Purely Fully Cancellation Fuzzy Modules

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Abstract: In this paper we introduce the concept of purely fully cancellation fuzzy modules and give some characterizations and properties of this concept.

Keywords: Fully cancellation Fuzzy modules, Invertible fuzzy ideal, Purely Fully-Cancellation fuzzy module

1. Introduction

Gilmer in [1] was introduced the concept of cancellation ideal, and Anderson in [2], studied the concept of cancellation ideals. In [3] A. S. Mijbash, gave some generalization of this concept namely cancellation module (weakly cancellation module). In [4], Buthyna Nijad Shihab, introduce and studied restricted (and weakly restricted) cancellation module.

Next, Dr. L.M. Salman and Buthyna Nigad Shihab introduced and studied Relatively cancellation module in[5]. In [6], Hatam Yahya Khalaf and Hadi G. Rashed, introduced the concept of Fully cancellation fuzzy modules, where a fuzzy module X of an R-module M is called fully cancellation fuzzy module if for each fuzzy ideal I of R and for each fuzzy submodules A and B of X such that IA=IB, implies A=B.

In this paper we will introduce the concept of Puerely-fully cancellation fuzzy module and gives some properties, examples of this concept.

2.1 Definition

Let X be a fuzzy module of an R-module M. X is called puerely fully cancellation fuzzy module if for all non-empty fuzzy pure ideal I of R and for all non-empty fuzzy submodules A, A, of X such that IA=IA, then A=A. And follow up to this same idea will offer the defition of puerely fully cancellation ideal. If for all non-empty pure fuzzy ideal J of R and for all non-empty fuzzy ideals A and B of R such that JA=JB, then A=B.

2.2 Proposition

Let X be a fuzzy module of an R-module M. X is purely-fully cancellation fuzzy module if and only if X is purely-fully cancellation module

Proof: (⇒) Let K, N be two submodules of an R-module M and let J be a pure ideal of R.

Let I: R→[0,1] such that I(x):\[ I(x) = \begin{cases} t & \text{if } x \in J \text{ } \forall t \in (0,1] \\ 0 & \text{otherwise} \end{cases} \]

It is clear that I is a fuzzy ideal of R

Let A: M→[0,1], B: M→[0,1] such that:

\[ A(x) = \begin{cases} t & \text{if } x \in K \text{ } \forall t \in (0,1] \\ 0 & \text{otherwise} \end{cases}, B(x) = \begin{cases} t & \text{if } x \in N \text{ } \forall t \in (0,1] \\ 0 & \text{otherwise} \end{cases} \]

It is clear that A and B are two fuzzy submodules of X and A=K, B=N and I=J.

Suppose that JA=JB, to prove A=B (IA)=IB, so IA=IB by[7].

Thus A=B, since X is purely-fully cancellation fuzzy module.

Therefore A=B.

Conversely, It is clear that X=M and M is purely-fully cancellation module.

Let A and B be two fuzzy submodules of a fuzzy module X and let I be a fuzzy ideal of R such that IA=IB, then (IA)=IB, ∀ t∈(0,1], which implies that A, B are submodules of X, but X is purely fully cancellation module, so IA=IB, implies A=B, hence A=B.

Thus X is purely fully cancellation fuzzy module.

Examples (2.3)

(1) Let M=Z_{10}, R=Z_{10} and let X: M→[0,1] such that X(x) = \begin{cases} 1 & \text{if } x \in \{5\} \text{ } \forall t \in (0,1] \\ 0 & \text{otherwise} \end{cases}

X is a fuzzy module of Z_{10} module.

Let I: Z_{10}→[0,1] such that I(x) = \begin{cases} t & \text{if } x \in \{5\} \text{ } \forall t \in (0,1] \\ 0 & \text{otherwise} \end{cases}

Let A: M→[0,1] such that A(x) = \begin{cases} t & \text{if } x \in \{1\} \text{ } \forall t \in (0,1] \\ 0 & \text{otherwise} \end{cases}

Let B: M→[0,1] such that B(x) = \begin{cases} t & \text{if } x \in \{12\} \text{ } \forall t \in (0,1] \\ 0 & \text{otherwise} \end{cases}

It is clear that A and B are fuzzy submodules of X.

X_1=M and A_1=1, B_1=12, I_1=5 it is pure ideal by[8].

Now, I_1A_1=I_1B_1 (since (5) \{18\} = (5) \{12\})

Thus A_1=B_1.

(2) Let M=Z_{12} and R=Z_{12}. Let X: M→[0,1] such that:

\[ X(x) = \begin{cases} 1 & \text{if } x \in Z_{12} \text{ } \forall t \in (0,1] \\ 0 & \text{otherwise} \end{cases} \]

Define I: R→[0,1] such that:

I(x) = \begin{cases} 1 & \text{if } x \in \{3\} \text{ } \forall t \in (0,1] \\ 0 & \text{otherwise} \end{cases}

Let A: M→[0,1] such that:

\[ A(x) = \begin{cases} t & \text{if } x \in \{3\} \text{ } \forall t \in (0,1] \\ 0 & \text{otherwise} \end{cases} \]

and B: M→[0,1] such that:

\[ B(x) = \begin{cases} t & \text{if } x \in \{2\} \text{ } \forall t \in (0,1] \\ 0 & \text{otherwise} \end{cases} \]

It is clear that I_1=\{3\}, A_1=\{2\} and B=\{6\}.

Then \{3\} is pure ideal of Z_{12} and I_1A_1=\{3\}, I_1B_1=\{3\}, B_1=\{6\}.

Thus A_1=B_1.

Therefore X_1=Z_{12} is not purely-fully cancellation module.
Then by proposition (2.2), X is not purely –fully cancellation fuzzy module.

**Remark: (2.4)**
Every fully cancellation fuzzy module is purely –fully cancellation fuzzy module.

The converse of this remark is not true in general for example:

For example (1) we get $I_1=\{5\}$ and $X_1=\{6\}$ is a $Z_{10}$-module, $X_1$ is purely –fully cancellation module and by proposition (2.2), we get $X$ is purely –fully cancellation fuzzy module.

Now, define $A$: $M \to [0, 1]$ where $M = Z_{10}$ by $A(x) = \begin{cases} t & \text{if } x \in \{6\} \\ 0 & \text{otherwise} \end{cases}$

Define $B$: $M \to [0, 1]$ by $B(x) = \begin{cases} t & \text{if } x \in \{0\} \\ 0 & \text{otherwise} \end{cases}$ Since $\{6\} = \{5\}$, but $\{0\} \neq \{5\}$. Thus $X_1$ is not fully cancellation module and by proposition (2.2), we get $X$ is not fully cancellation fuzzy module.

**Proposition: (2.5)**
Every fuzzy submodule of purely –fully cancellation fuzzy module is also purely –fully cancellation.

**Proposition: (2.6)**
Let $X_1$ and $X_2$ be two fuzzy submodules of an $R$-module $M_1$, $M_2$ respectively such that $M_1 \cong M_2$. Then X is purely –fully cancellation fuzzy module if and only if $X_1$ is purely –fully cancellation fuzzy module.

**Definition: (2.7)** Let $(x^{-1})$ be the invertible element of $x$ in $R$ then $(x^{-1})$ is an invertible of a fuzzy singleton in $A$ and $x(x^{-1}) = (xx^{-1}) = = 1 = (x^{-1})x$, where $1: R \to [0, 1]$ such that $1 = \begin{cases} t & \text{if } x = 1 \\ 0 & \text{otherwise} \end{cases}$

**Definition: (2.8)** Let $X$ be a fuzzy module of an $R$-module $M$ and every non-empty fuzzy ideal $I$ of $R$ is fuzzy invertible. Then $X$ over an $R$-module $M$ is purely –fully cancellation module.

**Definition: (2.9)**
Let $I$ be a fuzzy invertible ideal of a ring $R$, $I$ is called fuzzy invertible if there exist $I^{-1} = \lambda_I(x)$ where $\lambda_I(x) = 1$ if $x = 1$.

**Theorem: (2.10)**
Let $X$ be a fuzzy module of an $R$-module $M$. If $A$, $B$ are two non-empty fuzzy submodules of $X$ and $I$ be a pure fuzzy ideal of $R$. Then the following statements are equivalent:

1. $X$ is purely–fully cancellation fuzzy module.
2. If $IA \subseteq IB$, then $A \subseteq B$.
3. If $IA \subseteq IB$, then $a \subseteq A$, where $a \subseteq X \ \forall \ t \in [0, 1]$.
4. If $(IA:IB) = (A:B)$.

**Proof:**
(1)$\Rightarrow$(2) Let $IA \subseteq IB$, then we have $IB = IA + IB$. Therefore $A \subseteq B$ which is the end proof.

(2)$\Rightarrow$(3) If $IA \subseteq IB$, then (2) we get $a \subseteq B$. Thus by (1) we have $rIB \subseteq A \ \forall \ t \in [0, 1]$.

Therefore, $IB \subseteq IA$, and hence $rIB \subseteq A$ since (1) implies (2) Thus $IB \subseteq IA$, and hence we get $(IA:IB) = (A:B)$.

(3)$\Rightarrow$(4) Let $IA = IB$, then (IA:IB) = $\lambda x(x) = 1$ if $x = 1$.

Hence $(A:B) = (A:B)$, and so $B \subseteq A$.

Similarly $(IB:IA) = (B:A)$, then $(B:A) = (A:B)$.

Which implies that $A \subseteq B$. Therefore $A = B$.

Thus $X$ is purely–fully cancellation fuzzy modules.

**References**


