda \; \text{in} \; U \]
implies S has a property β at μ. Conversely assume that S has a property β at μ and let \( g_nU \to x \) be an analytic sequence such that
\[ \lim_{n \to \infty} (SR - \lambda)g_n(\lambda) = 0 \quad \text{in} \; U. \]

Then
\[ \lim_{n \to \infty} SRg_n(\lambda) = 0 \Leftrightarrow \lim_{n \to \infty} (S^2R - \lambda SR)g_n(\lambda) = (S - \lambda)SRg_n(\lambda) = 0 \quad \text{in} \; U \]
\[ \Rightarrow \lim_{n \to \infty} SRg_n(\lambda) = 0 \Rightarrow \lambda \lim_{n \to \infty} g_n(\lambda) = 0 \quad \text{in} \; U \; \Rightarrow \lim_{n \to \infty} g_n(\lambda) = 0 \quad \text{in} \; U \]
this implies that S has a property β

References