International Journal of Science and Research (IJSR)
ISSN (Online): 2319-7064
Index Copernicus Value (2015): 78.96 | Impact Factor (2015): 6.391

J- Semiprime Submodules

Nuhad Salim Al-Mothafar[1], Iman A. Athab[2]

Department of Mathematics, College of Science, Baghdad University, Baghdad, Iraq

Abstract: ***Let R be a commutative ring with identity and Nbe a proper submoduleN is called semiprime if whenever $r^n x \in N$; $r \in R, x \in M$, $n \in Z^+$, implies that $rx \in N$. In this paper we say that N is J-semiprime if whenever$r^n x \in N + J(M)$; $r \in R, x \in M, k \in Z^+$, implies that $rx \in N$. We prove some result of this type of submodules.***

Keywords: prime submodule, semiprimesubmodule, nearly semi prime submodule, weekly semi prime submodule

1. Introduction

Let R be a commutative ring with identity and M is an R-module, Danus [2] was named semiprime submodules that they are generalized of semiprime ideals, which get a big interest at last years, many students search are published about semiprime submodules by many people who care with the subject of commutative algebra and some of them are J. Dauns, R. L. Mcsland, C. p. Lu, P. F. Smith and M. E. Moore. The definition comes in [2] as follows: we say that a proper submodule N of an R-module M is semiprime submodule if whenever $r^n x \in N$; $r \in R, x \in M$ and $n \in Z^+$ implies that $rx \in N$. In [1], the most important result was proved which says that a proper submodule N of an R-module M is semi prime if whenever $r^2 x \in N; r \in R, x \in M$ implies that $rx \in N$. If N is a proper submodule of an R-module M, then the following statements are equivalent. $N$is semiprime submodule of M, then $[N:K]$ is semiprime ideal of R, for where $[N:K] = \{r \in R: rk \subseteq N\}$, [1], then $[N:(x)]$ is semiprime ideal of R, for all$x \notin N$. In this paper we introduce a new class of submodules which is called $J(M) = \cap \{L: L$ is a maximal submodule of $M\}$-semiprime submodule, for short we use J-semiprime submodule, a proper submodule N of an R-module M is said to be J-semiprime submodule of M if whenever $r^k m \in N + J(M)$; $r \in R, m \in M$ and $k \in Z^+$ implies that $rm \in N$. We prove important result that N is called J-semiprime if whenever $r^2 m \in N + J(M)$; $r \in R, m \in M$ implies that $rm \in N$, and we prove many new result. In this paper we give the following characterization, if$N$is a submodule of an R-mdoule M such that $J(M) = 0$, then the following statement are equivalent:

1) $N$is$J$-semiprimesubmodule of M.
2) $N$is semiprime submodule of M.
3) $[N:K]$is semiprime ideal of R for each submodule$K$ of M containg N properly.
4) $[N:(x)]$is semiprime ideal of R, for all $x \in M$ and $x \notin N$.

Also we introduce another characterization for J-semiprime submoduke of an R-module M and some properties of this class of submodules.

2. J-semiprime submodules

Recall that a proper submodule N of an R-module M is called semiprime submodule if $N \neq M$ and whenever $r^k m \in N$; $r \in R, m \in M$ and $k \in Z^+$ such that $r^k m \in N$ implies that $rm \in N$, [2].

We introduce the following definition:

Definition (2.1)
A proper submodule N of an R-module M is is said to be J-semiprime submodule of M if whenever $r^k m \in N + J(M)$; $r \in R, m \in M$ and $k \in Z^+$ implies that $rm \in N$

Remarks and examples (2.2)

1) Every J-semiprime submodule is semiprime. But the converse is not truein general for example:
   Let $M = Z_2 \oplus Z_4$ be a module over Z and N={(0,0), (1,0)} is a submodule of M. Now $N + J(M) =$ { (0,0),(1,0),(0,2),(1,2)}, then 9(0,2) $\in N + J(M)$ but 3(0,2) $\notin N$ which means that N is not J-semiprime while N is semiprime submodule.
2) Let P be a prime number the module Z_{P^∞} over Z has no J-semiprime submodule since Z_{P^∞} as Z-module has no semiprimesubmodule,[2]. Therefore by the previous remark Z_{P^∞} over Z has no J-semiprime submodule
3) Every maximal submodule of an R-module M is an J-semiprime submodule.

Proof:
Let N be a maximal submodule of an R-module M, then $[N:M]$ is a prime ideal of R,[5]. Now, for $r \in R$, $m \in M$ and $k \in Z^+$, suppose that $r^k m \in N + J(M)$, hence $r^k m \in N$. Since N is maximal submodule, then is prime submodule,.[5]. Therefore either $m \in N$ or$r^k \in [N:M]$, in each case we have $rm \in N$. This implies that N is J-semiprime submodule of M.

In particular the module Q over Z, the only$J$-semiprime submodule is 0.

4. Let M be an R-module. $Rad(M)$ is a submodule of Mdefine as follows:
$Rad(M) = \cap \{P: P$ is a prime submodule of $M\}$,[3].
If $J(M) = Rad(M)$, then every submodule of M is$J$-semiprime.

Proof:
Let N be a prime submodule of an R-module M, then $[N:M]$ is a prime ideal of R,.[5]. For $r \in R$, $m \in M$ and $k \in Z^+$, suppose that $r^k m \in N + J(M)$, hence $r^k m \in N$, but N is a prime submodule of M, therefore either $m \in N$ or $r^k \in [N:M]$, this implies that $rm \in N$.

The following proposition gives an important characterization for J-semiprime submodule.

Proposition (2.3):
A proper submodule N of an R-modue M is J-semiprime if and only if whenever $r^2m \in N + J(M)$,for $r \in R$, $m \in M$, then $rm \in N$.

Proof:
Since N is J-semiprimesubmodule, then by definition (2.1), we prove this direction. For the converse suppose that for $r \in R$, $m \in M$ and $k \in Z^+$, suppose that $r^k m \in N + J(M)$. Now $r^2(r^{k-2}m) \in N + J(M)$, by assumption this implies that $r^{k-1}m \in N$. After a finite number of steps, we get that $rm \in N$, this implies that N is J-seniprime submodule of M.
From the definition (2.1) we can define J-semiprime ideal I of a ring Ras a proper ideal satisfy that whenever $r^k \in I + J(R)$ for $r \in R$ and $k \in Z^+$, then $r \in I$.

Proposition (2.4):
A proper ideal I of a ring R is J-semiprime if and only if $\sqrt{I + J(R)} = I$.

Proof:
I is$J$-semiprime ideal of R, let $r \in \sqrt{I + J(R)}$; $r \in I, \exists k \in Z^+$ such that $r^k \in I + J(R)$. Since thus $r \in I$ and therefore $\sqrt{I + J(R)} = I$.For the converse, it is clear that from the properties of the radical of an ideal.
Consider the following proposition.

Proposition (2.5):
If N is J-semiprime submodule of an R-module M, then $[N{:}M]$ is J-semiprimeideal of R.

Proof:
It enough to show that $\sqrt{[N{:}M] + J(R)} \subseteq [N{:}M]$, let $r \in \sqrt{[N{:}M] + J(R)}$, then $\exists\, k \in Z^+$ such that $r^k \in \sqrt{[N{:}M] + J(R)}$, therefore $r^k = s + j$, for some $s \in [N{:}M]$ and $j \in J(R)$. Now for all $m \in M$,$r^k m = sm + jm \in N + J(M)$, but N is is J-semiprime submodule, then $rm \in N$ so that $r \in [N{:}M]$, this complete the proof.
The converse of the previous proposition not true in general as the following example show:
$M = Z \oplus Z$as Z-module, if $N =< (4,0) >$, then $[N{:}M] = 0$ is J-semiprimeideal, but N not$J$-semiprimesubmodule of M.

Proposition (2.6):
Let N be a submodule of an R-module M, then the following statement are equivalent:
1. $[N + J(M){:}K]$is$J$-semiprimeideal of R for all submodule K of Mcontaining$N + J(M)$ properly.
2.$[N + J(M){:}(x)]$is$J$-semiprimeideal of R for all $x \in M$ and $x \notin N + J(M)$.

Proof
1) $\Longrightarrow$ 2) We have to prove that $\sqrt{[N + J(M){:}(x)] + J(R)} \subseteq [N + J(M){:}(x)]$ for $x \in M$ and $x \notin N + J(M)$. Let $r \in \sqrt{[N + J(M){:}(x)] + J(R)}$, thus $\exists\, k \in Z^+$ such that $r^k \in [N + J(M){:}(x)] + J(R)$, $N + J(M) \subsetneq (N + J(M) + (x)) = K$, by (1), we get that $\sqrt{[N + J(M){:}K] + J(R)} = [N + J(M){:}K]$. Therefore $r \in [N + J(M){:}K]$ and hence $rx \in N + J(M)$, this implies that $r \in [N + J(M){:}(x)]$, then $\sqrt{[N + J(M){:}(x)] + J(R)} = [N + J(M){:}(x)]$ for $x \in M$ and $x \notin N + J(M)$
2) $\Longrightarrow$ 1)For a submodule K of M containning $N + J(M)$ properly, we want to show that $\sqrt{[N + J(M){:}K] + J(R)} = [N + J(M){:}K]$, let $r \in \sqrt{[N + J(M){:}K] + J(R)}$, then $\exists\, k \in Z^+$ such that $r^k \in [N + J(M){:}K] + J(R)$ thus $r^k = s + j$ for some $s \in [N + J(M){:}K]$ and $j \in J(R)$. Now for $x \in K$ and $x \notin N + J(M)$, from (2) $\sqrt{[N + J(M){:}(x)] + J(R)} = [N + J(M){:}(x)]$. It is clear that $r \in [N + J(M){:}(x)]$, hence $r \in [N + J(M){:}K]$ and therefore this complete the proof.
We can prove the following proposition:

Proposition (2.7):
If N is an J-semiprime submodule of an R-module M, then the ideal $[N + J(M){:}K]$ is J-semiprime in R for all submodules K of M containg $N + J(M)$ properly.

Proof:
let $r \in \sqrt{[N + J(M){:}K] + J(R)}$, then $\exists\, k \in Z^+$ such that $r^k \in [N + J(M){:}K] + J(R)$, thus $r^k = s + j$ for some $s \in [N{:}J(M){:}K]$ and $j \in J(R)$, then for all $x \in K$, $r^k x = sx + jx \in N + J(M)$, but N is is J-semiprime submodule, so that $rx \in N$ and hence $r \in [N{:}K] \subseteq [N + J(M){:}K]$, therefore $\sqrt{[N + J(M){:}K] + J(R)} = [N + J(M){:}K]$ this means that $[N + J(M) + K]$ is J-semiprime ideal of R.

From proposition (2.6) and proposition (2.7) we can prove the following corollary.

Corollary (2.8):
If a proper submodule N of an R-module M is J-semiprime, then the ideal $[N + J(M){:}(x)]$ is J-semiprime for all $x \in M$ and $x \notin N + J(M)$.
Now we are ready to prove the following characterization.

Proposition (2.9):
Let M be an R-module such that $J(M) = 0$ and Nbe a proper submodule of M, then the following statement equivalent:
1.) $N$$J$-semiprimesubmule of M.
2.) $N$is semiprime submodule of M.
3.) $[N{:}K]$is semiprime ideal of R, for each submodule K of M containg N properly.
4.) $[N{:}(x)]$is semiprime ideal of R, for all $x \in M$ and $x \notin N$

Proof:
(2)$\Leftrightarrow$ (3) $\Leftrightarrow$ (4)by (proposition (1.4),[1])
(2) $\Rightarrow$ (1) Suppose that $r^2m \in N + J(M)$ for $r \in R$ and $m \in M$. Thus $r^2m \in N$, but N is semiprime submodule of M, therefore either $m \in N$ or $r \in [N{:}M]$ in each case we conclude that $rm \in N$ and hence $N$is J-semiprimesubmule of M.

Recall that a submodule N of an R-module M is called injective envelope of Nin M, denoted by $E_m(N)$ and define as follows:
$E_m(N) = \{x = rm{:}r \in R, m \in M$ such that $r^k m \in N, k \in Z^+\}$. It is clear that $N \subseteq E_m(N)$, [4].

Proposition (2.10):
Let N be a proper submodule of an R-module M, then NJ-semiprime if and only if $N = E_m(N + J(M))$.

Proof:
N is J-semiprimesubmodule of M, let $x \in M$; $x \in E_m(N + J(M))$, thus $x = sm$; $s \in R, m \in M$ and there exist a positive integer k such that $s^k m \in N$, hence $s^k m \in N + J(M)$, therefore by assumption $sm \in N$, this implies that $N = E_m(N + J(M))$. For the converse, suppose that $s^k m \in N + J(M)$; $s \in R$, $m \in M$, then $sm \in N$. This implies that $N = E_m(N + J(M))$.
Compare the following proposition with proposition (2.1) in [1].

Proposition (2.11):
Let M and M' be R-modules and $\emptyset: M \to M'$ be an epimorphisim with $Ker\emptyset \ll M$.
1) If N is J-semiprimesubmodule of M with $Ker\emptyset \subseteq N$, then $\emptyset(N)$ is J-semiprimesubmodule of M'.
2) If N' is J-semiprimesubmodule of M', then $\emptyset^{-1}(N')$ is J-semiprimesubmodule of M.

Proof:
(1) $\emptyset(N)$ is a proper submodule of M', since if $\emptyset(N) = M'$, then $\emptyset(m) \in \emptyset(N)$ for all $m \in M$, therefore $\exists\, n \in N$ such that $\emptyset(m) = \emptyset(n)$, this implies that $m - n \in Ker\emptyset \subseteq N$, so that $M = N$ which is a contradiction and hence $\emptyset(N) \neq M'$. Now, let $r \in R$ and $m \in M'$ with $r^2 m \in \emptyset(N) + J(M')$, since $\emptyset$ is an epimorphisim therefore $\exists\, x \in M$ such that $\emptyset(x) = m$, then $\emptyset(r^2 x) = r^2\emptyset(x) = r^2 m \in \emptyset(N) + J(M')$,so that $\emptyset(r^2 x) \in \emptyset(N) + \emptyset(J(M))$, hence $\exists\, y \in N$ and $l \in J(M)$ such that $\emptyset(r^2 x) = \emptyset(y) + \emptyset(l)$, then $r^2 x - (y + l) \in Ker\emptyset \subseteq N$ therefore $r^2 x \in N + J(M)$, but N is J-semiprimesubmodule of M, this implies that $rx \in N$ and hence $rm = \emptyset(rx) \in \emptyset(N)$, this means that $\emptyset(N)$ is J-semiprimesubmodule ofM'.
(2) $\emptyset^{-1}(N')$ is a proper submodule of M', since if $\emptyset^{-1}(N') = M$, then $N' = \emptyset(M) = M'$, which is a contradiction. Now let $r \in R$ and $m \in M$ such that $r^2 m \in \emptyset^{-1}(N') + J(M)$, therefore $r^2 m \in \emptyset^{-1}(N') + \emptyset^{-1}(J(M'))$ thus $r^2\emptyset(m) \in N' + J(M')$ but N is J-semiprimesubmodule of M', then $r\emptyset(m) \in N$and hence $rm \in \emptyset^{-1}(N')$, this implies that $\emptyset^{-1}(N')$ is J-semiprimesubmodule of M.

References

[1] Athab, E.A.***1996.*** **Prime and semiprimesubmodules,** M.SC. Thesis, Department of Mathematics, College of Science, University of Baghdad, Baghdad, Iraq.
[2] Dauns, J.***1980***. **Prime modules and one sided** in Ring theory and Algebra Proceedings of the Third Oklahoma Conference Dekker, New York, 301-344.
[3] F. Kasch, **Modules and Rings**, Academic Press, London, ***1982***.
[4] D.W.Sharpe and P.Vamous, **Injective modules,** Lecture in pure Math. University of Sheffield, At the univ. Press, ***1972***.
[5] C.P.Lu, 1984. **Prime submodules of modules**, Comment Math. Univ. St. Paul,33,61-69.