Algebraic Structures on Multi Groups II

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Abstract: ***Cosets in the sense of multi-groups were discussed, also factor groups in the multi-group perspective.***

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1. Introduction

If G is any group and A is any subgroup of G we know that:

gA is a *Left Coset* of A in G where

$gA=\{ga : a\in A\}$ where $g\in G$. Also $Ag=\{ag : a\in A\}$ is said to be right Coset of A in G, any two left (right) Cosets either disjoint or identical and we mention some basic results in this subject.

Lemma 1.1: Let G be a group and A is a sub-group of G then we has:

(i) $xA=A$ iff $x\in A$

(ii) $xA=yA$ iff $y^{-1}x\in A$

Same results can be written with respect to right Cosets.

Now if A is normal subgroup of G, we can define an operation on Left Cosets of A in G as follows:

$xAyA=xyA$ and in this case, this operation is well defined and the set of all *Left Cosets* of A in G form a group under this operation called the Factor group or Quotient group and is denoted be G/A. In the next section we want to extend these results in the case of multi-groups.

2. Cosets in the Sense of Multi-Groups

In this section we introduce the concept of multi Cosets i.e. Cosets in multi-group perspective Recall the definition 3.5 in [16] that is if B is a multi-group over the group Γ and $\underset{B}{C}(\mathrm{e})=\zeta$, $B_\upsilon=\{g\in\Gamma : C_B(g)\geq\upsilon, \upsilon\in Z^+\}$

Where Z^+ is the set of all positive integers.

$B_{\zeta(e)}=\{g\in\Gamma : C_B(g)=\zeta=C_B(e)\}$

$B_0=\{g\in\Gamma : C_B(g)\geq 0\}$

In a previous note [16] we have proved that B_υ $B_{\zeta(e)}$ and B_0 are sub-groups of Γ and in the following theorem we will prove that they are normal sub-groups of Γ.

Theorem 2.1: let Γ be a group and let $B\in AMG(\Gamma)$ then $B_0, B_{\zeta(e)}$ and B_υ where $\upsilon\in Z^+$ are normal sub-groups of the group Γ.

Proof (1):To prove that B_0 is normal in Γ:

Since $B_0=\{g\in\Gamma : C_B(g)\geq 0\}$

We had shown that $B_0, B_{\zeta(e)}$ and B_υ (see [16]) are sub-groups and so it remains to show that they are normal in Γ.

Let $g\in\Gamma$ taken arbitrarily and $b\in B_0$, then $C_B(\mathrm{b})\geq 0$ and Recall theorem 3.15 in [16] that $\underset{B}{C}\left(gbg^{-1}\right)=\underset{B}{C}\left(b\right)\geq 0$ and hence we get $gbg^{-1}\in B_0$, then B_0 is a normal sub-groups of Γ.

(2) to prove that $B_{\zeta(e)}$ is normal in Γ, let $b\in B_{\zeta(e)}$ then $C_B(\mathrm{b})=C_B(e)$ and for $g\in\Gamma$ and again by theorem 3.15 in [16]

$\underset{B}{C}\left(gbg^{-1}\right)=\underset{B}{C}\left(b\right)=\underset{B}{C}(e)$ then $gbg^{-1}\in \underset{\zeta(e)}{B}$, then $\underset{\zeta(e)}{B}$ is normal in Γ.

(3) now we want to show that $\underset{\upsilon}{B}$ is normal in Γ, $\upsilon\in Z^+$, take $g\in\Gamma$, $b\in\underset{\upsilon}{B}$, since $b\in\underset{\upsilon}{B}$ then $\underset{B}{C}\left(b\right)\geq\upsilon$, also by theorem 3.15 in [16] $\underset{B}{C}\left(gbg^{-1}\right)=\underset{B}{C}\left(b\right)\geq\upsilon$ thus $gbg^{-1}\in\underset{\upsilon}{B}$ and so $\underset{\upsilon}{B}$ is normal in Γ which completes the proof of the theorem. Now we introduce the concept of singleton set in multi-set perspective.

Definition 2.2: Let Γ be any set (need not be a group) and $G=\{\upsilon/g\}$ that is $G=\{g\}_\upsilon$ and we write it as $\{g\}_\upsilon=\underset{g}{\upsilon}$.

Definition 2.3: Let Γ be a group with identity e, $B\in MG(\Gamma)$ and $\underset{B}{C}\left(e\right)=\underset{B}{\zeta}(e)=\zeta$ then the multi-set $\underset{g}{\zeta}\circ B$ is called a left *mCoset* of B in Γ and we write it as gB similarly $B \circ \zeta_g$ is called the right $mCoset$ of B in Γ, and can be written as $B \circ \zeta_g = Bg$.

Lemma 2.4:Let Γ be a group, $B \in MG(\Gamma)$, then

(1) $C_{gB}(a) = C_B(g^{-1}a)$ and

(2) $C_{Bg}(a) = C_B(ag^{-1})$

Proof (1): Let $C_B(e) = \zeta$

$$C_{gB}(a) = C_{\zeta_g \circ B}(a)$$

$$= \max_{\substack{g,b \in \Gamma \\ gb=a}} \left\{ \min \left\{ C_{\{g\}_\zeta}(g), C_B(b) \right\} \right\}$$

$$= \max_{b=g^{-1}a} \left\{ \min \left\{ \zeta_B(e), C_B(b) \right\} \right\}$$

since any element in B not exceeds in multiplicity the multiplicity of e then $C_{gB}(a) = \max_{gb=a} \left\{ C_B(b) \right\}$

$$C_{gB}(a) = C_B(g^{-1}a)$$

Proof (2): $C_{Bg}(a) = C_{B \circ \zeta_g}(a)$

$$= \max_{\substack{bg=a \\ b,g \in \Gamma}} \left\{ \min \left\{ C_B(b), C_{\{g\}_\zeta}(g) \right\} \right\}$$

$$= \max_{b=ag^{-1}} \left\{ \min \left\{ C_B(b), \zeta_B(g) \right\} \right\}$$

$$= \max_{b=ag^{-1}} \left\{ C_B(b) \right\}$$

$$= C_B(ag^{-1})$$

Hence our result follows.

Theorem 2.5: In $AMG(\Gamma)$, every left $mCoset$ is the same as right $mCoset$ corresponding to the same element, same sub-group of Γ i.e. $\forall B \in AMG(\Gamma)$, $\forall g \in \Gamma$ then: $gB = Bg$.

Proof: Let $B \in AMG(\Gamma)$, then $C_B(gh) = C_B(hg)$ $\forall g,h \in \Gamma$. see [16]

$$C_{gB}(a) = C_B(g^{-1}a) = C_B(ag^{-1}) = C_{Bg}(a) \forall a \in \Gamma$$

then:

$C_{gB}(a) = C_{Bg}(a) \forall a \in \Gamma$, and hence our result follows.

Theorem 2.6: Let B be an element of $MG(\Gamma)$ then $\forall g,h \in \Gamma$ $Bg = Bh$ iff $B_{\zeta(e)}g = B_{\zeta(e)}h$.

Proof (1)**"Necessity"**:

Assume that $Bg = Bh$ then $B \circ \zeta_g = B \circ \zeta_h$

$C_B(ag^{-1}) = C_B(ah^{-1}) \forall a \in \Gamma$ since a is taken arbitrary then if we put $a = h$

$$C_B(hg^{-1}) = C_B(hh^{-1}) = C_B(e) = \zeta(e) = \zeta$$

$C_B(hg^{-1}) = C_B(e) = \zeta(e) = \zeta$ which means that

$hg^{-1} \in B_{\zeta(e)}$ i.e. $B_{\zeta(e)}(hg^{-1}) = B_{\zeta(e)}$ (by Lemma 1.1) part (i)

$B_{\zeta(e)}h = B_{\zeta(e)}g$ (by Lemma 1.1) part (ii)

Proof (2)**"Sufficiency"**: Assume that $B_{\zeta(e)}g = B_{\zeta(e)}h$ which implies immediately to $gh^{-1} \in B_{\zeta(e)}$, $hg^{-1} \in B_{\zeta(e)}$

i.e. $C_B(gh^{-1}) = C_B(e)$, now let $a \in \Gamma$ taken arbitrary then:

$$C_B(ah^{-1}) = C_B(ag^{-1}gh^{-1})$$

$$= C_B((ag^{-1})(gh^{-1}))$$

$$\geq \min \left\{ C_B(ag^{-1}), C_B(gh^{-1}) \right\}$$

$$\geq \min \left\{ C_B(ag^{-1}), C_B(e) \right\}$$

$$C_B(ah^{-1}) \geq C_B(a g^{-1}) = C_{Bg}(a)$$

$$C_B(ah^{-1}) \geq C_{Bg}(a) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

Now $C_B(ag^{-1}) = C_B(a h^{-1}hg^{-1})$

$$= C_B((a h^{-1})(hg^{-1}))$$

$$C_B(ag^{-1}) \geq \min \left\{ C_B(ah^{-1}), C_B(hg^{-1}) \right\}$$

$$\geq \min \left\{ C_B(ah^{-1}), C_B(e) \right\}$$

$$\geq C_B(ah^{-1}) = C_{Bh}(a) \ldots\ldots\ldots\ldots\ldots\ldots\ldots (2)$$

$$(1) \Rightarrow C_B(ah^{-1}) \geq C_{Bg}(a) \ldots\ldots\ldots\ldots\ldots\ldots (1)$$

$$C_B(ag^{-1}) \geq C_{Bh}(a) \ldots\ldots\ldots\ldots\ldots\ldots (2)$$

$(1) \Rightarrow \underset{Bh}{C}(a) \geq \underset{Bg}{C}(a)$ .... (1)'

$(2) \Rightarrow \underset{Bg}{C}(a) \geq \underset{Bh}{C}(a)$ .... (2)'

which means that $Bg = Bh$ and hence the theorem is proved.

Remark 2.7: Theorem 2.6 is also true in the case of left $mCosets$ i.e. if $B \in MG(\Gamma)$ then $\forall g, h \in \Gamma$ $gB = hB$ iff $g \underset{\zeta(e)}{B} = h \underset{\zeta(e)}{B}$ the proof can be carried by a similar arguments as in theorem 2.6.

Theorem 2.8: In the space of all multi-groups over a group Γ if two right $mCosets$ Bg, Bh (where B is any member of that space) are identically equal, then g, h have the same multicity in B.

Proof: Let $B \in AMG(\Gamma)$ where Γ is a group and $Bg = Bh$, $g, h \in \Gamma$, then by theorem 2.6 $\underset{\zeta(e)}{B} g = \underset{\zeta(e)}{B} h$, $gh^{-1} \in \underset{\zeta(e)}{B}$ and $hg^{-1} \in \underset{\zeta(e)}{B}$, from the properties of abelianmulti-groups see [16], we can write

$\underset{B}{C}(g) = \underset{B}{C}(hgh^{-1})$

$\geq \min\{\underset{B}{C}(h), \underset{B}{C}(gh^{-1})\}$

$\geq \min\{\underset{B}{C}(h), \underset{B}{C}(e)\}$

$\underset{B}{C}(g) \geq \underset{B}{C}(h)$ .... (1)

also $\underset{B}{C}(h) = \underset{B}{C}(ghg^{-1})$

$\underset{B}{C}(h) \geq \min\{\underset{B}{C}(g), \underset{B}{C}(hg^{-1})\}$

$= \min\{\underset{B}{C}(g), \underset{B}{C}(e)\}$

$\underset{B}{C}(h) \geq \underset{B}{C}(g)$ .... (2)

From (1), (2) we deduce that

$\underset{B}{C}(g) = \underset{B}{C}(h)$

i.e. g, h have the same multiplicity in B, Which completes the proof of the theorem.

3. Quotient Group in the Multi-Group Perspective

In this section we want to extend the concept of quotient group in the multi-group. For this we restrict our salvesin the space of all abelian mulit-groups over a given group Γ i.e. $AMG(\Gamma)$.

Theorem 3.1: Assume that B is any member of $AMG(\Gamma)$, and $\Gamma / B = \{Bg : g \in \Gamma\}$, then $Bg \circ Bh = B(gh) \forall g, h \in \Gamma$, moreover this operation is well defined.

Proof:Assume that $g, h \in \Gamma$ then $Bg \circ Bh = B \circ \underset{g}{\zeta} \circ B \circ \underset{h}{\zeta}$, $\underset{g}{\zeta} = \{g\}_{\underset{B}{C}(e)}$, $\underset{h}{\zeta} = \{h\}_{\underset{B}{C}(e)}$

$Bg \circ Bh = (\underset{g}{\zeta} \circ B) \circ (B \circ \underset{h}{\zeta})$

$= \underset{g}{\zeta} \circ (B \circ B) \circ \underset{h}{\zeta}$

But $B \circ B = B$ see [16]

$Bg \circ Bh = (\underset{g}{\zeta} \circ B) \circ \underset{h}{\zeta}$

$= (B \circ \underset{g}{\zeta}) \circ \underset{h}{\zeta}$

$= B \circ (\underset{g}{\zeta} \circ \underset{h}{\zeta})$

$= B \circ \underset{gh}{\zeta}$

$= Bgh$

Now to show thatthis operation is well defined, let $Bg = Bg_1$, $Bh = Bh_1$, so we can write:

$\underset{B}{C}(ag^{-1}) = \underset{B}{C}(ag_1^{-1})$ and $\underset{B}{C}(ah^{-1}) = \underset{B}{C}(ah_1^{-1})$, now we have

$\underset{B}{C}\left(a(gh)^{-1}\right) = \underset{B}{C}\left(ah^{-1}g^{-1}\right)$, put $t = ah^{-1}$

$\underset{B}{C}\left(a(gh)^{-1}\right) = \underset{B}{C}\left(tg^{-1}\right) = \underset{B}{C}\left(tg_1^{-1}\right)$

$= \underset{Bg_1}{C}(t) = \underset{Bg_1}{C}\left(ah^{-1}\right) = \underset{Bg_1}{C}\left(ah_1^{-1}\right)$

$= \underset{Bg_1h_1}{C}(a)$

$\underset{B}{C}\left(a(gh)^{-1}\right) = \underset{Bg_1h_1}{C}(a)$

$\underset{Bgh}{C}(a) = \underset{Bg_1h_1}{C}(a) \ \forall a \in \Gamma$

$Bgh = Bg_1h_1$

Which means that the operation is well defined witch completes the proof of the theorem.

Theorem 3.2: Let Γ be a group and $B \in AMG(\Gamma)$, and $G = \Gamma / B = \{Bg : g \in \Gamma\}$ and let "$\circ$"be an operation on G such that $Bg \circ Bh = B(gh)$ then $\langle G, 0 \rangle$ is the group, moreover this group is isomorphic to $\underset{\zeta(e)}{G} = \Gamma / \underset{\zeta(e)}{B}$ i.e. $\Gamma / B \cong \Gamma / \underset{\zeta(e)}{B}$.

Proof: From the previous theorem (i.e. theorem 3.1) we have shown that the operation is closed and well defined and in our previous note (see [16]),we have shown that the operation "$\circ$" is associative.

Now it is easy to see that B is the identity element of $G = \Gamma / B = \{Bg : g \in \Gamma\}$

$B \circ Bg = Be \circ Bg = B(eg) = Bg \ \forall g \in \Gamma$, also for existence of the inverse $Bg \circ Bg^{-1} = B(gg^{-1}) = Be = B \ \forall g \in \Gamma$, and hence $G = \Gamma / B$ is a group. Since $B \in AMG(\Gamma)$, then $\Gamma / B_{\zeta(e)}$ is a group (since $B_{\zeta(e)}$ is normal subgroup of Γ).

Now to prove that Γ / B and $\Gamma / B_{\zeta(e)}$ are isomorphic, let us consider the mapping $\theta : \Gamma / B \to \Gamma / B_{\zeta(e)}$ where $\theta(Bg) = B_{\zeta(e)}(g)$, θ is injection, since if $\theta(Bg) = \theta(Bh)$, thus

$B_{\zeta(e)}(g) = B_{\zeta(e)}(h)$, and from theorem 2.6, we get $Bg = Bh$, so θ is a injection, θ preserves the operation:

$\theta(Bg \circ Bh) = \theta(\mathrm{B}(gh))$

$= B_{\zeta(e)}(gh) = B_{\zeta(e)} g . B_{\zeta(e)} h$

$= \theta(Bg).\theta(Bh)$

The surjection of θ is clear and hence θ is an isomorphism and so $\Gamma / B, \Gamma / B_{\zeta(e)}$ are isomorphic, hence our required result follows.

4. Conclusion

Cosets and quotient groups in the multi-group perspective and some of their properties were discussed.

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