A Study of Some Geometric Properties of Meromorphic Univalent Functions Associated with Ruscheweyh Derivative

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Abstract: ***We presented in this paper a new class S(λ,α,β) containing meromorphic univalent in functions in the punctured unit disk. We obtained many geometric properties, likes coefficient inequality, distortion and growth theorems, radii of starlikeness and convexity, extreme points, convex linear combination, arithmetic mean, hadamard product, closure theorems, neighborhood and partial sums results.***

Keywords: Univalent functions, meromorphic functions, starlike and convex functions.

1. Introduction

Let S denote the class of functions of the from

$$f(z) = z^{-1} + \sum_{n=1}^{\infty} a_n z^n, a_n \geq 0, n \in N = \{1, 2, \ldots\} \quad (1.1)$$

which are meromorphic univalent functions in the punctured unit disk $U^*=\{z\in\mathbb{C};\ 0<|z|<1\}$ .The hadamard product (or convolution) of two functions f and g

$$(f * g)(z) = z^{-1} + \sum_{n=1}^{\infty} a_n b_n z^n, \quad (1.2)$$

where $g(z) = z^{-1} + \sum_{n=1}^{\infty} b_n z^n$, $b_n \geq 0$

Also, a function $f\in S$ is meromorphic starlike function of order δ ($0 \leq\delta< 1$) if

$$-Re\left\{\frac{z f'(z)}{f(z)}\right\} > \delta \ \ ,\text{for } z \in U^*, \quad (5)$$

and a function $f\in S$ is meromorphic convex of order δ ($0 \leq\delta< 1$) if

$$-Re\left\{1+\frac{z f''(z)}{f'(z)}\right\} > \delta \ ,\text{for } z \in U^*.$$

The Ruscheweyh derivative of order λ is denoted by $D^\lambda f$ and is defined as follows

$$D^\lambda f(z) = \frac{1}{z(1-z)^{\lambda+1}} * f(z) =$$

$$z + \sum_{n=p}^{\infty} D_n(\lambda) a_n z^n, \lambda > -1, z \in U^*.$$

where

$$D_n(\lambda) = \frac{(\lambda+1)(\lambda+2)\ldots.(\lambda+n+1)}{(n+1)!}$$

Our aim of thispaper is to study the class S(λ,α,β) containing of function f and satisfying

$$\left|\frac{\frac{z(D^\lambda f(z))'}{D^\lambda f(z)}+1}{\frac{2z(D^\lambda f(z))'}{D^\lambda f(z)}+2\alpha}\right| < \beta \quad (1.3)$$

Where

$$D^\lambda f(z) = z^{-1} + \sum_{n=1}^{\infty} D_n(\lambda) a_n z^n. \quad (1.4)$$

Many authors studied classes of meromorphic functions which are univalent and multivalent like W. G. Atshan in [1,2] , B. A. Frasin and M. Darus in [3],A. R. S. Juma and H. Zirar in [4] ,L. Liu in [5], L. Liu and M. Srivastava in [6], J. E. Miller in [7] , and M. L. Mogra in [8],R. K. Raina and H. M. Srivastava in [9] and N. Xu and D. Yang in [10].

2. Coefficient Inequality

Theorem (2.1) Let the function f defined by (1.1). Then $f\in$S(λ,α,β) if and only if

$$\sum_{n=1}^{\infty}[n+1+2\beta(n+\alpha)] D_n(\lambda) a_n < 2\beta(1-\alpha) \quad (2.1)$$

where , $0 \leq\alpha< 1$, $0 <\beta\leq 1$.

Proof. Assume the condition (2.1) is satisfied, therefore

$$\left|z(D^\lambda f(z))' + (D^\lambda f(z))\right| - \beta\left|2z(D^\lambda f(z))' + 2\alpha(D^\lambda f(z))\right|$$

$$= \left|-z^{-1} + \sum_{n=1}^{\infty} D_n(\lambda) n a_n z^n + z^{-1} + \sum_{n=1}^{\infty} D_n(\lambda) a_n z^n\right| - \beta\left|-2z^{-1} + 2\sum_{n=1}^{\infty} D_n(\lambda)\ n a_n z^n + 2\alpha z^{-1} + 2\alpha \sum_{n=1}^{\infty} D_n(\lambda) a_n z^n\right|$$

$$=\left|\sum_{n=1}^{\infty} D_n(\lambda)\ (n+1) a_n z^n\right| - \beta\left|2(1-\alpha)z^{-1} - 2\sum_{n=1}^{\infty} D_n(\lambda)\ (n+\alpha) a_n z^n\right|$$

$$\leq \sum_{n=1}^{\infty} D_n(\lambda)(n+1)a_n + 2\beta\sum_{n=1}^{\infty} D_n(\lambda)(n+\alpha)a_n - 2\beta(1-\alpha)$$

$$\leq \sum_{n=1}^{\infty} D_n(\lambda)[n+1+2\beta(n+\alpha)]a_n - 2\beta(1-\alpha) \leq 0.$$

Then by maximum modulus theorem, we get the result.

For the Converse, assume that $\left|\frac{\frac{z(D^\lambda f(z))'}{D^\lambda f(z)}+1}{\frac{2z(D^\lambda f(z))'}{D^\lambda f(z)}+2\alpha}\right| < \beta$ is satisfied.

Therefore,

$\left|\frac{\sum_{n=1}^{\infty} D_n(\lambda)(n+1)a_n z^n}{2(1-\alpha)z^{-1} - 2\sum_{n=1}^{\infty} D_n(\lambda)(n+\alpha)a_n z^n}\right| < \beta$.Since $|Re(z)|\leq|z|$ for all z, we have

$Re\left\{\frac{\sum_{n=1}^{\infty} D_n(\lambda)(n+1)a_n z^n}{2(1-\alpha)z^{-1} - 2\sum_{n=1}^{\infty} D_n(\lambda)(n+\alpha)a_n z^n}\right\} < \beta$.Then by choosing the value of z on the real axis and letting $z \to 1-$ through values , we get

$$\sum_{n=1}^{\infty}[n+1+2\beta(n+\alpha)] D_n(\lambda) a_n < 2\beta(1-\alpha).$$

Corollary (2.1) If $f \in S(\lambda,\alpha,\beta)$, then

$a_n \leq \frac{2\beta(1-\alpha)}{[n+1+2\beta(n+\alpha)]D_n(\lambda)}$ (2.2)

where, $0 \leq \alpha < 1$, $0 < \beta \leq 1$.

3. Distortion and Growth Theorem

Theorem (3.1) If the function f defined by (1.1) is in the class $S(\lambda,\alpha,\beta)$, then for $0<|z|=r<1$, we have

$$|f(z)| \leq \frac{1}{r} + \frac{\beta(1-\alpha)}{1+\beta(1+\alpha)} r \text{ and } |f(z)| \geq \frac{1}{r} - \frac{\beta(1-\alpha)}{1+\beta(1+\alpha)} r \quad (3.1)$$

Proof

Since $f(z)$ be a function in $S(\lambda,\alpha,\beta)$, then we have from Theorem(2.1)

$$\sum_{n=1}^{\infty} [n+1+2\beta(n+\alpha)] D_n(\lambda) a_n < 2\beta(1-\alpha).$$

Then

$$|f(z)| = \left|z^{-1} + \sum_{n=1}^{\infty} a_n z^n\right| \leq |z^{-1}| + \sum_{n=1}^{\infty} a_n |z|^n$$

$\leq \frac{1}{r} + r\sum_{n=1}^{\infty} a_n$,

By theorem (2.1),

we get $\sum_{n=1}^{\infty} |a_n| \leq \frac{2\beta(1-\alpha)}{[1+\beta(1+\alpha)](\lambda+1)(\lambda+2)}$.

Then

$$|f(z)| \leq \frac{1}{r} + \frac{2\beta(1-\alpha)}{[1+\beta(1+\alpha)](\lambda+1)(\lambda+2)} r$$

Also,

$$|f(z)| \geq \frac{1}{r} - \frac{2\beta(1-\alpha)}{[1+\beta(1+\alpha)](\lambda+1)(\lambda+2)} r .$$

Corollary (3.2) If the function f defined by (1.1) is in the class $S(\lambda,\alpha,\beta)$, then for $0<|z|=r<1$, we have

$\frac{1}{r^2} - \frac{2\beta(1-\alpha)}{[1+\beta(1+\alpha)](\lambda+1)(\lambda+2)} \leq |f'(z)| \leq \frac{1}{r^2} + \frac{2\beta(1-\alpha)}{[1+\beta(1+\alpha)](\lambda+1)(\lambda+2)}$ (3.2)

4. Radii of Star Likeness and Convexity

The following results giving the radii of starlikeness and convexity of the function $f(z) \in S(\lambda,\alpha,\beta)$.

Theorem (4.1) If a function $f \in S(\lambda,\alpha,\beta)$, then f is meromorphically starlike function of order δ , $0 \leq \delta < 1$ in the disk $|z| < R_1$, where

$$R_1 = \inf_n \left[\frac{[n+1+2\beta(n+\alpha)]D_n(\lambda)(1-\delta)}{2\beta(1-\alpha)(n+\delta+2)}\right]^{\frac{1}{n+1}} \quad (4.1)$$

Proof

It is sufficient to show that

$$\left|\frac{zf'(z)}{f(z)} + 1\right| \leq 1-\delta ,$$

$$\left|\frac{zf'(Z)}{f(Z)} + 1\right| = \left|\frac{Zf'(Z) + f(Z)}{f(Z)}\right|$$

$$= \left|\frac{-Z^{-1} + \sum_{n=1}^{\infty} n a_n z^n + Z^{-1} + \sum_{n=1}^{\infty} a_n z^n}{Z^{-1} + \sum_{n=1}^{\infty} an z^n}\right|$$

$$= \left|\frac{\sum_{n=1}^{\infty}(n+1) a_n z^{n+1}}{1 + \sum_{n=1}^{\infty} a_n z^{n+1}}\right| \leq \frac{\sum_{n=1}^{\infty}(n+1) a_n |z|^{n+1}}{1 - \sum_{n=1}^{\infty} a_n |z|^{n+1}}$$

The last expression must bounded by 1- δ if

$$\sum_{n=1}^{\infty} (n+1) a_n |z|^{n+1} \leq 1-\delta-(1-\delta)\sum_{n=1}^{\infty} a_n |z|^{n+1},$$

$$\sum_{n=1}^{\infty} (n+\delta+2) a_n |z|^{n+1} \leq (1-\delta).$$

Then by theorem(2.1), we have

$$\sum_{n=1}^{\infty} \frac{[n+1+2\beta(n+\alpha)]D_n(\lambda)}{2\beta(1-\alpha)} a_n < 1 .$$

Hence

$$|z|^{n+1} \leq \frac{[n+1+2\beta(n+\alpha)]D_n(\lambda)(1-\delta)}{2\beta(1-\alpha)(n+\delta+2)},$$

Also,

$$|z| \leq \left[\frac{[n+1+2\beta(n+\alpha)]D_n(\lambda)(1-\delta)}{2\beta(1-\alpha)(n+\delta+2)}\right]^{\frac{1}{n+1}}.$$

Therefore, we get the result.

Theorem (4.2). If a function $f \in S(\lambda,\alpha,\beta)$, then f is meromorphically convex function of order δ ,($0 \leq \delta < 1$) in the disk $|z| < R_2$, where

$$R_2 = \inf_n \left[\frac{[n+1+2\beta(n+\alpha)](1-s)}{2n\beta(1-\alpha)(n+\delta+2)}\right]^{n+1} \quad (4.2)$$

Proof. It is sufficient to show that

$$\left|\frac{zf''(z)}{f'(z)} + 2\right| \leq 1-\delta ,$$

$$\left|\frac{zf''(z) + 2f'(z)}{f'(z)}\right| = \left|\frac{2Z^{-2} + \sum_{n=1}^{\infty} n(n-1)a_n Z^{n-1} - 2Z^{-2} + 2\sum_{n=1}^{\infty} n a_n Z^{n-1}}{-Z^{-2} + \sum_{n=1}^{\infty} n a_n z^{n-1}}\right|$$

$$= \left|\frac{\sum_{n=1}^{\infty} n(n+1)a_n z^{n+1}}{-1+\sum_{n=1}^{\infty} n a_n z^{n+1}}\right| \leq \frac{\sum_{n=1}^{\infty} n(n+1)a_n |z|^{n+1}}{1-\sum_{n=1}^{\infty} n a_n |z|^{n+1}}.$$

The last expression must bounded by $1-\delta$.Therefore,

$$\sum_{n=1}^{\infty} n(n+1) a_n |z|^{n+1} \leq 1-\delta - (1-\delta)\sum_{n=1}^{\infty} n a_n |z|^{n+1}$$

$$\sum_{n=1}^{\infty} n(n+\delta+2) a_n |z|^{n+1} \leq 1-\delta$$

Then by theorem(2.1), we have

$$\sum_{n=1}^{\infty} \frac{[n+1+2\beta(n+\alpha)]D_n(\lambda)}{2\beta(1-\alpha)} a_n < 1 .$$

Hence

$$|z|^{n+1} \leq \frac{[n+1+2\beta(n+\alpha)]D_n(\lambda)(1-\delta)}{2\beta(1-\alpha)n(n+\delta+2)},$$

Also,

$$|z| \leq \left[\frac{[n+1+2\beta(n+\alpha)]D_n(\lambda)(1-s)}{2\beta(1-\alpha)n(n+\delta+2)}\right]^{n+1}$$

Therefore, we get the result.

5. Convex Linear Combination

In the following theorem, the class $S(\lambda,\alpha,\beta)$ is closed under convex linear combination.

Theorem (5.1) .The class S (λ ,α ,β) is closed under convex linear combination.

Proof: We want to show that function k(z) = (1-μ)f_1(z) + μ f_2(z), 0 ≤μ≤ 1is in the class S(λ,α,β) wheref_1(z) , f_2(z)∈S(λ,α,β) and

$$f_1(z) = z^{-1}+ \sum_{n=1}^{\infty} a_n z^n, f_2(z) = z^{-1}+ \sum_{n=1}^{\infty} b_n z^n, \quad (5.1)$$

where $a_n, b_n \geq 0$.

By theorem (2.1) we have

$$\sum_{n=1}^{\infty}[n+1+2\,\beta\,(n+\alpha)]\,D_n(\lambda)a_n \leq 2\,\beta\,(1\text{-}\alpha)$$

and

$$\sum_{n=1}^{\infty}[n+1+2\,\beta\,(n+\alpha)]D_n(\lambda)\,b_n \leq 2\,\beta\,(1\text{-}\alpha).$$

Therefore,

k(z) = (1-μ)f_1(z) + μ f_2(z)

$= (1-\mu)(z^{-1}+\sum_{n=1}^{\infty} a_n z^n)+ \mu(z^{-1}+\sum_{n=1}^{\infty} b_n z^n)$

$= z^{-1}+\sum_{n=1}^{\infty}[(1-\mu)a_n + \mu b_n]\, z^n$. Further

$$\sum_{n=1}^{\infty}[(n+1)+2\,\beta\,(n+\alpha)]\,[(1-\mu)a_n + \mu b_n]D_n(\lambda)$$

$=\sum_{n=1}^{\infty}[n+1+2\,\beta\,(n+\alpha)]\,(1-\mu)a_n + \sum_{n=1}^{\infty}[n+1+2\,\beta(n+\alpha)\mu bn$

≤2(1- μ) β (1-α) + 2μ β(1- α) = 2β(1-α),

Therefore the result is follows that k(z) is in the class S(λ,α,β)

6. Arithmetic Mean

In the following, we shall prove that class S(λ,α,β)is closed under arithmetic mean.

Theorem (6.1). Let f_1(z) ,f_2(z) ….. f_l(z) defined by f_i(z) = $z^{-1}+ \sum_{n=1}^{\infty} a_{n,i} z^n$, ($a_{n,i} \geq 0$, i = 1, 2…… l)be in the class S(λ,α,β). Then the Arithmetic mean of f_i(z) , (i = 1 ….. l) defined by

$$h\,(z) = \frac{1}{l}\sum_{i=1}^{l} f_i\,(z) \quad (6.1)$$

is also in the class S(λ,α,β).

Proof

By hypothesis we get

$h\,(z) = \frac{1}{l}\sum_{i=1}^{l}(z^{-1}+\sum_{n=1}^{\infty} a_{n,i} z^n) = z^{-1} + \sum_{n=1}^{\infty}\left(\frac{1}{l}\sum_{i=1}^{l} a_{n,i}\right) z^n$

since f_i (z)∈S(λ,α,β) for every i = 1,2,…. l.So by using theorem (2.1), we get

that $\sum_{n=1}^{\infty}[n+1+2\,\beta\,(n+\alpha)]\,D_n(\lambda)\left(\frac{1}{l}\sum_{i=1}^{l} a_{n,i}\right)$

$= \frac{1}{l}\sum_{i=1}^{l}(\sum_{n=1}^{\infty}[n+1+2\,\beta\,(n+\alpha)]D_n(\lambda)a_{n,i})$

$\leq \frac{1}{l}\sum_{i=1}^{l} 2\,\beta(1-\alpha)$= 2β(1-α)

7. Hadamard Product

In the following theorem, we obtain the convolution result for function belongs to the class S(λ,α,β).

Theorem (7.1). Let the functions f , g of the form (1.1) be in the class S(λ,α,β). Then ($f * g$) ∈ S(λ,α,β) .Then ($f * g$) ∈ S(λ,α,l) ,where

$$l \geq \frac{2\beta^2\,(1-\alpha)(n+1)}{[n+1+2\beta(n+\alpha)]^2 D_n(\lambda) - 4\beta^2\,(1-\alpha)(n+\alpha)} \quad (7.1)$$

Proof

Let f , g ∈S(λ,α,β) and so

$$\sum_{n=1}^{\infty}\frac{[n+1+2\beta(n+\alpha)]D_n(\lambda)}{2\beta(1-\alpha)}a_n \leq 1$$

and

$$\sum_{n=1}^{\infty}\frac{[n+1+2\beta(n+\alpha)]D_n(\lambda)}{2\beta(1-\alpha)}b_n \leq 1 \quad (7.2)$$

We have to find the smallest number l such that

$$\sum_{n=1}^{\infty}\frac{[n+1+2l(n+\alpha)]D_n(\lambda)}{2l(1-\alpha)}a_n b_n \leq 1 \quad (7.3)$$

By the Cauchy-Schwartz inequality, we get

$$\sum_{n=1}^{\infty}\frac{[n+1+2\beta(n+\alpha)]D_n(\lambda)}{2\beta(1-\alpha)}\sqrt{a_n b_n} \leq 1 \quad (7.4)$$

It is sufficient to show that

$$\frac{[n+1+2l(n+\alpha)]D_n(\lambda)}{2l(1-\alpha)}a_n b_n \leq \frac{[n+1+2\beta(n+\alpha)]D_n(\lambda)}{2\beta(1-\alpha)}\sqrt{a_n b_n} \quad (7.5)$$

That is,

$$\sqrt{a_n b_n} \leq \frac{l\,[n+1+2\beta(n+\alpha)]}{\beta[n+1+2l(n+\alpha)]}$$

But from (7.4)

$$\sqrt{a_n b_n} \leq \frac{2\beta\,(1-\alpha)}{[n+1+2\beta(n+\alpha)]D_n(\lambda)}$$

Thus it is enough to show that

$$\frac{2\beta\,(1-\alpha)}{[n+1+2\beta(n+\alpha)]D_n(\lambda)} \leq \frac{l\,[n+1+2\beta(n+\alpha)]}{\beta[n+1+2l(n+\alpha)]}$$

which simplifies to

$$l \geq \frac{2\beta^2\,(1-\alpha)(n+1)}{[n+1+2\beta(n+\alpha)]^2 D_n(\lambda) - 4\beta^2\,(1-\alpha)(n+\alpha)}$$

which proves theorem(7.1)

Theorem (7.2). Let the functions f , g of the form (1.1) be in the class S(λ,α,β).Then the function $h(z) = z^{-1} + \sum_{n=1}^{\infty}(a_n^2 + b_n^2)\,z^n$ is in the class S(λ,α,γ) , where

$$\gamma = \frac{4\beta^2(n+1)(1-\alpha)^2}{(1-\alpha)[n+1+2\beta\,(n+\alpha)]^2 D_n(\lambda) - 8\beta^2(n+\alpha)(1-\alpha)^2} \quad (7.6)$$

Proof. Since f , g ∈S(λ,α,β) , therefore , by theorem (2.1) yields

$$\sum_{n=1}^{\infty}\left[\frac{[n+1+2\beta(n+\alpha)]D_n(\lambda)}{2\beta(1-\alpha)}\right]^2 a_n^2 \leq 1 \quad (7.7)$$

and

$$\sum_{n=1}^{\infty}\left[\frac{[n+1+2\beta(n+\alpha)]D_n(\lambda)}{2\beta(1-\alpha)}\right]^2 b_n^2 \leq 1 \quad (7.8)$$

We obtain from the last two inequalities

$$\sum_{n=1}^{\infty}\frac{1}{2}\left[\frac{[n+1+2\beta(n+\alpha)]D_n(\lambda)}{2\beta(1-\alpha)}\right]^2 (a_n^2+b_n^2) \leq 1$$

But $h(z)$ ∈S(λ,α,γ) if and only if

$$\sum_{n=1}^{\infty} \frac{[n+1+2\gamma\,(n+\alpha)]D_n(\lambda)}{2\gamma\,(1-\alpha)}(a_n^2+b_n^2) \leq 1 \qquad (7.9)$$

Therefore, the inequality (7.9) satisfied if

$$\frac{[n+1+2\gamma\,(n+\alpha)]D_n(\lambda)}{2\gamma\,(1-\alpha)} \leq \frac{1}{2}\left[\frac{[n+1+2\beta(n+\alpha)]D_n(\lambda)}{2\beta\,(1-\alpha)}\right]^2$$

which is imply

$$\gamma \geq \frac{4\beta^2(n+1)(1-\alpha)^2}{(1-\alpha)[n+1+2\beta\,(n+\alpha)]^2D_n(\lambda)-8\beta^2(n+\alpha)(1-\alpha)^2}$$

which is complete the proof.

8. Closure Theorem

We shall prove the following closure theorems for the class S (λ,α,β).

Theorem (8.1). Let f_j∈S(λ,α,β) ,j = 1 , 2 … s , then

$$g(z) = \sum_{j=1}^{s} c_j\, f_j(z) \in S(\lambda,\alpha,\beta), \qquad (8.1)$$

for $f_j(z) = z^{-1} + \sum_{n=1}^{\infty} a_{n,j}\, z^n$, where $\sum_{j=1}^{s} c_j = 1$.

Proof. Suppose that $g(z) = \sum_{j=1}^{s} c_j\, f_j(z)$. Since $f_j(z) = z^{-1} + \sum_{n=1}^{\infty} a_{n,j}\, z^n$,

Therefore, $g(z) = z^{-1} + \sum_{n=1}^{\infty} e_n z^n$, where $e_n = \sum_{j=1}^{s} c_j\, a_{n,j}$.

Thus

$g(z) \in S(\lambda,\alpha,\beta)$ if $\sum_{n=1}^{\infty} \frac{[n+1+2\beta(n+\alpha)]D_n(\lambda)}{2\beta(1-\alpha)} e_n \leq 1$. That is, if

$$\sum_{n=1}^{\infty}\sum_{j=1}^{s} \frac{[n+1+2\beta(n+\alpha)]D_n(\lambda)}{2\beta(1-\alpha)} c_j\, a_{n,j} = \sum_{j=1}^{s} c_j \sum_{n=1}^{\infty} \frac{[n+1+2\beta(n+\alpha)]D_n(\lambda)}{2\beta(1-\alpha)} a_{n,j}$$

$\leq \sum_{j=1}^{\infty} c_j = 1$. Therefore the proof is completed.

9. Neighborhoods

Now, we define (n,γ)-neighborhood of a function f∈ S by

$$N_{n,\gamma} = \{g \in S : g(z) = z^{-1} + \sum_{n=1}^{\infty} b_n\, z^n \text{ and } \sum_{n=1}^{\infty} n|a_n - b_n| \leq \gamma, 0 \leq \gamma < 1\} \qquad (9.1)$$

For the identity function $e(z)=z$, we have

$N_{n,\gamma}(e) = \{g \in S : g(z) = z^{-1} + \sum_{n=1}^{\infty} b_n\, z^n$
and $\sum_{n=1}^{\infty} n\,|b_n| \leq \gamma\,\}$

Definition (9.1). A function f∈ S issaid to be in the class $S^{\mu}(\lambda,\alpha,\beta)$ if there exists a function $g \in S(\lambda,\alpha,\beta)$ such that

$$\left|\frac{f(z)}{g(z)} - 1\right| < 1-\mu\,, (z \in U\,, 0 \leq \mu < 1)$$

Theorem (9.2) If g ∈S(λ,α,β) and

$$\mu = 1 - \frac{\gamma[1+\beta(1+\alpha)](\lambda+1)(\lambda+2)}{[1+\beta(1+\alpha)](\lambda+1)(\lambda+2)-2\beta(1-\alpha)} \qquad (9.2)$$

Then $N_{n,\gamma}(g) \subset S^{\mu}(\lambda,\alpha,\beta)$.

Proof. Let f∈ $N_{n,\gamma}(g)$,we want to find from (9.1) that

$$\sum_{n=1}^{\infty} n\,|a_n - b_n| \leq \gamma$$

Which readily implies the following coefficient in equality

$$\sum_{n=1}^{\infty} |a_n - b_n| \leq \gamma\ ,\ \ (n \in N\,)$$

Since g ∈S(λ,α,β) , hence we have from Theorem(2.1)

$$\sum_{n=1}^{\infty} b_n \leq \frac{\beta\,(1-\alpha)}{[1+\beta(1+\alpha)](\lambda+1)(\lambda+2)}$$

So that

$$\left|\frac{f(z)}{g(z)} - 1\right| = \left|\frac{f(z)-g(z)}{g(z)}\right| = \left|\frac{\sum_{n=1}^{\infty}|a_n-b_n|\, z^{n+1}}{1+\sum_{n=1}^{\infty} b_n\, z^{n+1}}\right|$$

$$\leq \frac{\sum_{n=1}^{\infty}|a_n-b_n|}{1-\sum_{n=1}^{\infty} bn} \leq \frac{\gamma}{1-\frac{\beta(1-\alpha)}{[1+\beta(1+\alpha)](\lambda+1)(\lambda+2)}}$$

$$= \frac{\gamma[1+\beta(1+\alpha)](\lambda+1)(\lambda+2)}{[1+\beta(1+\alpha)](\lambda+1)(\lambda+2)-\beta(1-\alpha)}$$

Then we get

$$\left|\frac{f(z)}{g(z)} - 1\right| \leq \frac{\gamma[1+\beta(1+\alpha)](\lambda+1)(\lambda+2)}{[1+\beta(1+\alpha)](\lambda+1)(\lambda+2)-\beta(1-\alpha)} = 1-\mu$$

Therefore, by definition(9.1), $f \in S^{\mu}(\lambda,\alpha,\beta)$

Thus by definition , $f \in S^{\mu}(\lambda,\alpha,\beta)$ for μ given by (9.2)

10. Partial Sums

Theorem(10.1). Let $f(z)$∈S(λ,α,β) is given by (1.1) .Also $f_1(z)$ and $f_k(z)$ defined by

$$f_1(z) = z^{-1} \quad \text{and}$$

$$f_k(z) = z^{-1} + \sum_{n=1}^{k} a_n z^n\ , (n \in N/\{1\}). \qquad (10.1)$$

Suppose that

$$\sum_{n=1}^{\infty} d_n |a_n| \leq 1, \qquad (10.2)$$

where $\left(d_n := \frac{[n+1+2\beta(n+\alpha)]D_n(\lambda)}{2\beta(1-\alpha)}\right)$.

Furthermore,

$$\operatorname{Re}\left\{\frac{f(z)}{f_k(z)}\right\} > 1 - \frac{1}{d_{k+1}}\ , z \in U, n \in N \qquad (10.3)$$

And

$$\operatorname{Re}\left\{\frac{f_k(z)}{f(z)}\right\} > \frac{d_{k+1}}{1+d_{k+1}}. \qquad (10.4)$$

Proof. For the coefficients d_n given by (10.2)

$$d_{n+1} > d_n > 1\,. \qquad (10.5)$$

Therefore,

$$\sum_{n=1}^{k} a_n + d_{k+1}\sum_{n=k+1}^{\infty} a_n \leq \sum_{n=1}^{\infty} d_n a_n \leq 1 \qquad (10.6)$$

By setting

$$g_1(z) = d_{k+1}\left\{\frac{f(z)}{f_k(z)} - \left(1 - \frac{1}{d_{k+1}}\right)\right\}$$

$$= d_{k+1}\left\{\frac{f(z)}{f_k(z)} - 1 + \frac{1}{d_{k+1}}\right\}$$

$$= 1 + \frac{d_{k+1}\sum_{n=k+1}^{\infty} a_n z^{n+1}}{1+\sum_{n=1}^{k} a_n z^{n+1}}$$

Applying (10.6), we get that

$$\left|\frac{g_1(z)-1}{g_1(z)+1}\right| \leq \frac{d_{k+1}\sum_{n=k+1}^{\infty} a_n}{2-2\sum_{n=1}^{k} a_n - d_{k+1}\sum_{n=k+1}^{\infty} a_n} \leq 1 \qquad (10.7)$$

Which readily yields the assertion (10.3) of Theorem(10.1). If the function

$$f(z) = z^{-1} + \frac{z^{k+1}}{d_{k+1}} \qquad (10.8)$$

$\frac{f(z)}{f_k(z)} = 1 - \frac{z^{k+1}}{d_{k+1}} \to 1 - \frac{1}{d_{k+1}}$ as $z \to 1^-$.This show the bound in (10.3).

Similarly, if we take

$$g_2(z) = (1 + d_{k+1})\left\{\frac{f_k(z)}{f(z)} - \frac{d_{k+1}}{1+d_{k+1}}\right\}$$

$$= 1 - \frac{(1+d_{k+1})\sum_{n=k+1}^{\infty} a_n z^{n+1}}{1+\sum_{n=1}^{k} a_n z^{n+1}}$$

and making use of (10.6), we can deduce that

$$\left|\frac{g_2(z)-1}{g_2(z)+1}\right| \le \frac{(1+d_{k+1})\sum_{n=k+1}^{\infty} a_n}{2-2\sum_{n=1}^{k} a_n - (1+d_{k+1})\sum_{n=k+1}^{\infty} a_n} \le 1, \ z \in U. \qquad (10.9)$$

Which leads to assertion (10.4) of Theorem (10.1).Therefore, the proof of Theorem(4.1) is complete.

Theorem(10.2). If $f(z)$ of the form (1.1) satisfies the Theorem (2.1). Then

$$\text{Re}\left\{\frac{f'(z)}{f_k'(z)}\right\} \ge 1 - \frac{k+1}{d_{k+1}} \qquad (10.10)$$

Proof. By setting

$$g(z) = \frac{d_{k+1}}{k+1}\left\{\frac{f'(z)}{f_k'(z)} - \left(1 - \frac{k+1}{d_{k+1}}\right)\right\}$$

$$= \frac{1 + \frac{d_{k+1}}{k+1}\sum_{n=k+1}^{\infty} n a_n z^{n+1} + \sum_{n=1}^{k} n a_n z^{n+1}}{1+\sum_{n=1}^{k} n a_n z^{n+1}}$$

$$= 1 + \frac{\frac{d_{k+1}}{k+1}\sum_{n=k+1}^{\infty} n a_n z^{n+1}}{1+\sum_{n=1}^{k} n a_n z^{n+1}}, \text{ Therefore,}$$

$$\left|\frac{g(z)-1}{g(z)+1}\right| \le \frac{\frac{d_{k+1}}{k+1}\sum_{n=k+1}^{\infty} n a_n}{2-2\sum_{n=1}^{k} n\, a_n - \frac{d_{k+1}}{k+1}\sum_{n=k+1}^{\infty} n a_n} \le 1.$$

Now $\left|\frac{g(z)-1}{g(z)+1}\right| \le 1$ if

$$\sum_{n=1}^{k} n\,|a_n| + \frac{d_{k+1}}{k+1}\sum_{n=k+1}^{\infty} n|a_n| \le 1 \qquad (10.11)$$

The result is sharp for the function

$$f(z) = z + \frac{z^{k+1}}{d_{k+1}}.$$

Theorem (10.3). If $f(z)$ of the form (1.1) satisfies the Theorem (2.1) then

$$\text{Re}\left\{\frac{f_k'(z)}{f'(z)}\right\} \ge \frac{d_{k+1}}{k+1+d_{k+1}}. \qquad (10.12)$$

Proof: By setting

$$g(z) = \frac{[(k+1) + d_{k+1}]}{k+1}\left\{\frac{f_k'(z)}{f'(z)} - \frac{d_{k+1}}{[(k+1) + d_{k+1}]}\right\}$$

$$= 1 - \frac{\left(1+\frac{d_{k+1}}{k+1}\right)\sum_{n=k+1}^{\infty} n a_n z^{n+1}}{1+\sum_{n=1}^{\infty} n a_n z^{n+1}},$$

and making use

$$\sum_{n=1}^{k} n a_n + \left(1 + \frac{d_{k+1}}{k+1}\right)\sum_{n=k+1}^{\infty} n a_n \le 1.$$

We can deduce that

$$\left|\frac{g(z)-1}{g(z)+1}\right| \le \frac{\left(1+\frac{d_{k+1}}{k+1}\right)\sum_{n=k+1}^{\infty} n a_n}{2-2\sum_{n=1}^{k} n a_n - \left(1+\frac{d_{k+1}}{k+1}\right)\sum_{n=k+1}^{\infty} n a_n} \le 1.$$

Therefore, the result of Theorem (10.3) holds.

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