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Modification on Two Parameters Kappa with Methods of Estimation

Rafid I. Ahmed

Al – Turath University College

Abstract: ***This paper deals with transforming two parameters Kappa (α, β) into three parameters Kappa (α, β, λ)using exponential of given CDF of Kappa by a new parameter (λ)thenthe new generated three parameters Kappa is obtained, also its cumulative distribution function (CDF), and then the derivation of rthmoment formula about origin is also derived after application of certain formulas of integral, and then we derive the formulas for maximum likelihood estimators of three parameters (α, β, λ), all the derivation required are explain.***

Keywords: Modified three parameters Kappa, Moments estimators, Maximum likelihood Estimators.

1. Introduction

Kappa probability distribution was introduced by Mielke and Mielke and Johson, it is one of family distribution that represent data with positive skewed model[2].Many research work, on powering the given CDF by a power, from parameter to obtain a new exponential class family,Gupta et al[1], (1998) work on generalization of standard exponential distribution to obtain a new exponentiated exponential family, Gupta and Kundu[5] in (2001) discuss the properties ofexponential family, Nadarajah, S.[8] (2005) work on transforming Gumbel distribution to exponentiated one, Nadarajah, S . and Gupta, A. K.[9,11] (2007), derive exponentiated gamma with application, Samir, K. A. , El – Sayed, A. Elsherpieny and Yassmen, Y. A.*[11]* (2009) workon maximum likelihood estimators (MLE's) for the unknown parameters and the corresponding asymptotic variance covariance matrix of the three-parameter Kappa distribution will be obtained under type II censored sample[11]. AlsoD. S. Hassun, Inam A. N. and Layla M. N[3] (2014) work on introducing three methods, for estimating the two parameters (α, β), Kappa distribution, these methods are maximum likelihood, maximum entropy, and L – moments.

The r^{th}non-central moments are derived, all the required formula for integrals needed explained, especially for maximum entropy method, since it requires four steps to beapplied, using Lagrange multipliers and certain constraints.

In our research, we work on extending two parameters Kappa to one with three parameters, using exponentiated. The P.D.F is obtained which have three parameters (β) scale parameter and two shapes parameters (α, λ), also CDF is derived and r^{th} Moments about origin and maximum likelihood estimators are obtained.

2. Definition

The two-parameter Kappa distribution (α, β)is[13];

$$f(x) = \frac{\alpha}{\beta}[\alpha + (\frac{x}{\beta})^{\alpha}]^{-(\frac{\alpha+1}{\alpha})} \qquad x, \alpha, \beta > 0 \quad \ldots(1)$$

$$F(x) = \frac{x}{\beta}[\alpha + (\frac{x}{\beta})^{\alpha}]^{-\frac{1}{\alpha}} \qquad x > 0 \quad \ldots(2)$$

A new family can be obtained from;

$$G(x) = [F(x)]^{\lambda} \qquad \ldots(3)$$

This implies;

$$G(x) = \frac{x^{\lambda}}{\beta^{\lambda}}[\alpha + (\frac{x}{\beta})^{\alpha}]^{-\frac{\lambda}{\alpha}} \qquad x, \alpha, \beta, \lambda > 0 \quad \ldots(4)$$

New generated P.D.F is;

$$g(x) = \frac{x^{\lambda}}{\beta^{\lambda}}\left[\left(\frac{-\lambda}{\alpha}\right)\left(\alpha + \left(\frac{x}{\beta}\right)^{\alpha}\right)\right]^{\frac{-\lambda}{\alpha}-1}\left(\alpha\frac{x^{\alpha-1}}{\beta^{\alpha}}\right) + [\alpha + (\frac{x}{\beta})^{\alpha}]^{\frac{-\lambda}{\alpha}}\frac{\lambda x^{\lambda-1}}{\beta^{\lambda}}$$

$$g(x) = \frac{\lambda}{\beta^{\lambda}}x^{\lambda-1}\left[\alpha + \left(\frac{x}{\beta}\right)^{\alpha}\right]^{\frac{-\lambda}{\alpha}} - \frac{\lambda x^{\lambda+\alpha-1}}{\beta^{\lambda+\alpha}}[\alpha + (\frac{x}{\beta})^{\alpha}]^{\frac{-\lambda}{\alpha}-1}$$

$$g(x) = \frac{\lambda}{\beta^{\lambda}}x^{\lambda-1}\left[\alpha + \left(\frac{x}{\beta}\right)^{\alpha}\right]^{\frac{-\lambda}{\alpha}}\left[1 - (\frac{x}{\beta})^{\alpha}(\alpha + (\frac{x}{\beta})^{\alpha})^{-1}\right] \quad \ldots(5)$$

This new generated three parameters kappa (α, β, λ);

$$m'r = e(x^r) = \int_0^{\infty} x^r g(x)dx \qquad x, \alpha, \beta, \lambda > 0 \quad \ldots(6)$$

$$= \int_0^{\infty} \frac{\lambda}{\beta^{\lambda}} x^{r+\lambda-1}[\alpha + (\frac{x}{\beta})^{\alpha}]^{-\frac{\lambda}{\alpha}}dx - \frac{\lambda}{\beta^{\lambda+\alpha}}\int_0^{\infty} x^{r+\lambda+\alpha-1}\left[\alpha + \left(\frac{x}{\beta}\right)^{\alpha}\right]^{-\frac{\lambda}{\alpha}-1} dx \quad \ldots(7)$$

Let $Z = \alpha + \left(\frac{x}{\beta}\right)^{\alpha} \quad \rightarrow z - \alpha = \left(\frac{x}{\beta}\right)^{\alpha}$

$$z - \alpha = \left(\frac{x}{\beta}\right)^{\alpha} \rightarrow dz = \frac{\alpha x^{\alpha-1}}{\beta^{\alpha}}\, dx$$

$$m'r = \frac{\lambda}{\beta^{\lambda}}I_1 - \frac{\lambda}{\beta^{\lambda+\alpha}}I_2 \qquad \ldots(8)$$

$$I_1 = \int_0^{\infty} x^{r+\lambda-1}[\alpha + (\frac{x}{\beta})^{\alpha}]^{-\frac{\lambda}{\alpha}}dx \qquad \ldots(9)$$

$$I_2 \Rightarrow \int_0^{\infty} x^{r+\lambda+\alpha-1}[\alpha + (\frac{x}{\beta})^{\alpha}]^{\frac{\lambda}{\alpha}-1}dx$$

Now

$$(z - \alpha)^{\frac{1}{\alpha}} = \frac{x}{\beta}$$

$$x = \beta(z-\alpha)^{\frac{1}{\alpha}}$$
$$dx = \frac{\beta}{\alpha}(z-\alpha)^{\frac{1}{\alpha}-1}dz$$
$$\therefore\ I_1 = \frac{\lambda}{\beta^{\lambda}}\int_0^{\infty} \beta[\,(z-\alpha)^{\frac{1}{\alpha}}]^{r+\lambda-1} z^{\frac{-\lambda}{\alpha}}\frac{\beta}{\alpha}(z-\alpha)^{\frac{1}{\alpha}-1}dz$$
$$\frac{\lambda}{\alpha\beta^{\lambda-2}}\int_0^{\infty}[\,(z-\alpha)]^{\frac{r+\lambda-1}{\alpha}} z^{\frac{-\lambda}{\alpha}}(z-\alpha)^{\frac{1}{\alpha}-1}dz$$
$$\frac{\lambda}{\alpha\beta^{\lambda-2}}\int_0^{\infty}[\,(z-\alpha)]^{\frac{r+\lambda}{\alpha}-1} z^{\frac{-\lambda}{\alpha}}dz$$
$$\frac{\lambda}{\alpha\beta^{\lambda-2}}(\alpha^{\frac{r+\lambda}{\alpha}-1})\int_0^{\infty} z^{\frac{-\lambda}{\alpha}}\,[\,(z-\alpha)]^{\frac{r+\lambda}{\alpha}-1}dz$$
$$\frac{\lambda}{\beta^{\lambda-2}}(\alpha^{\frac{r+\lambda}{\alpha}-2}(-1)^{\frac{r+\lambda}{\alpha}-1}\int_0^{\infty} u^{\frac{-\lambda}{\alpha}}\,[\,(1-u)]^{\frac{r+\lambda}{\alpha}-1}du$$
$$\frac{\lambda}{\beta^{\lambda-2}}(\alpha^{\frac{r+\lambda}{\alpha}-2}(-1)^{\frac{r+\lambda}{\alpha}-1}\frac{d\left(1-\frac{\lambda}{\alpha}\right)d\left(\frac{r+\lambda}{\alpha}\right)}{d\left(1+\frac{r}{\alpha}\right)} \qquad ...(10)$$
$$I_2 = \frac{\lambda}{\beta^{\lambda+\alpha}}\int_0^{\infty} x^{r+\lambda+\alpha-1}[\alpha+(\frac{x}{\beta})^{\alpha}]^{\frac{\lambda}{\alpha}-1}dx \qquad ...(11)$$
$$z = \alpha + \left(\frac{x}{\beta}\right)^{\alpha}$$
$$(z-\alpha)^{\frac{1}{\alpha}} = \frac{x}{\beta}$$
$$dx = \frac{\beta}{\alpha}(z-\alpha)^{\frac{1}{\alpha}-1}dz$$
Let $u = \frac{z}{\alpha}$
$$dz = \alpha du$$
$$I_2 = \frac{\lambda}{\beta^{\lambda+\alpha}}\int_0^{\infty} \beta[\,(z-\alpha)^{\frac{1}{\alpha}}]^{r+\lambda+\alpha-1} z^{\frac{-\lambda}{\alpha}-1}\frac{\beta}{\alpha}(z-\alpha)^{\frac{1}{\alpha}-1}dz$$
$$\frac{\lambda}{\beta^{\lambda+\alpha}}\frac{\beta^{r+\lambda+\alpha}}{\alpha}\int_0^{\infty}(z-\alpha)^{\frac{r+\lambda}{\alpha}} z^{\frac{-\lambda}{\alpha}-1}dz$$
$$u = \frac{z}{\alpha} \rightarrow \alpha du = dz$$
$$\frac{\lambda}{\alpha}\beta^{r}\int_0^{\infty}(\alpha u-\alpha)^{\frac{r+\lambda}{\alpha}}(\alpha u)^{\frac{-\lambda}{\alpha}-1}\alpha\, du$$
$$\lambda\beta^{r}(\alpha)^{\frac{r}{\alpha}+\frac{\lambda}{\alpha}-\frac{\lambda}{\alpha}-1}\int_0^{\infty}(u-1)^{\frac{r+\lambda}{\alpha}} u^{\frac{-\lambda}{\alpha}-1}\,du$$
$$\lambda\beta^{r}\alpha^{\frac{r}{\alpha}-1}(-1)^{\frac{r+\lambda}{\alpha}}\int_0^{\infty}(1-u)^{\frac{r+\lambda}{\alpha}} u^{\frac{-\lambda}{\alpha}-1}\,du \qquad ...(12)$$
$$I_2 = \lambda\beta^{r}\alpha^{\frac{r}{\alpha}-1}(-1)^{\frac{r+\lambda}{\alpha}}\frac{\Gamma\left(\frac{r+\lambda}{\alpha}+1\right)\Gamma\left(1-\left(\frac{\lambda}{\alpha}+1\right)\right)}{\Gamma\left(\frac{r}{\alpha}\right)} \qquad (13)$$

Therefore, the general formula of r^{th} moment for new modified three parameters kappa is;
$$m' r = I_1 + I_2$$

Which represent equation (10) and equation (13) knowing that $(\alpha > \lambda)$.

3. Estimation by Maximum Likelihood Method

Let $x_1, x_2 \dots x_n$ be a random sample from $g(x)$ defined by equation (5), then[7];
$$L = \prod_{i=1}^{n} g(x_i)$$

$$L = \lambda^{n}\beta^{-n\lambda}\prod_{i=1}^{n} x_i^{\lambda-1}\prod_{i=1}^{n}[\left(\alpha+\left(\frac{x_i}{\beta}\right)^{\alpha}\right)]^{-\frac{\lambda}{\alpha}}\prod_{i=1}^{n}[1-$$
xiβα(α+xiβ)α−1] ...(14)

$$\log l = n\log\lambda - n\lambda\log\beta + (\lambda-1)\sum_{i=1}^{n}\log x_i - \frac{\lambda}{\alpha}\sum_{i=1}^{n}\log\left(\alpha+\left(\frac{x_i}{\beta}\right)^{\alpha}\right) + \sum_{i=1}^{n}\log\left(1-\frac{\left(\frac{x_i}{\beta}\right)^{\alpha}}{\alpha+\left(\frac{x_i}{\beta}\right)^{\alpha}}\right)$$
$$\log l = n\log\lambda - n\lambda\log\beta + (\lambda-1)\sum_{i=1}^{n}\log x_i - \frac{\lambda}{\alpha}\sum_{i=1}^{n}\log\left(\alpha+\left(\frac{x_i}{\beta}\right)^{\alpha}\right) + \sum_{i=1}^{n}\log\left(\alpha+\frac{\alpha}{\alpha+\left(\frac{x_i}{\beta}\right)^{\alpha}}\right) \dots(15)$$
$$\Rightarrow n\log\lambda - n\lambda\log\beta + (\lambda-1)\sum_{i=1}^{n}\log x_i - \frac{\lambda}{\alpha}\sum_{i=1}^{n}\log(\alpha + \left(\frac{x_i}{\beta}\right)^{\alpha}) + n\log\alpha - \sum_{i=1}^{n}\log\left(\alpha+\left(\frac{x_i}{\beta}\right)^{\alpha}\right)$$
$$= n\log\lambda - n\lambda\log\beta + (\lambda-1)\sum_{i=1}^{n}\log x_i + n\log\alpha - (1 + \frac{\lambda}{\alpha})\sum_{i=1}^{n}\log\left(\alpha+\left(\frac{x_i}{\beta}\right)^{\alpha}\right)$$

Then;
$$\frac{\partial \log l}{\partial\lambda} = \frac{n}{\lambda} - n\log\beta + \sum_{i=1}^{n}\log x_i - \frac{1}{\alpha}\sum_{i=1}^{n}\log\left(\alpha+\left(\frac{x_i}{\beta}\right)^{\alpha}\right)$$
$$\frac{\partial \log l}{\partial\beta} = \frac{-n\lambda}{\beta} - [(1+\frac{\lambda}{\alpha})\sum_{i=1}^{n}\frac{x_i^{-\alpha}\beta^{-\alpha}(-1)\log\beta}{\left(\alpha+\left(\frac{x_i}{\beta}\right)^{\alpha}\right)}$$
$$\frac{\partial \log l}{\partial\beta} = \frac{n}{\alpha} - [(1+\frac{\lambda}{\alpha})\sum_{i=1}^{n}\frac{1+\left(\frac{x_i}{\beta}\right)^{\alpha}(1)\log\left(\frac{x_i}{\beta}\right)}{\left(\alpha+\left(\frac{x_i}{\beta}\right)^{\alpha}\right)} + \sum_{i=1}^{n}\log(\alpha+\left(\frac{x_i}{\beta}\right)^{\alpha}(\frac{-\lambda}{\alpha^2})$$
$$\Rightarrow \frac{n}{\alpha} - [(1+\frac{\lambda}{\alpha})\sum_{i=1}^{n}\frac{1+\left(\frac{x_i}{\beta}\right)^{\alpha}\log\left(\frac{x_i}{\beta}\right)}{\left(\alpha+\left(\frac{x_i}{\beta}\right)^{\alpha}\right)} + \frac{\lambda}{\alpha^2}\sum_{i=1}^{n}\log(\alpha+\left(\frac{x_i}{\beta}\right)^{\alpha}$$
$$\frac{\partial \log l}{\partial\lambda} = 0 \Rightarrow$$
$$\frac{n}{\lambda} = n\log\beta - \sum_{i=1}^{n}\log x_i + \frac{1}{\alpha}\sum_{i=1}^{n}\log(\alpha+\left(\frac{x_i}{\beta}\right)^{\alpha})$$
$$\hat{\lambda}_{MLE} = \frac{n}{n\log\beta - \sum_{i=1}^{n}\log x_i + \frac{1}{\alpha}\sum_{i=1}^{n}\log(\alpha+\left(\frac{x_i}{\beta}\right)^{\alpha})}$$
$$\frac{\partial \log l}{\partial\beta} = 0 \Rightarrow$$

$$\frac{n\lambda}{\beta} = (1+\frac{\lambda}{\alpha})\sum_{i=1}^{n}\frac{x_i^{-\alpha}\beta^{-\alpha}\log\beta}{\left(\alpha+\left(\frac{x_i}{\beta}\right)^{\alpha}\right)}$$

Finally

$$\frac{n}{\hat{\alpha}} = [(1+\frac{\lambda}{\alpha})\sum_{i=1}^{n}\frac{(1+\left(\frac{x_i}{\beta}\right)^{\alpha})\log\left(\frac{x_i}{\beta}\right)}{\left(\alpha+\left(\frac{x_i}{\beta}\right)^{\alpha}\right)} - \frac{\lambda}{\alpha^2}\sum_{i=1}^{n}\log(\alpha+\left(\frac{x_i}{\beta}\right)^{\alpha})$$

Now we explain different terms of $p.d.f$ and $c.d.f$ according to different sets of parameters that is necessary to explain the shape and behavior of $p.d.f$ and $c.d.f$ as shown in the two figures below;

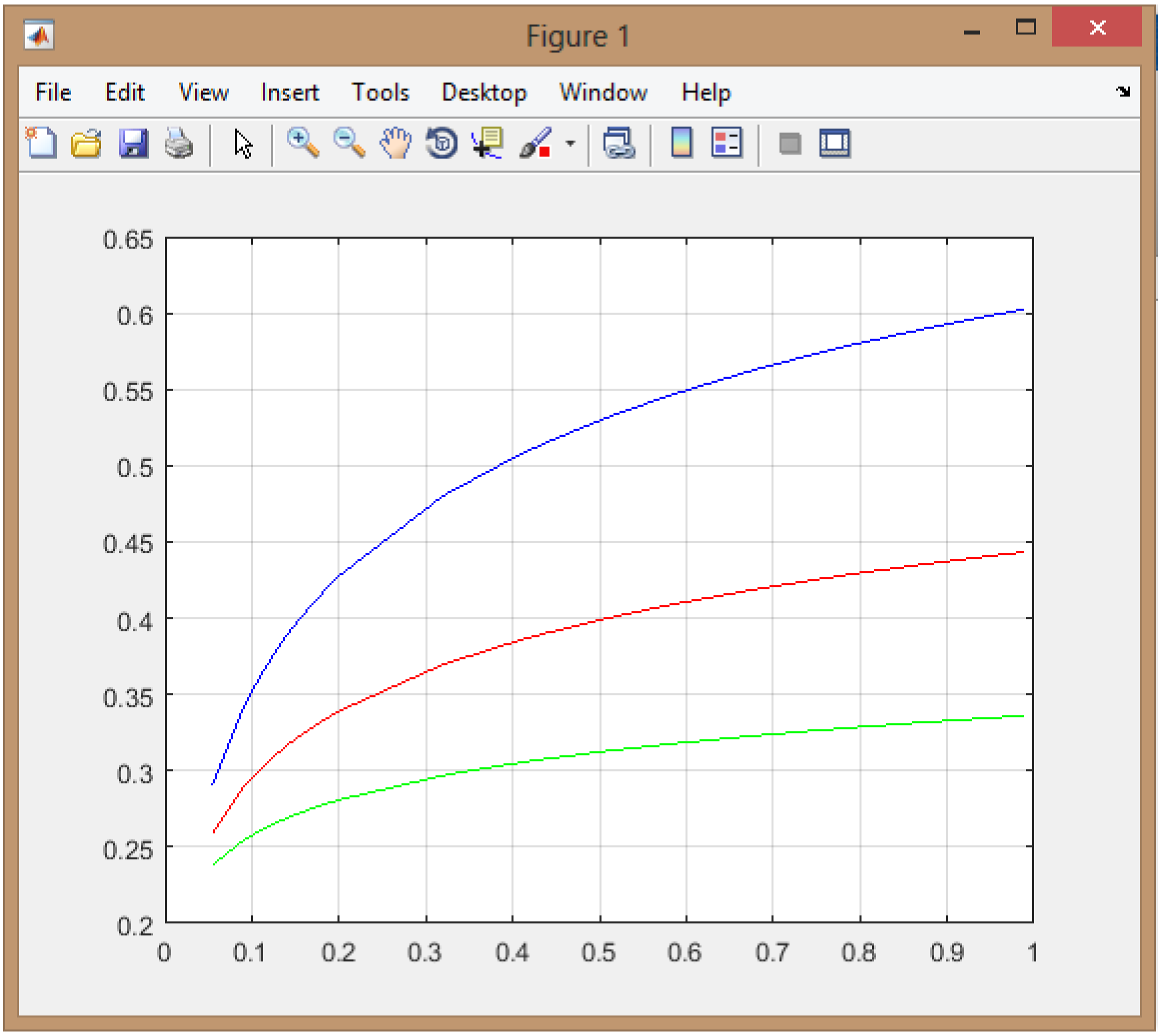

Figure 1: The C.D.F result is obtained have three parameters (β) scale parameter and two shapes parameters (α, λ).

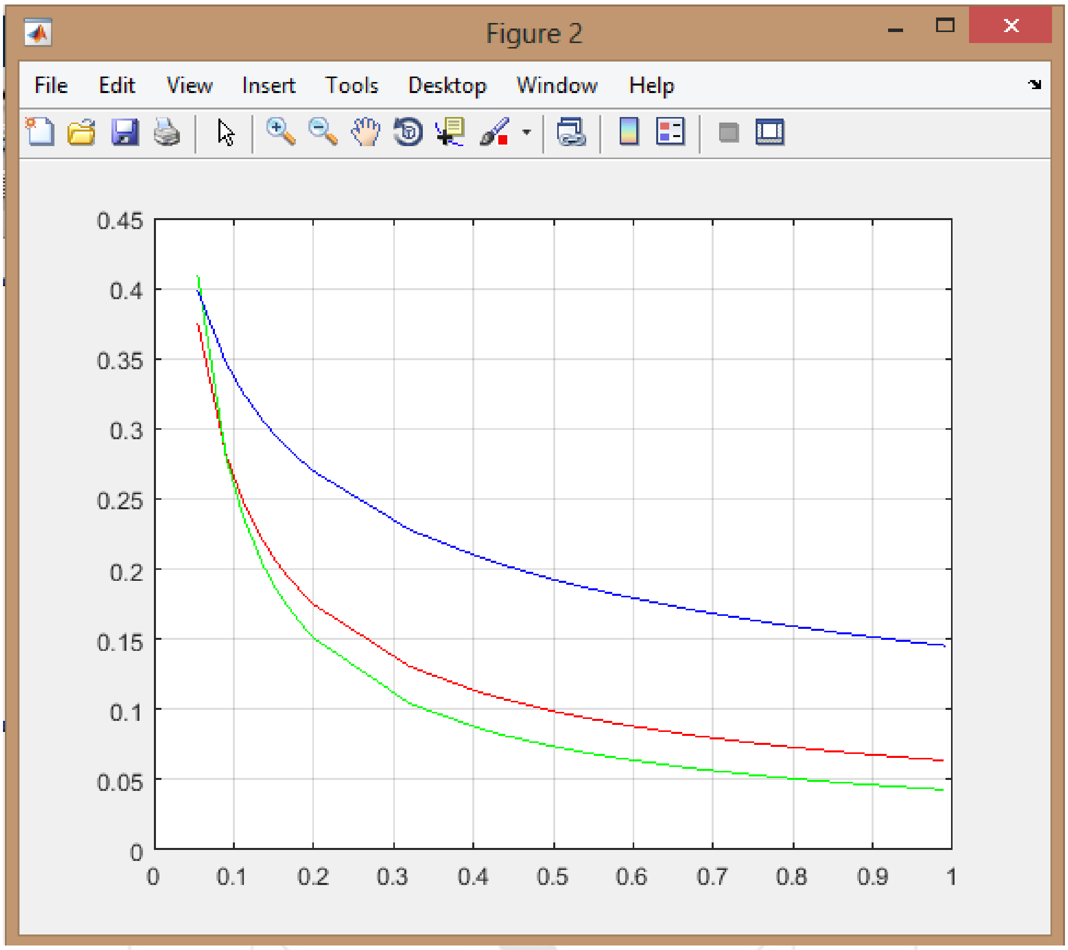

Figure 2: The P.D.F result is obtained

4. Conclusion

We explain different figures representing the $p.d.f$ of modified Kappa, and also c.d.f, using different of value.We see that all the curves are flexible, this is due to modification of Kappa.

Modification through exponent gives new $p.d.f$ which is necessary when the values of random variable lie between zero and one.

We suppose the variation of estimatorsα=0.2, $\beta = 1.8$, $\lambda = 0.8$,the results can show in Fig. (1) and Fig. (2)which is in red color.

Then suppose the variation of estimatorsα =0.1, $\beta = 0.6$, $\lambda = 1.2$,the results can show in Fig. (1) and Fig. (2)which is in green color.

We suppose the variation of estimators areα=0.4, $\beta = 1$, $\lambda = 0.6$,the results can show in Fig. (1) and Fig. (2)which is in blue color.

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