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Fixed Point Theorems for Fuzzy Mappings on Closed Subset of Hilbert Spaces for Rational Expression

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Abstract: ***The aim of this paper is to obtain a common fixed point for fuzzy mapping on closed subset of Hilbert spaces for rational expression. The proof rely on the parallelogram law in Hilbert spaces.***

Keywords: fuzzy mapping, fixed point, approximate quantity and Hilbert space

1. Introduction

The concept of fuzzy set was introduce by L.Zadeh [2]in 1965.After that a lot of work has been done regarding fuzzy set and fuzzy mappings. The concept of fuzzy mapping was first introduced by Heilpern [3], he proved fixed point theorem for fuzzy contraction mappings which is a fuzzy analogue of the fixed point theorem for multy valued mapping of Nadler[4]. In 2015 [1] Jagdish C. Chaudhary, Jatin, Patel and Chirag proved some fixed point theorems in fuzzy mappings containing the rational expressions. In this paper we are proving some fixed point for fuzzy mapping on closed subset of Hilbert spaces for rational expression.

2. Preliminaries

In the following discussions mainly follow the definition and notions due to Helipern [3]. Let H be a Hilbert space and F(H) be a collection of all fuzzy sets in H . Let A∈ F(H)and $\alpha \in [0, 1]$ the $\alpha -$ level set of A, denoted by A_α is defined by

$$A_\alpha = \{x : A(x) \geq \alpha\} \text{ if } \alpha \in [0,1]$$

$$A_0 = \overline{\{x : A(x) > \alpha\}}$$

Where $\overline{B}$ denotes the closure of a set B.

Definition 2.1[3]: A fuzzy set A is said to be an approximate quantity if and only if A_α is compact and convex for each $\alpha \in [0,1]$, and $\sup_{x\in X} A(x) = 1$ .When A is an approximate quantity and$A(x_0)=1$ for some$x_0 \in H$, A is identified with an approximate of x_0 .

The collection of all fuzzy sets in H is denoted by F(H) and W(H) is the sub collection of all approximate quantities.

Definition 2.2[3]: Let A, B $\in$ W(H) and $\alpha \in [0,1]$. Then

i. $P_\alpha(A, B) = \inf_{x\in A_\alpha, y\in B_\alpha} \|x - y\|$
ii. $D_\alpha(A, B) = \mathrm{dis}(A_\alpha, B_\alpha)$, where "dis" is the Hausdorff distance
iii. $D(A, B) = \sup_\alpha D_\alpha(A, B)$
iv. $P(A, B) = \sup_\alpha P_\alpha(A, B)$.

It is to be noted that for any 'α', P_α is a non decreasing as well as continuous function.

Definition 2.3[2]. Let A, B $\in$ W(H). An approximate quantity A is said to be more accurate than B (denoted by A⊂ B) if and only if $A(x) \leq B(x), \forall x \in H$.

Definition 2.4[3]: A mapping T from the set H into W(H) is said to be fuzzy mapping.

Definition 2.5[3]: The point $x \in H$ is called fixed point for the fuzzy mapping T if and only if $\{x\} \subset T(x)$. We shall use the following lemmas due to Helipern.

Lemma 2.6[3]: $P_\alpha(x, B) \leq \|x - y\| + P_\alpha(y, B), \forall x, y \in H$.

Lemma 2.7[3]: If $\{x_0\} \subset A$, then $P_\alpha(x_0, B) \leq D_\alpha(A, B), \forall B \in W(H)$.

Lemma 2.8[3]: Let $A \in W(H)$ and $x_0 \in H$, if$\{x_0\} \subset A$ then Pα(x0 , A)=0, for each α∈0, 1.

Lemma 2.9[3]: Let H be a Hilbert space and T fuzzy mapping from H into W(H) and $x_0 \in H$, then there exist $x_1 \in H$ such that$\{x_1\} \subset T(x_0)$.

3. Main Results

The proof rely on the parallelogram law in Hilbert spaces.

Theorem 3.1: Let C be a closed subset of a Hilbert space H . Let T, S be a fuzzy mapping from C into W(C) satisfying :

$$D^2(T(x), S(y)) \leq \alpha P_\alpha^2(x, T(x)) + \beta P_\alpha^2(y, S(y)) + \gamma\|x - y\|^2 + \delta \min\{P_\alpha^2(x, S(y)), P_\alpha^2(y, T(x))\} + m(x,y)$$

For all x, y∈ Cwhere $\gamma, \alpha, \beta, \delta, \sigma, \varepsilon \geq 0$ with

$$\alpha + \beta + \varepsilon + \gamma + \delta + \sigma < 1 \text{ and}$$

$$m(x,y) = \sigma \frac{P_\alpha^2(y, S(y))}{1 + P_\alpha^2(x, S(x))P_\alpha^2(x, T(y))} + \varepsilon \frac{P_\alpha^2(x, S(x))}{1 + P_\alpha^2(y, S(x))P_\alpha^2(x, T(y))}.$$

Then T and S have a fixed point $z \in C$ such that $\{z\} \subset T(z) \cap S(z)$.

Proof: Let $x_0 \in C$, we construct the sequence $\{x_n\} \in C$ as follows $\{x_1\} \subset T(x_0)$, $\{x_2\} \subset S(x_1)$, ………, $\{x_{2n+1}\} \subset T(x_{2n})$, $\{x_{2n+2}\} \subset S(x_{2n+1})$ and $\| x_{2n} - x2n+12 \leq D12T(x2n-1), S(x2n)$.

Now

$\| x_{2n+1} - x_{2n}\|^2 \leq D_1^2(S(x_{2n-1}), T(x_{2n}))$, for all $n \in N$.

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$$\| x_{2n+1} - x_{2n}\|^2 \leq D^2(S(x_{2n-1}),T(x_{2n}))$$
$$\leq \alpha P_\alpha^2(x_{2n-1},S(x_{2n-1})) + \beta P_\alpha^2(x_{2n},T(x_{2n})) + \gamma\|x_{2n}-x_{2n-1}\|^2$$
$$+\delta \min\{P_\alpha^2(x_{2n},S(x_{2n-1})),P_\alpha^2(x_{2n-1},T(x_{2n}))\}$$
$$+\sigma\frac{P_\alpha^2(x_{2n},S(x_{2n-1}))}{1+P_\alpha^2(x_{2n},T(x_{2n}))P_\alpha^2(x_{2n},S(x_{2n-1}))}$$
$$+\varepsilon\frac{P_\alpha^2(x_{2n-1},S(x_{2n-1}))}{1+P_\alpha^2(x_{2n},T(x_{2n}))P_\alpha^2(x_{2n},S(x_{2n-1}))}$$
$$\leq \alpha\|x_{2n}-x_{2n-1}\|^2 + \beta\|x_{2n}-x_{2n+1}\|^2 + \gamma\|x_{2n}-x_{2n-1}\|^2$$
$$+\delta\min\{\|x_{2n}-x_{2n}\|^2,\|x_{2n+1}-x_{2n-1}\|^2\}$$
$$+\sigma\frac{\|x_{2n}-x_{2n}\|^2}{1+\|x_{2n}-x_{2n+1}\|^2\|x_{2n}-x_{2n}\|^2}$$
$$+\varepsilon\frac{\|x_{2n}-x_{2n-1}\|^2}{1+\|x_{2n}-x_{2n+1}\|^2\|x_{2n}-x_{2n}\|^2}$$
$$(1-\beta)\|x_{2n+1}-x_{2n}\|^2 \leq (\alpha+\gamma+\varepsilon)\|x_{2n-1}-x_{2n}\|^2$$
$$\|x_{2n+1}-x_{2n}\|^2 \leq \frac{\alpha+\gamma+\varepsilon}{1-\beta}\|x_{2n-1}-x_{2n}\|^2$$

Putting $q=\frac{\alpha+\gamma+\varepsilon}{1-\beta}<1$

Then, we have
$$\|x_{2n+1}-x_{2n}\|^2 \leq q\|x_{2n-1}-x_{2n}\|^2$$
Now, for any positive integer p
$$\|x_n-x_{n+p}\|^2 \leq q\|x_n-x_{n+1}\|^2 + \|x_{n+1}-x_{n+2}\|^2+.+\|x_{n+p-1}-x_{n+p}\|^2$$
$$\leq (q^n+q^{n+1}+q^{n+2}+\cdots+q^{n+p-1}\|x_1-x_0\|^2$$
$$\leq \frac{q^n}{1-q}\|x_1-x_0\|^2$$
Which implies that $\|x_n-x_{n+p}\|^2 \to 0 \ as \ n\to\infty$

Hence
$\{x_n\}$ *is cauchy sequance in H, but H is a Hilbert space, so* $\{x_n\}$ *is converge to z and since C is closed and* $\{x_n\}$ *is sequence in C*

Then $z\in C$*, such that* $\lim_{n\to\infty} x_n = z$
$$P_\alpha^2(z,T(z)) \leq \|z-x_{2n}\|^2 + P_\alpha^2(x_{2n},T(z))$$
$$\leq \|z-x_{2n}\|^2 + D_\alpha^2(S(x_{2n-1}),T(z))$$
$$\leq \|z-x_{2n}\|^2 + D^2(S(x_{2n-1}),T(z))$$
$$\leq \|z-x_{2n}\|^2 + \alpha P_\alpha^2(x_{2n-1},S(x_{2n-1})) + \beta P_\alpha^2(z,T(z)) +\gamma\|x_{2n-1} - z\|^2 + \delta\min\{P_\alpha^2(x_{2n-1},T(z)),P_\alpha^2(z,S(x_{2n-1}))\}$$
$$+\sigma\frac{P_\alpha^2(z,S(x_{2n-1}))}{1+P_\alpha^2(x_{2n-1},S(x_{2n-1}))P_\alpha^2(x_{2n-1},T(z))}$$
$$+\varepsilon\frac{P_\alpha^2(x_{2n-1},S(x_{2n-1}))}{1+P_\alpha^2(z,S(x_{2n-1}))P_\alpha^2(x_{2n-1},T(z))}$$
$$\leq \|z-x_{2n}\|^2 + \alpha\|x_{2n-1}-x_{2n}\|^2 + \beta P_\alpha^2(z,T(z)) + \gamma\|x_{2n-1}-z\|^2$$
$$+\delta\min\{P_\alpha^2(x_{2n-1},T(z)),\|x_{2n}-z\|^2\}$$
$$+\sigma\frac{\|z-x_{2n}\|^2}{1+\|x_{2n}-x_{2n-1}\|^2P_\alpha^2(x_{2n-1},T(z))}$$
$$+\varepsilon\frac{\|x_{2n}-x_{2n-1}\|^2}{1+\|x_{2n}-z\|^2P_\alpha^2(x_{2n-1},T(z))}$$
$$P_\alpha^2(z,T(z)) \leq \alpha\|x_{2n-1}-x_{2n}\|^2 + \beta P_\alpha^2(z,T(z)) + \varepsilon\|x_{2n}-x_{2n-1}\|^2$$
$$(1-\beta)P_\alpha^2(z,T(z)) \leq (\alpha+\varepsilon)\|x_{2n}-x_{2n-1}\|^2$$
$$P_\alpha^2(z,T(z)) \leq \frac{\alpha++\varepsilon}{1-\beta}\|x_{2n-1}-x_{2n}\|^2$$

Putting $q=\frac{\alpha+\varepsilon}{1-\beta}<1$

Then, we have
$$P_\alpha^2(z,T(z)) \leq q\|x_{2n-1}-x_{2n}\|^2 \leq q^n\|x_{2n-1}-x_{2n}\|^2$$
As $\to\infty$*, then* $P_\alpha^2(z,T(z))=0$

Hence we get {z}⊂T(z).

Similarly, {z}⊂S(z).

Theorem 3.2: Let C be a closed subset of a Hilbert space H . Let T, S be a fuzzy mapping from C into W(C) satisfying :
$$D^2(S(x),T(y)) \leq \alpha\frac{P_\alpha^2(y,T(x))+P_\alpha^2(x,S(x))+P_\alpha^2(y,S(x))}{1+P_\alpha^2(y,S(x))P_\alpha^2(x,T(y))} + \beta[P_\alpha^2(y,T(y))+P_\alpha^2(y,S(x))]+\gamma\|x-y\|^2$$
For all x, y∈ C where $\gamma,\alpha,\beta\geq 0$ with $2\alpha+\beta+\gamma<1$. Then there is $z\in C$ such that z is a common fixed point for T and S.

Proof: Let $x_0\in C$, we construct the sequence $\{x_n\}\in C$ as follows $\{x_1\}\subset T(x_0)$, $\{x_2\}\subset S(x_1)$, ………, $\{x_{2n+1}\}\subset T(x_{2n})$, $\{x_{2n+2}\}\subset S(x_{2n+1})$ and $\| x_{2n} -$ x2n+12≤D12T(x2n−1), S(x2n) .

Now
$$\| x_{2n+1} - x_{2n}\|^2 \leq D_1^2(S(x_{2n-1}),T(x_{2n})), \ for \ all \ n\in N.$$
$$\| x_{2n+1} - x_{2n}\|^2 \leq D^2(S(x_{2n-1}),T(x_{2n}))$$
$$\leq \alpha\frac{P_\alpha^2(x_{2n},T(x_{2n}))+P_\alpha^2(x_{2n-1},S(x_{2n-1}))+P_\alpha^2(x_{2n},S(x_{2n-1}))}{1+P_\alpha^2(x_{2n},S(x_{2n-1}))P_\alpha^2(x_{2n-1},T(x_{2n}))}$$
$$+ \beta[P_\alpha^2(x_{2n},T(x_{2n}))+P_\alpha^2(x_{2n},S(x_{2n-1}))] +\gamma\|x_{2n-1}-x_{2n}\|^2$$
$$\leq\alpha\frac{\|x_{2n-1}-x_{2n}\|^2+\|x_{2n-1}-x_{2n}\|^2+\|x_{2n}-x_{2n}\|^2}{1+\|x_{2n}-x_{2n}\|^2\|x_{2n-1}-x_{2n+1}\|^2}$$
$$+\beta[\|x_{2n-1}-x_{2n}\|^2+\|x_{2n}-x_{2n}\|^2]+\gamma\|x_{2n-1}-x_{2n}\|^2$$
Then $\|x_{2n+1}-x_{2n}\|^2 \leq (2\alpha+\beta+\gamma)\|x_{2n-1}-x_{2n}\|^2$

Putting $q=2\alpha+\beta+\gamma<1$

Then, we have
$$\|x_{2n+1}-x_{2n}\|^2 \leq q\|x_{2n-1}-x_{2n}\|^2$$
Now, for any positive integer p
$$\|x_n-x_{n+p}\|^2 \leq q\|x_n-x_{n+1}\|^2 + \|x_{n+1}-x_{n+2}\|^2+\ldots+\|x_{n+p-1} - x_{n+p}\|^2$$
$$\leq (q^n+q^{n+1}+q^{n+2}+\cdots+q^{n+p-1}\|x_1-x_0\|^2$$
$$\leq\frac{q^n}{1-q}\|x_1-x_0\|^2$$
Which implies that $\|x_n-x_{n+p}\|^2\to 0 \ as \ n\to\infty$

Hence,
$\{x_n\}$ *is cauchy sequance in H, but H is a Hilbert space. so* $\{x_n\}$ *is converge to z and since C is closed and* $\{x_n\}$ *is sequence in C.*

Then $z\in C$*, such that* $\lim_{n\to\infty} x_n = z$
$$P_\alpha^2(z,T(z)) \leq \|z-x_{2n}\|^2 + P_\alpha^2(x_{2n},T(z))$$
$$\leq \|z-x_{2n}\|^2 + D_\alpha^2(S(x_{2n-1}),T(z))$$
$$\leq \|z-x_{2n}\|^2 + D^2(S(x_{2n-1}),T(z))$$

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$$\leq \|z - x_{2n}\|^2 + \alpha \frac{P_\alpha^2(z, T(z)) + P_\alpha^2(x_{2n-1}, S(x_{2n-1})) + P_\alpha^2(z, S(x_{2n-1}))}{1 + P_\alpha^2(z, S(x_{2n-1}))P_\alpha^2(x_{2n-1}, T(z))}$$

$$+ \beta[P_\alpha^2(z, T(z)) + P_\alpha^2(z, S(x_{2n-1}))] + \gamma\|x_{2n-1} - z\|^2$$

$$\leq \|z - x_{2n}\|^2 + \alpha \frac{P_\alpha^2(z, T(z)) + \|x_{2n-1} - x_{2n}\|^2 + \|z - x_{2n}\|^2}{1 + \|z - x_{2n}\|^2 P_\alpha^2(x_{2n-1}, T(z))} + \beta[P_\alpha^2(z, T(z)) + \|z - x_{2n}\|^2] + \gamma\|x_{2n-1} - z\|^2$$

Then $(1\text{-}\alpha - \beta)P_\alpha^2(z, T(z)) \leq \alpha\|x_{2n-1} - x_{2n}\|^2$

$$P_\alpha^2(z, T(z)) \leq \frac{\alpha}{1 - \alpha - \beta}\|x_{2n-1} - x_{2n}\|^2$$

Putting $q = \frac{\alpha}{1-\alpha-\beta} < 1$

Then, we have

$$P_\alpha^2(z, T(z)) \leq q\|x_{2n-1} - x_{2n}\|^2 \leq q^n\|x_{2n-1} - x_{2n}\|^2$$

As $\to \infty$, *then* $P_\alpha^2(z, T(z)) = 0$

Hence we get $\{z\} \subset T(z)$.

Similarly, $\{z\} \subset S(z)$.

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