FI-Semihollow and FI- Semilifting Modules

Dr.Sahira M. Yaseen[1], Layla H. Helal[2]

[1]Department of Math., College of Science, University of Baghdad, Baghdad, Iraq

[2]Department of Civil Eng. College of Eng.University of Baghdad, Baghdad, Iraq

Abstract: ***Let R be a commutative ring with identity and let M be a left unital Rmodule. In this paper we also give a generalization of semihollow and modules, namely FI- semihollow, FI- semilifting modules respectivly. We study the properties of this concept.***

Keywords: semihollow module, semilifting module, fully invariant submodule.

1. Introduction

Let R be a commutative ring with identity and M be a left unital R-module. A submodule N of R-module M is called small in M (denoted by N $\ll$ M) if N + K $\neq$ M for each proper submodule K of Mand M is called a hollow module if every proper submodule is small in M, [2]. A submodule N of R-module M is called fully invariant submodule of M if f(N)$\subseteq$N, for every f$\in$Hom(M,M).Clearly 0and M are fully invariant submodule of M.The R-module is called duo module if every submodule of M is fully invariant [3].

In [4] there was given the concepts of semismall submodule semihollow modules and semilifting modules as a generalization of the concepts of small submodule, hollow modules and lifting modules. Where a submodule N of R-module M I,is called semismall (N$\ll_s$ M) if N=0 or for each nonzero submodule Kof M, then N/K$\ll$ M/K.

In this paper we introduce the concept of FI-semihollow module and FI-semilifting module as a generalization of semihollow and semilifting modules and study the basic properties of this type of modules and give some characterizations for such modules.

Lemma (1):[4]

Let N be a submodule of R-module M, N$\ll_s$ M iff N+L=M for all L$\leq$ M implies K+L=M For all K$\leq$ N, K $\neq$(0).

Lemma (2):[4]

1) IF N$\ll_s$ M and A<N then A$\ll_s$ M.
2) IF X, Y are submodule of M such that X$\ll_s$Y then X$\ll_s$ M.
3) IF N$\ll_s$M and K<M such that K$\subseteq$N then N/K$\ll_s$ M/K.
4) Let M=$M_1 \oplus M_2$ and N< M such that N=$N_1 \oplus N_2$, IF N$\ll_s$ M then $N_1 \ll_s M_1$ and $N_2 \ll_s M_2$.
5) Let N< N_1< M if N$\ll_s$ M and N_1 is direct summand then N $\ll_s$ N_1.

1- **FI-Semi Hollow module**

An non-zero R-module M is called semihollow iff every proper submodule of M is Semismall, [4].In this section we introduce the concept of FI-semihollow module and we study the properties of this concept.

Definition (1.1):-Anon zero R-module M is called FI-semihollow if every fully invariant proper submodule of M is Semismall.

A ring R is called FI-semihollow if R is FI-semihollow as an R-module , Equivalently every two sided ideal of R is semismall.

Remarks and Examples (1.2):-

1) Every hollow and semihollow is FI-semihollow module.
2) IF M is Duo module then semihollow and FI-semihollow are equivelent.
3) Z as Z-module is not semihollow and not FI-semihollow since 2Z is fully invariant proper submodule and 2Z+3Z=Z but 6Z+3Z$\neq$ M , 6Z< 2Z
4) $Z_2 \oplus Z_8$ as Z-module is FI-semihollow which is not Duo module.
5) Every simple R-module is FI-semihollow

Proposition (1.3):

FI-semihollow is closed under isomorphism.

Proof:

Let M, M′ be R-modules and M is FI-semihollow, let f: M→M′ be an R-isomorphism, we have to show that M′ is FI-semihollow, let N be fully invariant proper Submodule of M′, Now,f^{-1}(N) is proper submodule of M, If f^{-1}(N) =M then f (f^{-1}(N)) =N= M′ which is comtoduction. Thus f^{-1}(N) is proper submodule if M to show f^{-1}(N) is fully invariant of M.Let g: M→M, since N is fully invariant of M′, thus fgf^{-1}(N) $\subseteq$N and $f^{-1}f$ $(g(f^{-1}(N)) \subseteq f^{-1}(N)$, then $g(f^{-1}(N)) \subseteq f^{-1}(N)$. Thus f^{-1}(N) is fully invariant of M, since M is FI-semihollow , thus f^{-1}(N) $\ll_s$ M and by [4] prop. (1.3)and(N$\ll_s$ M′).

Proposition (1.4):

Let M be a FI- semihollow and N be a submodule of M with N/K is direct summand of M/K for each proper submodule K of N then N is FI-semihollow.

Proof: Let L be a proper fully invariant submodule of N then L$\ll_s$M, by Lemma(2).

Let K be a proper submodule of L then L/K$\ll$M/K and by hypothesis L/K is a direct summund of M/K then L/K$\ll_s$ N/K [4], hence L$\ll_s$N

Remark (1.5): FI-semihollow need not be indecomposable , Z_6 as Z-module is semihollow then FI-semihollow which is decomposable.

Recall that a submodule N of an R-module M is called coclosed in M if whenever $N/K' \ll M/K$ then N=K ∀ K submodule of M contained in N[5]

This means N is coclosed if whenever K<N, N/K is not small in M/K , it is known that the only proper coclosed submodule in semihollow module is simple submodule.

We have the following Remark.

Remark: Every proper fully invariant coclosed submodule of a semihollow module is a simple submodule.
Proof: let N be fully proper coclosed submodule of FI-semihollow module M , since M is FI-semihollow then N/K ≪ M/K ∀ K≤N since N is coclosed , N=K then N issimple.

Corollary: Every non-zero coclosed fully invariant submodule of FI-semihollow module is FI-semihollow.

Proof: Since every simple is FI-semihollow.

2. FI-Semilifting modules

An R-module M is called semilifting if for any submodule N of M there exist submodules K, K′ 0f M such that M=K⊕ K′ with K≤ K and N∩ $K' \ll_s$ N (equivalently N∩ $K' \ll_s$ M). In this section we introduce the notion of FI-semilifting modules and discus some properties of this kind of modules which are generalization of semilifting module.

Definition (2.1) :
An R-module M is called FI-semilifting if for any fully invariant submodule N of M there exist submodules K, K′ of M such that M=K⊕ K′ with K⊆ N and N⋂ K′ $\ll_s$ N.[4] The following theorem given a characterization of FI-semilifting modules.

Theorem (2.2) : Let M be an R-module,then the following statement are equivalent :
1) M is FI-semilifting
2) Every fully invariant submodule N of M, N can be written as N=A⊕ B where A is direct summand of M and $B \ll_s$ M.
3) For every fully invariant submodule N of M, there exist a direct summand K of M such that K≤ N and $N/K \ll_s M/K$.

Proof: It is clearsame like theorem 3.3 in [4].
It is known that every hollow is lifting, but the converse is not true (see Remark 1.1.7. in 11).

Remark (2.3) :
1) Every FI-semihollow is FI-semilifting
Proof: - let N be fully invariant submodule of M if N≠ M, $N \ll_s$ M, N= (0) ⊕N the result follows directly by (theorem2.2)
2) Every semisimple module is lifting hence semilifting and then FI- semilifting.

Proposition (2.4) :
An indecomposable R-module is FI-semihollow if and only if FI-semilifting.

Proof: Let M be FI-semihollow then M is FI-semilifting by Remark (2.3)

Conversely suppose that M is FI-semilifting and A fully invariant proper submodule of M, by (Theorem.2.2),We have A= N⊕ D where N is a direct summand of M and $D \ll_s$ M but M is indecomposable.Then either N= (0) or N=M then M=N⊆ A which implies that A=M which is contradiction, thus N= (0), So $A=D \ll_s$ M, hence M is FI-semihollow.

Proposition (2.6) :
If an R-module M is FI- semihollow then M/N is FI-semihollow for every fully invariant proper submodule N of M.

Proof: Assume that M is FI-semihollow and let N be fully invariant proper submodule of M

Let K/N be fully invariant proper submodule of M/N,Such that $\frac{M}{N} = \frac{K}{N} + \frac{H}{N}$ where H⊆ M and N⊂H. Then $\frac{M}{N} = \frac{K+H}{N}$, So it implies M=K+H, now since K/N is fully invariant of M/N and N is fully invariant proper of M thus K is fully invariant by (lemma 1.2.23[6]) then $K \ll_s$ M, So K+H=M, H≤ M then K′+H=M, ∀ K′ ⊂K (Lemma1).Hence we have $\frac{K'+H}{N} = \frac{M}{N}$ then $\frac{M}{N} = \frac{K'}{N} + \frac{H}{N}$ ∀ $\frac{K'}{N} \subseteq \frac{K}{N}$, K′ ≠ N, thus $\frac{M}{N}$ is FI- semihollow module.

References

[1] D.Kasch,Modules and rings ,Acadimic Press,London,1982
[2] P.Feury, "Hollow Modules and Local Endomorphism Rings", pac.J.Math., 53 (1974), 379-385.
[3] A.C. Ozcan,Duo Modules,Glasgow math.J.Trust 48,2006,pp.533-545.
[4] I.M.Ali and L.S.Mahmood, "Semismall submodules and Semi-lifting Modules", Proc. of 3rd Scientific Conference of College of Science, University of Baghdad 24-26 March(2009), 385-393.
[5] Y.Talebi and T.amoozegar fully invariant TM- Lifting Module,AlbanianJ.of Math. V.3(2009), 49-53,ISSN 1930-1235.
[6] Y. Talebi and T.AmoozeGar, strong FI-Lifting module.International Electronic J.of Algebra 3(2008), 75-82.