Generalized Radical_g Lifting Modules

Wasan Khalid¹, Adnan S. Wadi²

Department of Mathematics, College of Science, Baghdad University, Baghdad - Iraq

Abstract: This research deals with types of modules called $Radical_g$ lifting (generalized $Radical_g$ lifting) module as a generalization of lifting modules, some of properties of these types of modules will be studied including direct sum, direct summand, and quotient of $Radical_g$ lifting (generalized $Radical_g$ lifting) modules.

Keywords: lifting, Radical_g lifting module, generalized radical_g lifting module

1. Introduction

In this paper all ring are associated with identity and all modules are unital left R – Module. A submodule N of M is called small in M and (briefly N<< M) if whenever M = N + L for L \leq M implies M= L. Rad (M) is the intersection of all maximal submodules of M. Equivalently Rad (M) is the sum of all small submodules of M [1]. A submodule L of M is called essential submodule of M if L \cap N \neq 0 for every non-zero submodule N of M [2]. A submodule N of M is called generalized small submodule (briefly N <<g M) if for every essential submodule L of M with M = N + L lmplies M = L [3, 7]. It is clear that every small submodule of M is g – small but the converse is not true in general.

A submodule N of an R- module M is called generalized maximal submodule of M if it is maximal and essential in M, Recall that the intersection of all maximal essential submodule of M is called generalized Radical_g of M and (briefly Rad_g (M)). Equivalently Rad_g (M) is the sum of all g-small of M, i.e. Rad_g (M) = $\sum_{N << gM} N$ [3]. If M has no generalized maximal submodule then Rad_g (M) = M. [1, 4].

It is clear that Rad (M) \leq Rad_g (M), but the converse is not true in general. Consider Z₆ module as Z-module Rad (Z₆) = 0 but Rad_g (Z₆) = Z₆.

A module M is called lifting module if for every submodule N of M there exists a direct summand K of M such that $M = K \oplus K', K \le N, K' \le M$ and $N \cap K' << M$.[2].

In this paper Radical, lifting module and generalized Radical_g lifting module will be introduced as a generalization of lifting module. An R - module M is called Radical_g lifting module (briefly Rad_g - lifting) module if for every submodule N of M there exists a submodule K of N such that $M = K \oplus K'$, $K' \leq M$ and $N \cap K' \leq Rad_g$ (M). Amodule M is called a generalized Radical_g lifting module (briefly G - Rad_g - lifting) if for every submodule N of M with $\operatorname{Rad}_{g}(M) \leq N$ there exists a submodule K of N such that $M=K\oplus K'$, $K'\leq M$ and $N\,\cap\,K'\leq Rad_g$ (M) , some properties of this types of modules will be studied, it will be prove that the direct sum of two Radg-lifting module (G -Rad_g lifting) is again a Rad_g- lifting module (G - Rad_g lifting). And under certain condition a quotient of Rad_g lifting module (G - Rad_g – lifting) will be Rad_g lifting module (G - Rad_g lifting). Another properties of these types of modules were investigated in this paper.

2. Radical_g Lifting Modules

In this section Rad_g - lifting module ,will be introduced , and some of properties of this types of modules will be proved .

The following gives some properties of $\operatorname{Rad}_g(M)$ that appeared in [5].

Lemma 2.1

The following assertions are holds:

1. If M be an R – module , then $R_m <<_g M$ for every $m \in Rad (\ M$) .

2.If $f: M \longrightarrow N$ is an R – module homomorphism , then $f(Rad_{g}(M)) \leq Rad_{g}(N)$.

3.If $N \leq M$, then $\operatorname{Rad}_{\mathbf{g}}(N) \leq \operatorname{Rad}_{\mathbf{g}}(M)$.

4. If K, $L \le M$, then $\operatorname{Rad}_{g}(K) + \operatorname{Rad}_{g}(L) \le \operatorname{Rad}_{g}(K+L)$ 5. If K, $L \le M$, then $\operatorname{Rad}_{g}\frac{K+L}{L} \le \frac{\operatorname{Rad}_{g}(K+L)}{L}$.

6. If $M = \bigoplus_{i \in I} M_i$, then $Rad_g(M) = \bigoplus_{i \in I} Rad_g(M_i)$.

Lemma 2.2

Let N be a direct summand submodule of M . Then $Rad_{g}\left(N\right)$ = $Rad_{g}\left(\,M\,\right)\cap N$.

Proof:

Since N is a direct summand of M , then there exists a submodule $K \leq M$ such that $M = N \oplus K$, hence by lemma (2.1. (6)) $Rad_g(M) = (Rad_g(N) \oplus Rad_g(K))$. $Rad_g(M) \cap N = (Rad_g(N) \oplus Rad_g(K)) \cap N$. $= Rad_g(N)$ Hence $Rad_g(N) = Rad_g(M) \cap N$

Definition 2.3:

Let M be an R – module , M is called Radical_g lifting module (briefiy Rad_g -liftlng) module if for every submodule N of M there exists a direct summand K of M , K \leq N such that M = K \oplus K', K' \leq M and N \cap K' \leq Rad_g (M).

It is clear that every lifting module, semi – simple is Rad_g – lifting module but the converse is not true in general. Q as Z - module is Rad_g lifting but not lifting and not sime –simple.

Proposition 2.4:

Let M be an R - module. If $Rad_g(\ M\)=M$, then M is Rad_g- lifting module .

Volume 6 Issue 7, July 2017

www.ijsr.net Licensed Under Creative Commons Attribution CC BY

Proof:

Let N be a submodule of M. Then there exists $0 \le N$ such that M=0+M, and $N \cap M = Rad_{g}(M)$.

Theorem 2.5:

Let M be an R – module, then the following statements are equivalent:

1. M is Rad_g - lifting.

2. Every submodule N of M can be written as $N = A \oplus S$ where A is a direct summand of M and $S \leq Rad_g (M)$.

Proof . (1) = (2):

Let N be an submodule of M , then by (1) there exists a direct summand K of M , $K \leq N$ such that $M = K \oplus K'$, $K' \leq M$ and $N \cap K' \leq Rad_g$ (M). Hence $N = N \cap M = N \cap$ (K \oplus K') = K \oplus N \cap K', take A = K and S = K' \cap N \leq Rad_g (M) \blacksquare .

Proof . (2) 📫 (1):

Let N be any submodule of M , then By (2), N can be written as $N = A \oplus S$ where A is a direct summand of M and $S \leq Rad_g$ (M) i.e. $M = A \oplus K'$ and $K' \cap N = K' \cap (A \oplus S) = K' \cap S \leq S \leq Rad_g$ (M).

Proposition 2.6:

Let M be a R – module , and let $M = M_1 \oplus M_2$. If M_1 , M_2 are Rad_g – lifting . Then M is Rad_g - lifting .

Proof:

Let N be a submodule of M , then $M_1\cap N\leq M_1$ and $M_2\cap N\leq M_2$, hence there exist K_1 , K_2 in $M_1\cap N$ and $M_2\cap N$ respectively such that $M_1=K_1\oplus K_1',\,K_1'\leq M_1$, $M_2=K_2\oplus K_2'$, $K_2'\leq M_2$ and $K'_1\cap$ ($M_1\cap N$) $\leq Rad_g$ (M_1), $K_2'\cap$ ($M_2\cap N$) $\leq Rad_g$ (M_2). Now $M=M_1$

 $\oplus M_2 = K_1 \oplus K_1' \oplus K_2 \oplus K_2' = K_1 \oplus K_2 \oplus K_1' \oplus K_2'$. Therefore $K_1 \oplus K_2 \leq M_1 \oplus M_2$, $K_1 \oplus K_2$

is a direct summand of M, and $K_1 \oplus K_2 \leq M_1 \cap N \oplus M_2 \cap N = N$, $(K_1^{'} \oplus K_2^{'}) \cap N = K_1^{'} \cap N \oplus K_2^{'} \cap N \leq \text{Rad}_g(M_1) \oplus \text{Rad}_g(M_2) = \text{Rad}_g(M)$ [lemma2.1. (6)].

Corollary 2.7:

Let M be an R-module, and let $M=M_1\oplus M_2\oplus\ldots\oplus M_n$. If for all i=1, 2, \ldots , n . M_i is a Rad_g lifting , then M is Rad_g lifting .

Recall that a submodule N of M is called fully invariant if f (N) \leq N \forall f \in End (M) [1 (6.4)]. And R – module M is called a duo module if every submodule M is fully invariant [6].

Lemma 2.8:

Let M be an R – module , if M is Rad_g lifting module , then $\frac{M}{N}$ is Rad_g lifting for every fully invariant submodule N of M .

Proof:

Let N be a fully invariant submodule of M. Let $\frac{K}{N}$ be a submodule of $\frac{M}{N}$. Hence $K \le M$, then there exist a direct

summand L of M , $L{\leq}~K$. .i,e. $M=L{\oplus}~L',~L'{\leq}~M$ and , $L'{\cap}~K{\leq}~Rad_{\tt g}(~M~)$

Thus
$$\frac{M}{N} = \frac{L \oplus L'}{N} = \frac{L+N}{N} \oplus \frac{L'+N}{N}$$
. Since N is fully

invariant submodule .

Therefore	N =	N∩L€	$\mathbb{N} \cap L'$.	Hence –	$\frac{L+N}{N}$ ($\int \frac{L+N}{N}$	- =
0,therefore	$\frac{M}{N} =$	$\frac{L+N}{N}$	$\oplus \frac{L'+N}{N}$	- Thus -	$\frac{L+N}{N}$	is a dire	ect
summand	of	$\frac{M}{N}$ and	$\frac{L'+N}{N}$	$\cap \frac{\kappa}{N} =$	$\frac{L' \cap K}{l}$	$\frac{K+N}{N}$	\leq
$\frac{Rad_{g}(M)}{N}$	<u>(</u>)+	N = Rad	l _g (<u>M</u>) . [b	y lemma	12.1(5)].	

Corollary 2.9:

If $M=M_1\oplus M_2$ and M is a duo Rad_g lifting module , then M_1 and M_2 are Rad_g lifting .

Corollary 2.10:

Every direct summand of duo Rad_g lifting is again a Rad_g lifting.

Corollary 2.11:

The homomorphic image of a duo Rad_g lifting is a Rad_g lifting .

Proof:

Since every homomorphic image isomorphic to quotient modul. \blacksquare

Proposition 2.12:

Let M be a Rad_g lifting module , Let $L \leq M$ with $L \, \cap \, Rad_g$ (M) = 0 . Then L is semi –simple .

Proof :

Let $N\leq L$,then $N\leq M$ Since M is a Rad_g - lifting , then threre exists a direct summand K of M, $K\leq N$,such that $M=K\oplus K'$, $N\cap K'\leq Rad_g$ (M).Therefore $N\cap \,K'\leq Rad_g$ (M) \cap L=0. Thus $N\cap \,K'=0$, Hence $K'\oplus N=M$, hence M is semi–simple

Corollary 2.13:

Let M be an R – module . If M is a Rad_g - lifting with Rad_g (M) = 0 , Then M is semi – simple .

3. Generalization Radical_g Lifting Module

In this section a generalized ${\rm Radiccal_g}$ lifting module will be introduced as a generalization of ${\rm Rad_g}$ -lifting module it will be proved some properties of this type of modules .

Difinition 3.1:

Let M be an R– module . M is called Generalized $Radical_g$ lifting module (briefly G- Rad_g lifting) if for every submodule N of M with Rad_g (M) \leq N ,there exists a direct summand K of M such that M = K \oplus K', K \leq N , K' \leq M , and N \cap K' \leq Rad_g (M).

Volume 6 Issue 7, July 2017

<u>www.ijsr.net</u>

Licensed Under Creative Commons Attribution CC BY

DOI: 10.21275/ART20175657

Every lifting , semi – simple , and Rad_g lifting module is a G- Rad_g lifting module.

It is clear that every $Rad_g-lifting$ is a generalized $Rad_g-lifting$, but the converse in general is not true. It is easy to see that Z_{12} as Z-module is $G-Rad\ _g-lifting$, but not $Rad_{g_}$ lifting.

Theorem 3.2:

Let M be any R – module, then the following statement are equivalent:

1. M is $G - Rad_g$ lifting module.

2. Every submodule N of M with Rad_g (M) \leq N can be written as N = A \oplus S. Where A is a direct summand of M and S \leq Rad_g (M).

Proof . (1) = (2):

Let N be an submodule of M, with $Rad_g (M) \le N$, then by (1) there exists a direct summand K of M, $K \le N$, such that $M = K \oplus K'$, $K' \le M$, and $N \cap K' \le Rad_g (M)$.

Proof . (2) 📫 (1):

Let N be a submodule of M . By (2) N = A \oplus S , where A is a direct summand of M and S $\leq Rad_g$ (M) . A is a direct summand of M , therefore M = A \oplus L , L \leq M , L \cap N= L \cap (A \oplus S) =L \cap A \oplus L \cap S = L \cap S \leq S \leq Rad_g (M).

Proposition 3.3:

Let M be an R- module . Let $M=M_1\oplus M_2$, if M_1 and M_2 are $G-Rad_g$ lifting , then M is $G-Rad_g$ lifting .

Proof

Let N be a submodule of M such that $Rad_g (M) \leq N$, then $Rad_g (M_1) \leq N \cap M_1$ and $Rad_g (M_2) \leq N \cap M_2$. Then By theorem (**3.2**) $N \cap M_1 = A_1 \oplus S_1$, where A_1 is a direct summand of M_1 and $S_1 \leq Rad_g (M_1)$, and $N \cap M_2 = A_2 \oplus S_2$ where A_2 is a direct summand of M_2 and $S_2 \leq Rad_g (M_2)$. $N = N \cap M_1 \oplus N \cap M_2 = (A_1 \oplus A_2) \oplus (S_1 \oplus S_2)$, where $A_1 \oplus A_2$ is a direct summand of M and $S_1 \oplus S_2 \leq Rad_g (M_1)$ $\oplus Rad_g (M_2) = Rad_g (M)$ by [lemma2.1.(6)]

Corollary 3.4:

Let M be an R- module , Let $M = M_1 \oplus M_2 \oplus \ldots \oplus M_n$, if for all $i = 1, 2, \ldots, n$, M_i is a $G - Rad_g$ - lifting . Then M is a $G_g - Rad_g$ - lifting module.

Proposition 3.5:

Let M be an R – module, M be a G – Rad_g - lifting module , then for every fully invariant submodule N of M . Then $\frac{M}{N}$ is G. Rad_g - lifting module.

Proof:

Let $\frac{K}{N}$ be a submodule of $\frac{M}{N}$, with $\operatorname{Rad}_{g}\left(\frac{M}{N}\right) \leq \frac{K}{N}$. Since $\frac{\operatorname{Rad}_{g}\left(M\right) + N}{N} \leq \operatorname{Rad}_{g}\left(\frac{M}{N}\right)$ then $\operatorname{Rad}_{g}\left(M\right) \leq K$, Since

M is a $G-Rad_{\rm g}$ lifting then $K=A\oplus S$, where A is a direct

summand of M , and S
$$\leq$$
 Rad_g (M) . Now $\frac{\kappa}{N} = \frac{A+N}{N} \oplus$

$$\frac{S+N}{N}$$
, since N is fully invarian therefore $\frac{A+N}{N}$ is a

direct summand of
$$\frac{\kappa}{N}$$
 and $\frac{S+N}{N} \leq \frac{Rad_g(M)+N}{N} \leq \frac{Rad_g(M)+N}{N}$

$$\operatorname{Rad}_{g}\left(\frac{M}{N}\right)$$
. Thus $\frac{M}{N}$ is a G Rad_{g} – lifting.

Corollary 3.6:

Let M be an R – module, Let $M = M_1 \oplus M_2$ a duo G – Rad_g - lifting. Then M_1 and M_2 are G – Rad_g – lifting module.

Corollary 3.7:

The homomorphic image of a duo G $-Rad_g$ - lifting is again a G $-Rad_g$ - lifting.

Remark 3.8:

Not every submodule of $G - Rad_g - lifting$ is a $G - Rad_g - lifting$.

Consider Q as Z – module is a G – Rad_g – lifting . Since the only submodule contains Rad_g (Q) is Q and Z is n ot G – Rad_g –lifting . $0 \le 2Z$, $Z = Z \oplus 0$, $Z \cap 2Z \not\subset Rad_g$ (Z)=0

However under certain condition we have the following. .

Proposition 3.9:

Every direct summand of a $G - Rad_g - lifting$ module is a $G - Rad_g - lifting$ module.

Proof:

Let $K \leq M$, and let $U \leq K$ such that $Rad_g (K) \leq U$, Then $Rad_g (M) \leq U + Rad_g (M)$ Since M is a $G - Rad_g$ - lifting . Then there exists a submodule $N \leq U + Rad_g (M)$ with $M = N \oplus L$ and $L \cap U + Rad_g (M) \leq Rad_g (M)$. Now $K \cap M = K \cap N \oplus K \cap L$, $K \cap N \leq K \cap (U + Rad_g (M)) = U + Rad_g (K) = U$, and $K \cap L \cap U \leq K \cap L \cap (U + Rad_g (M)) \leq K \cap Rad_g (M) = Rad_g (K)$ by lemma(2.2).

References

- [1] Wisbauer R , **1991**, *Foundations of module and Ring Theory*, Gordon and Breach, Philadelphia,
- [2] Clark j., C. Lomp, N.Vanaja., R.Wisbauer, 2006, Lifting modules. Supplements and projectivity in module theory, series Frontiers in Mathematics, 406 ,publisher=Birkhäuser, address= Basel.
- [3] Kosar B., Nebiyev C. and Sökmez N., 2015, G. Supplemented Modules, Ukrainian Mathematical journal, 67 No. 6, 861 – 864.
- [4] Anderson F. W. and Fuller K. R., **1974**, *Rings and Cotegories of Modules*, Springer- Verlag, New York.
- [5] Celel Nebiyev, **2016**, G-Radical supplemented Modules, Dept.of Mathematics, Onokuz Mayis University, 1603.05517y2, [*Math. AC*].
- [6] Ozcan A. C., Harmanci A. Smith P.F., 2006, Duo Modules, *Glagow Math. J.* Trust, 48, 533-545.
- [7] Sökmez N., Kosar B, Nebiyev C. ,
 2010,Genellestirilmis Kücük Alt Modüller, XXIII. Ulusal Matematik Sempozyumu , Erciyes Universitesi, Kayseri,.

Volume 6 Issue 7, July 2017

<u>www.ijsr.net</u>

Licensed Under Creative Commons Attribution CC BY

- [8] Wang Y. and Ding N., 2006, Generalized Supplemented Modules, Taiwanese journal of Mathematics, 10 No. 6, 1589 – 1601.
- [9] Kash F., 1982, Modules and Rings, London New York
- [10] Xue W., **1996**, *Characterizations of Semiperfect and perfect Rings*, Publications Matematiques, 40, 115-125

Volume 6 Issue 7, July 2017 <u>www.ijsr.net</u> Licensed Under Creative Commons Attribution CC BY