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Generalized Radical$_g$ Lifting Modules

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Abstract: ***This research deals with types of modules called Radical$_g$ lifting (generalized Radical$_g$ lifting) module as a generalization of lifting modules , some of properties of these types of modules will be studied including direct sum , direct summand , and quotient of Radical$_g$ lifting (generalized Radical$_g$ lifting) modules.***

Keywords: lifting, Radical$_g$ lifting module, generalized radical$_g$ lifting module

1. Introduction

In this paper all ring are associated with identity and all modules are unital left R – Module. A submodule N of M is called small in M and (briefly N<< M) if whenever M = N + L for L ≤ M implies M= L. Rad (M) is the intersection of all maximal submodules of M. Equivalently Rad (M) is the sum of all small submodules of M [1]. A submodule L of M is called essential submodule of M if L ∩ N ≠ 0 for every non - zero submodule N of M [2]. A submodule N of M is called generalized small submodule (briefly N $<<_g$ M) if for every essential submodule L of M with M = N + L lmplies M = L [**3, 7**]. It is clear that every small submodule of M is g – small but the converse is not true in general.

A submodule N of an R- module M is called generalized maximal submodule of M if it is maximal and essential in M, Recall that the intersection of all maximal essential submodule of M is called generalized Radical$_g$ of M and (briefly Rad$_g$ (M)). Equivalently Rad$_g$ (M) is the sum of all g-small of M, i.e. Rad$_g$ (M) = $\sum_{N<<gM}$ N [3]. If M has no generalized maximal submodule then Rad$_g$ (M) = M. [1, 4] .

It is clear that Rad (M) ≤ Rad$_g$ (M), but the converse is not true in general. Consider Z_6 module as Z-module Rad (Z_6) = 0 but Rad$_g$ (Z_6) = Z_6 .

A module M is called lifting module if for every submodule N of M there exists a direct summand K of M such that M = K ⊕ K′, K ≤ N , K′ ≤ M and N ∩ K′ << M .[2].

In this paper Radical$_g$ lifting module and generalized Radical$_g$ lifting module will be introduced as a generalization of lifting module. An R – module M is called Radical$_g$ lifting module (briefly Rad$_g$ - lifting) module if for every submodule N of M there exists a submodule K of N such that M = K⊕K′ , K′ ≤ M and N ∩ K′ ≤ Rad$_g$ (M). Amodule M is called a generalized Radical$_g$ lifting module (briefly G - Rad$_g$ - lifting) if for every submodule N of M with Rad$_g$ (M) ≤ N there exists a submodule K of N such that M = K ⊕ K′ , K′ ≤ M and N ∩ K′ ≤ Rad$_g$ (M) , some properties of this types of modules will be studied , it will be prove that the direct sum of two Rad$_g$-lifting module (G - Rad$_g$ lifting) is again a Rad$_g$- lifting module (G - Rad$_g$ lifting) . And under certain condition a quotient of Rad$_g$ lifting module (G - Rad$_g$ – lifting) will be Rad$_g$ lifting module (G - Rad$_g$ - lifting) . Another properties of these types of modules were investigated in this paper.

2. Radical$_g$ Lifting Modules

In this section Rad$_g$ - lifting module ,will be introduced , and some of properties of this types of modules will be proved .

The following gives some properties of Rad$_g$ (M) that appeared in [5] .

Lemma 2.1
The following assertions are holds:
1. If M be an R – module , then R_m $<<_g$ M for every m ∈ Rad (M) .
2.If f : M ⟶ N is an R – module homomorphism , then f (Rad$_g$ (M)) ≤ Rad$_g$ (N) .
3.If N ≤ M , then Rad$_g$ (N) ≤ Rad$_g$ (M) .
4. If K , L ≤ M , then Rad$_g$ (K) + Rad$_g$ (L) ≤ Rad$_g$ (K + L)
5 . If K , L ≤ M , then $Rad_g \frac{K+L}{L} \leq \frac{Rad_g(K+L)}{L}$.
6. If M = $\oplus_{i\in I}$ M_i **,** then Rad$_g$ (M) = $\oplus_{i\in I}$ Rad$_g$ (M_i) **.**

Lemma 2.2
Let N be a direct summand submodule of M .Then Rad$_g$ (N) = Rad$_g$ (M) ∩ N .

Proof:
Since N is a direct summand of M , then there exists a submodule K ≤ M such that M = N ⊕ K , hence by lemma (2.1. (6)) Rad$_g$ (M) = (Rad$_g$ (N) ⊕ Rad$_g$ (K) .
Rad$_g$ (M) ∩ N = (Rad$_g$ (N) ⊕ Rad$_g$ (K)) ∩ N.
= Rad$_g$ (N)
Hence Rad$_g$(N) = Rad$_g$ (M) ∩ N ■

Definition 2.3:
Let M be an R – module , M is called Radical$_g$ lifting module (briefiy Rad$_g$ -liftlng) module if for every submodule N of M there exists a direct summand K of M , K ≤ N such that M = K ⊕ K′ , K′ ≤ M and N ∩ K′ ≤ Rad$_g$ (M).

It is clear that every lifting module, semi – simple is Rad$_g$ – lifting module but the converse is not true in general. Q as Z - module is Rad$_g$ lifting but not lifting and not sime –simple.

Proposition 2.4:
Let M be an R - module. If Rad$_g$(M) = M , then M is Rad$_g$ – lifting module .

Proof:
Let N be a submodule of M. Then there exists $0 \leq N$ such that M= 0 + M, and N ∩ M = Rad_g (M).

Theorem 2.5:
Let M be an R – module, then the following statements are equivalent:

1. M is Rad_g - lifting.
2. Every submodule N of M can be written as N = A ⊕ S where A is a direct summand of M and S ≤ Rad_g (M) .

Proof . (1) ⇨ (2):
Let N be an submodule of M , then by (1) there exists a direct summand K of M , K ≤ N such that M = K ⊕ K′ , K′ ≤ M and N ∩ K′ ≤ Rad_g (M) . Hence N = N ∩ M = N ∩ (K ⊕ K′) = K ⊕ N ∩ K′ , take A = K and S = K′ ∩ N ≤ Rad_g (M) ■ . .

Proof . (2) ⇨ (1):
Let N be any submodule of M , then By (2) , N can be written as N = A ⊕ S where A is a direct summand of M and S ≤ Rad_g (M) i.e. M = A ⊕ K′ and K′ ∩ N = K′ ∩ (A ⊕ S) = K′ ∩ S ≤ S ≤ Rad_g (M) . ■

Proposition 2.6:
Let M be a R – module , and let M = $M_1 \oplus M_2$. If M_1 , M_2 are Rad_g – lifting . Then M is Rad_g - lifting .

Proof:
Let N be a submodule of M , then $M_1 \cap N \leq M_1$ and $M_2 \cap N \leq M_2$,hence there exist K_1 , K_2 in $M_1 \cap N$ and $M_2 \cap N$ respectively such that $M_1 = K_1 \oplus K_1'$, $K_1' \leq M_1$, $M_2 = K_2 \oplus K_2'$, $K_2' \leq M_2$ and $K'_1 \cap (M_1 \cap N) \leq Rad_g (M_1)$, $K_2' \cap (M_2 \cap N) \leq Rad_g (M_2)$. Now M =M_1

$\oplus M_2 = K_1 \oplus K_1' \oplus K_2 \oplus K_2' = K_1 \oplus K_2 \oplus K_1' \oplus K_2'$. Therefore $K_1 \oplus K_2 \leq M_1 \oplus M_2$, $K_1 \oplus K_2$ is a direct summand of M , and $K_1 \oplus K_2 \leq M_1 \cap N \oplus M_2 \cap N = N$, $(K_1' \oplus K_2') \cap N = K_1' \cap N \oplus K_2' \cap N \leq Rad_g (M_1) \oplus Rad_g (M_2) = Rad_g (M)$ [lemma2.1. (6)] . ■

Corollary 2.7:
Let M be an R–module, and let M = $M_1 \oplus M_2 \oplus \ldots \oplus M_n$. If for all i = 1 , 2 , … , n . M_i is a Rad_g lifting , then M is Rad_g lifting .

Recall that a submodule N of M is called fully invariant if f (N) ≤ N ∀ f ∊ End (M) [1 (6.4)] . And R – module M is called a duo module if every submodule M is fully invariant [6] .

Lemma 2.8:
Let M be an R – module , if M is Rad_g lifting module , then $\frac{M}{N}$ is Rad_g lifting for every fully invariant submodule N of M .

Proof:
Let N be a fully invariant submodule of M . Let $\frac{K}{N}$ be a submodule of $\frac{M}{N}$. Hence K ≤ M , then there exist a direct summand L of M , L≤ K . .i,e. M = L⊕ L′, L′≤ M and , L′∩ K≤ Rad_g (M)

Thus $\frac{M}{N} = \frac{L \oplus L'}{N} = \frac{L+N}{N} \oplus \frac{L'+N}{N}$ .Since N is fully invariant submodule .

Therefore N = N ∩ L ⊕ N ∩ L′ .Hence $\frac{L+N}{N} \cap \frac{L'+N}{N} =$ 0,therefore $\frac{M}{N} = \frac{L+N}{N} \oplus \frac{L'+N}{N}$ Thus $\frac{L+N}{N}$ is a direct summand of $\frac{M}{N}$ and $\frac{L'+N}{N} \cap \frac{K}{N} = \frac{L' \cap K+N}{N} \leq \frac{Rad_g(M)+N}{N} \leq Rad_g (\frac{M}{N})$. [by lemma 2.1(5)] .

Corollary 2.9:
If M = $M_1 \oplus M_2$ and M is a duo Rad_g lifting module , then M_1 and M_2 are Rad_g lifting .

Corollary 2.10:
Every direct summand of duo Rad_g lifting is again a Rad_g lifting.

Corollary 2.11:
The homomorphic image of a duo Rad_g lifting is a Rad_g lifting .

Proof:
Since every homomorphic image isomorphic to quotient modul . ■

Proposition 2.12:
Let M be a Rad_g lifting module , Let L ≤ M with L ∩ Rad_g (M) = 0 . Then L is semi –simple .

Proof :
Let N ≤ L,then N ≤ M Since M is a Rad_g - lifting , then threre exists a direct summand K of M , K ≤ N,such that M = K ⊕ K′ , N∩K′ ≤ Rad_g (M) .Therefore N∩ K′ ≤ Rad_g (M) ∩ L = 0. Thus N ∩ K′ = 0 , Hence K′ ⊕ N = M , hence M is semi – simple

Corollary 2.13:
Let M be an R – module . If M is a Rad_g - lifting with Rad_g (M) = 0 , Then M is semi – simple .

3. Generalization Radical$_g$ Lifting Module

In this section a generalized Radiccal$_g$ lifting module will be introduced as a generalization of Rad_g - lifting module it will be proved some properties of this type of modules .

Difinition 3.1:
Let M be an R– module . M is called Generalized Radical$_g$ lifting module (briefly G- Rad_g lifting) if for every submodule N of M with Rad_g (M) ≤ N ,there exists a direct summand K of M such that M = K ⊕ K′ , K ≤ N , K′ ≤ M , and N ∩ K′ ≤ Rad_g (M).

Every lifting , semi – simple , and Rad_g lifting module is a G- Rad_g lifting module.

It is clear that every Rad_g – lifting is a generalized Rad_g – lifting , but the converse in general is not true . It is easy to see that Z_{12} as Z – module is G – Rad $_g$ – lifting , but not Rad_g – lifting.

Theorem 3.2:
Let M be any R – module, then the following statement are equivalent:
1. M is G – Rad_g lifting module.
2. Every submodule N of M with Rad_g (M) ≤ N can be written as N = A ⊕ S . Where A is a direct summand of M and S ≤ Rad_g (M).

Proof . (1) ⇒ (2):
Let N be an submodule of M , with Rad_g (M) ≤ N , then by (1) there exists a direct summand K of M , K ≤ N , such that M = K ⊕ K′ , K′ ≤ M , and N ∩ K′ ≤ Rad_g (M).

Proof . (2) ⇒ (1):
Let N be a submodule of M . By (2) N = A ⊕ S , where A is a direct summand of M and S ≤ Rad_g (M) . A is a direct summand of M , therefore M = A ⊕ L , L ≤ M , L∩N= L ∩ (A ⊕ S) =L ∩ A ⊕ L ∩ S = L ∩ S ≤ S ≤ Rad_g (M).

Proposition 3 .3:
Let M be an R- module . Let M = M_1 ⊕ M_2 , if M_1 and M_2 are G – Rad_g lifting , then M is G – Rad_g lifting .

Proof
Let N be a submodule of M such that Rad_g (M) ≤ N , then Rad_g (M_1) ≤ N ∩ $M_1$and Rad_g (M_2) ≤ N ∩ M_2 . Then By theorem (**3.2**) N ∩ M_1 = A_1 ⊕ S_1, where A_1 is a direct summand of M_1 and S_1 ≤ Rad_g (M_1) , and N ∩ M_2 = A_2 ⊕ S_2 where A_2 is a direct summand of M_2 and S_2 ≤ Rad_g (M_2) . N = N ∩ M_1 ⊕ N ∩ M_2 = (A_1 ⊕ A_2) ⊕ (S_1 ⊕ S_2) , where A_1 ⊕ A_2 is a direct summand of M and S_1 ⊕ S_2 ≤ Rad_g (M_1) ⊕ Rad_g(M_2) = Rad_g (M) by [lemma2.1.(6)]

Corollary 3.4:
Let M be an R– module , Let M = M_1 ⊕ M_2 ⊕ … ⊕ M_n , if for all i = 1 , 2 , … , n , M_i is a G – Rad_g - lifting . Then M is a G_g – Rad_g - lifting module.

Proposition 3 .5:
Let M be an R – module, M be a G – Rad_g - lifting module , then for every fully invariant submodule N of M . Then $\frac{M}{N}$ is G. Rad_g - lifting module.

Proof:
Let $\frac{K}{N}$ be a submodule of $\frac{M}{N}$, with Rad_g $(\frac{M}{N})$ ≤ $\frac{K}{N}$. Since $\frac{Rad_g(M)+N}{N}$ ≤ Rad_g ($\frac{M}{N}$) then Rad_g (M) ≤ K , Since M is a G – Rad_g lifting then K = A ⊕ S , where A is a direct summand of M , and S ≤ Rad_g (M) . Now $\frac{K}{N}$= $\frac{A+N}{N}$ ⊕ $\frac{S+N}{N}$, since N is fully invariant therefore $\frac{A+N}{N}$ is a direct summand of $\frac{K}{N}$and $\frac{S+N}{N}$ ≤ $\frac{Rad_g(M)+N}{N}$ ≤ Rad_g ($\frac{M}{N}$). Thus $\frac{M}{N}$is a G Rad_g – lifting.

Corollary 3 .6:
Let M be an R – module, Let M = M_1 ⊕ M_2 a duo G – Rad_g - lifting. Then M_1 and M_2 are G – Rad_g – lifting module.

Corollary 3 .7:
The homomorphic image of a duo G –Rad_g– lifting is again a G – Rad_g – lifting.

Remark 3.8:
Not every submodule of G – Rad_g – lifting is a G – Rad_g – lifting.

Consider Q as Z – module is a G – Rad_g – lifting . Since the only submodule contains Rad_g (Q) is Q and Z is n ot G – Rad_g –lifting . 0 ≤ 2Z , Z = Z ⊕ 0 , Z ∩ 2Z ⊄ Rad_g (Z)=0

However under certain condition we have the following. .

Proposition 3.9:
Every direct summand of a G – Rad_g – lifting module is a G – Rad_g – lifting module.

Proof:
Let K ≤ M, and let U ≤ K such that Rad_g (K) ≤ U, Then Rad_g (M) ≤ U+ Rad_g (M) Since M is a G – Rad_g - lifting . Then there exists a submodule N ≤ U + Rad_g (M) with M = N ⊕ L and L ∩ U + Rad_g (M) ≤ Rad_g (M) . Now K ∩ M = K ∩ N ⊕ K ∩ L, K ∩ N ≤ K ∩ (U+ Rad_g (M)) = U + Rad_g (K) = U , and K ∩ L ∩ U≤ K∩L ∩ (U+Rad_g(M)) ≤ K ∩ Rad_g (M) = Rad_g (K) by lemma(**2.2** **).**

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