Generalized Radical_\(g\) Lifting Modules

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Abstract: This research deals with types of modules called Radical_\(g\) lifting (generalized Radical_\(g\), lifting ) module as a generalization of lifting modules , some of properties of these types of modules will be studied including direct sum , direct summam , and quotient of Radical_\(g\) lifting ( generalized Radical_\(g\), lifting ) modules.

Keywords: lifting, Radical_\(g\) lifting module, generalized radical_\(g\) lifting module

1. Introduction

In this paper all ring are associated with identity and all modules are unital left \(R\) – Module. A submodule \(N\) of \(M\) is called small in \(M\) and (briefly \(N\) iff \(M\) ) if whenever \(M = N + L\) for some submodules \(N\) and \(L\) of \(M\). Equivalently \(Rad\) (\(M\)) is the intersection of all maximal submodules of \(M\). Equivalently \(Rad\) (\(M\)) is the sum of all submodules of \(M\) [1]. A submodule \(L\) of \(M\) is called an essential submodule of \(M\) if \(L \neq 0\) for every non - zero submodule \(N\) of \(M\) [2]. A submodule \(N\) of \(M\) is called a small submodule of \(M\) (briefly \(N\) iff \(M\)) if for every essential submodule \(L\) of \(M\) with \(M = N + L\). Implies \(M = L\) [3, 7]. It is clear that every small submodule of \(M\) is a \(g\) – small but the converse is not true in general.

A submodule \(N\) of an \(R\) - module \(M\) is called a generalized maximal submodule of \(M\) if it is maximal and essential in \(M\). Recall that the intersection of all maximal essential submodules of \(M\) is called generalized Radical_\(g\) of \(M\) and (briefly \(Rad\) (\(M\))). Equivalently \(Rad\) (\(M\)) is the sum of all \(g\)-small of \(M\), i.e. \(Rad\) (\(M\)) = \(\sum_{i=1}^{\infty} N_i\) [3]. If \(M\) has no generalized maximal submodule then \(Rad\) (\(M\)) = \(M\). [1, 4].

It is clear that \(Rad\) (\(M\)) \(\leq Rad\) (\(M\)), but the converse is not true in general. Consider \(Z_2\) module as \(Z\)-module \(Rad\) (\(Z_2\)) = 0 but \(Rad\) (\(Z_2\)) = \(Z_2\).

A module \(M\) is called lifting module if for every submodule \(N\) of \(M\) there exists a direct summand \(K\) of \(M\) such that \(M = K \oplus \cap\), \(K \leq M\), \(K \leq M\), and \(\cap \leq M\) [2].

In this paper Radical_\(g\) lifting module and generalized Radical_\(g\) lifting module will be introduced as a generalization of lifting module. An \(R\) – module \(M\) is called Radical_\(g\) lifting module (briefly \(Rad\) (\(g\), lifting ) module if for every submodule \(N\) of \(M\) there exists a submodule \(K\) of \(N\) such that \(M = K \oplus \cap\), \(K \leq M\) and \(\cap \leq Rad\) (\(M\)).

2. Radical_\(g\) Lifting Modules

In this section \(Rad\) (\(g\), lifting module ,will be introduced , and some of properties of this types of modules will be proved .

The following gives some properties of \(Rad\) (\(g\), (\(M\))) that appeared in [5].

Lemma 2.1

The following assertions are holds :

1. If \(M\) be an \(R\) – module , then \(R_m\) \(\leq g\) \(M\) for every \(m \in Rad\) (\(M\)).

2. If \(f : M \rightarrow N\) be an \(R\) – module homomorphism , then \(f \cup (Rad\) (\(M\))) \(\leq Rad\) (\(N\)).

3. If \(N \leq M\), then \(Rad\) (\(N\)) \(\leq Rad\) (\(M\)).

4. If \(K, L \leq M\), then \(Rad\) (\(K\)) \(\cup Rad\) (\(L\)) \(\leq Rad\) (\(K + L\))\(\leq\) \(Rad\) (\(K\)).

5. If \(K, L \leq M\), then \(Rad\) (\(K\) \(\cup Rad\) (\(L\)) \(\leq Rad\) (\(K + L\)).

6. If \(M = \cap_{i=1}^{\infty} M_i\), then \(Rad\) (\(M\)) \(\leq \cap_{i=1}^{\infty} Rad\) (\(M_i\)).

Lemma 2.2

Let \(N\) be a direct summand submodule of \(M\). Then \(Rad\) (\(g\), (\(N\)) \(\leq Rad\) (\(M\)) \(\cap N\).

Proof:

Since \(N\) is a direct summand of \(M\), then there exists a submodule \(K \leq M\) such that \(M = N \oplus K\), hence by lemma (2.1) \(Rad\) (\(M\)) \(= (Rad\) (\(N\)) \(\oplus Rad\) (\(K\))).

\(Rad\) (\(M\)) \(\cap N\) \(= (Rad\) (\(N\)) \(\oplus Rad\) (\(K\))) \(\cap N\).

\(= Rad\) (\(N\)).

Hence \(Rad\) (\(N\)) \(= Rad\) (\(M\)) \(\cap N\).

Definition 2.3:

Let \(M\) be an \(R\) – module , \(M\) is called Radical_\(g\) lifting module (briefly \(Rad\) (\(g\), lifting ) module if for every submodule \(N\) of \(M\) there exists a direct summand \(K\) of \(M\), \(K \leq N\) such that \(M = K \oplus \cap\), \(K \leq M\) and \(\cap \leq Rad\) (\(M\)).

It is clear that every lifting module, semi – simple is Radical_\(g\) lifting module but the converse is not true in general. Q as \(Z\) - module is Radical_\(g\) lifting but not lifting and not sine – simple.

Proposition 2.4:

Let \(M\) be an \(R\) – module. If \(Rad\) (\(g\), (\(M\)) \(= M\), then \(M\) is Radical_\(g\) lifting module.
Proof:
Let N be a submodule of M. Then there exists 0 ≤ N such that M = 0 + M, and N ∩ M = Radg (M).

Theorem 2.5:
Let M be an R – module, then the following statements are equivalent:

1. M is Radg - lifting.
2. Every submodule N of M can be written as N = A ⊕ S where A is a direct summand of M and S ≤ Radg (M).

Proof (1) ⇒ (2):
Let N be a submodule of M, then by (1) there exists a direct summand K of M, K ≤ N such that M = K ⊕ K', K' ≤ M and N ∩ K' ≤ Radg (M). Hence N = N ∩ M = N ∩ (K ⊕ K') = K ∩ N ⊕ K', take A = K and S = K' ∩ N ≤ Radg (M).

Proof (2) ⇒ (1):
Let N be any submodule of M, then By (2), N can be written as N = A ⊕ S where A is a direct summand of M and S ≤ Radg (M) i.e., M = A ⊕ K' and K' ∩ N = K' ∩ (A ⊕ S) = K' ∩ S ≤ S ≤ Radg (M).

Proposition 2.6:
Let M be a R – module, and let M = M1 ⊕ M2. If M1, M2 are Radg – lifting, then M is Radg – lifting.

Proof:
Let N be a submodule of M, then M1 ∩ N ≤ M1, and M2 ∩ N ≤ M2, hence there exist K1, K2 in M1 ∩ N and M2 ∩ N respectively such that M1 = K1 ⊕ K1', K1' ≤ M1, M2 = K2 ⊕ K2', K2' ≤ M2, and K1' ∩ (M1 ∩ N) ≤ Radg (M1), K2' ∩ (M2 ∩ N) ≤ Radg (M2). Now M = M1 ⊕ M2.

Proposition 2.7:
Let M be an R–module, and let M = M1 ⊕ M2 ⊕ … ⊕ Mn. If for all i = 1, 2, …, n, Mi is a Radg lifting, then M is a Radg lifting.

Recall that a submodule N of M is called fully invariant if f (N) ≤ N ∀ f ∈ End (M) [1 (6.4)]. And R – module M is called a duo module if every submodule M is fully invariant [6].

Lemma 2.8:
Let M be an R – module, if M is a Radg lifting module, then \( \frac{M}{N} \) is Radg lifting for every fully invariant submodule N of M.

Proof:
Let N be a fully invariant submodule of M. Let \( \frac{K}{N} \) be a submodule of \( \frac{M}{N} \). Hence K ≤ M, then there exist a direct summand L of M, L ≤ K. i.e., M = L ⊕ L', L' ≤ M and \( \frac{K}{N} \) ≤ Radg (M).

Thus \( \frac{M}{N} = \frac{L + L'}{N} = \frac{L + N}{N} \oplus \frac{L' + N}{N} \). Since N is fully invariant submodule.

Therefore N = N ∩ L ⊕ N ∩ L'. Hence \( \frac{L + N}{N} \oplus \frac{L' + N}{N} \) is a direct summand of \( \frac{M}{N} \) and \( \frac{L' + N}{N} \cap \frac{K + N}{N} \leq \frac{Radg (M) + N}{N} \). [by lemma 2.1 (5)].

Corollary 2.9:
If M = M1 ⊕ M2 and M is a duo Radg lifting module, then M1 and M2 are Radg lifting.

Corollary 2.10:
Every direct summand of duo Radg lifting is again a Radg lifting.

Corollary 2.11:
The homomorphic image of a duo Radg lifting is a Radg lifting.

Proof:
Since every homomorphic image isomorphic to quotient module.

Proposition 2.12:
Let M be a Radg lifting module, Let L ≤ M with L ∩ Radg (M) = 0. Then L is semi–simple.

Proof:
Let N ≤ L, then N ≤ M. Since M is a Radg - lifting, then there exists a direct summand K of M, K ≤ N, such that M = K ⊕ K', N ∩ K' ≤ Radg (M). Therefore N ∩ K' ≤ Radg (M) ∩ L = 0. Thus N ∩ K' = 0, Hence K' ⊕ N = M, hence M is semi–simple.

Corollary 2.13:
Let M be an R – module. If M is a Radg - lifting with Radg (M) = 0, Then M is semi – simple.

3. Generalization Radicalg Lifting Module

In this section a generalized Radicalg lifting module will be introduced as a generalization of Radg lifting module it will be proved some properties of this type of modules.

Definition 3.1:
Let M be an R– module. M is called Generalized Radicalg lifting module (briefly G- Radg lifting) if for every submodule N of M with Radg (M) ≤ N, there exists a direct summand K of M such that M = K ⊕ K’, K ≤ N, K’ ≤ M, and \( \frac{N}{K} \leq Radg (M) \).
Every lifting, semi – simple, and $\text{Rad}_g$ lifting module is a $G$- $\text{Rad}_g$ lifting module.

It is clear that $\text{every } \text{Rad}_g$ – lifting is a generalized $\text{Rad}_g$ – lifting, but the converse in general is not true. It is easy to see that $Z_g$ as $Z$ module is $G$ – $\text{Rad}_g$ – lifting, but not $\text{Rad}_g$ – lifting.

**Theorem 3.2:**
Let $M$ be any $R$ – module, then the following statement are equivalent:
1. $M$ is $G$ – $\text{Rad}_g$ lifting module.
2. Every submodule $N$ of $M$ with $\text{Rad}_g (M) \leq N$ can be written as $N = A \oplus S$. Where $A$ is a direct summand of $M$ and $S \leq \text{Rad}_g (M)$.

**Proof:** (1) $\Rightarrow$ (2):
Let $N$ be an submodule of $M$, with $\text{Rad}_g (M) \leq N$, then by (1) there exists a direct summand $K$ of $M$, $K \leq N$, such that $M = K \oplus K'$, $K' \leq N$, and $N \cap K' \leq \text{Rad}_g (M)$.

**Proof. (2) $\Rightarrow$ (1):**
Let $N$ be a submodule of $M$. By (2) $N = A \oplus S$, where $A$ is direct summand of $M$ and $S \leq \text{Rad}_g (M)$. $A$ is an direct summand of $M$, therefore $M = A \oplus L \oplus M \leq M$, $L \cap N = \emptyset \cap (A \oplus S) = L \cap A \oplus L \cap S = L \cap S \leq S \leq \text{Rad}_g (M)$.

**Proposition 3.3:**
Let $M$ be an $R$- module. Let $M = M_1 \oplus M_2$, if $M_1$ and $M_2$ are $G$ – $\text{Rad}_g$ lifting, then $M$ is $G$ – $\text{Rad}_g$ lifting.

**Proof:**
Let $N$ be a submodule of $M$ such that $\text{Rad}_g (M) \leq N$, then $\text{Rad}_g (M_1) \leq N \cap M_1$ and $\text{Rad}_g (M_2) \leq N \cap M_2$. Then By theorem (3.2) $N \cap M_1 = A_1 \oplus S_1$, where $A_1$ is a direct summand of $M_1$ and $S_1 \leq \text{Rad}_g (M_1)$, and $N \cap M_2 = A_2 \oplus S_2$ where $A_2$ is a direct summand of $M_2$ and $S_2 \leq \text{Rad}_g (M_2)$.

Consider $Q$ as $Z$ – module is $G$ – $\text{Rad}_g$ – lifting. Since the only submodule contains $\text{Rad}_g (Q)$ is $Q$ and $Z$ is not $G$ – $\text{Rad}_g$ – lifting. 0 $\leq 2Z, Z = Z \oplus 0, Z \cap 2Z \subset \text{Rad}_g (Z) = 0$

However under certain condition we have the following.

**Proposition 3.9:**
Every direct summand of a $G$ – $\text{Rad}_g$ – lifting module is a $G$ – $\text{Rad}_g$ – lifting module.

**Proof:**
Let $K \leq M$, and let $U \leq K$ such that $\text{Rad}_g (K) \leq U$. Then $\text{Rad}_g (M) \leq U + \text{Rad}_g (K)$. Since $M$ is $G$ – $\text{Rad}_g$ – lifting, then there exists a submodule $N \leq U + \text{Rad}_g (M)$ with $M = N \oplus L \oplus U + \text{Rad}_g (M) \leq \text{Rad}_g (M)$. Now $K \cap M = K \cap N \oplus K \cap L \cap \text{Rad}_g (N) \leq \text{Rad}_g (K) \leq U$ and $K \cap L \cap U \leq K \cap L \cap (U + \text{Rad}_g (M)) \leq K \cap \text{Rad}_g (M) = \text{Rad}_g (K)$ by lemma(2.2).

**References**

