Fixed Point Theorems for Fuzzy Mappings on Closed Subset of Hilbert Spaces for Rational Expression

Buthainah A. A. Ahmed¹, Manar Falih Dheyab²

Department of Mathematics, College of Science, University of Baghdad, Iraq

Abstract: The aim of this paper is to obtain a common fixed point for fuzzy mapping on closed subset of Hilbert spaces for rational expression. The proof rely on the parallelogram law in Hilbert spaces.

Keywords: fuzzy mapping, fixed point, approximate quantity and Hilbert space

1. Introduction

The concept of fuzzy set was introduce by L.Zadeh [2]in 1965. After that a lot of work has been done regarding fuzzy set and fuzzy mappings. The concept of fuzzy mapping was first introduced by Heilpern [3], he proved fixed point theorem for fuzzy contraction mappings which is a fuzzy analogue of the fixed point theorem for mulity valued mapping of Nadler[4]. In 2015 [1] Jagdish C. Chaudhary, Jatin, Patel and Chirag proved some fixed point theorems in fuzzy mappings containing the rational expressions. In this paper we are proving some fixed point for fuzzy mapping on closed subset of Hilbert spaces for rational expression.

2. Preliminaries

In the following discussions mainly follow the definition and notions due to Heilpern [3]. Let H be a Hilbert space and F(H) be a collection of all fuzzy sets in H. Let \( A \in F(H) \) and \( \alpha \in [0, 1] \) the \( \alpha \) – level set of \( A \), denoted by \( A_\alpha \), is defined by

\[
\begin{align*}
A_\alpha &= \{ x \in A | \alpha \}} \\
A_{\overline{\alpha}} &= \{ x \in A | \alpha \}
\end{align*}
\]

Where \( \overline{B} \) denotes the closure of a set \( B \).

Definition 2.1[3]: A fuzzy set \( A \) is said to be an approximate quantity if and only if \( A_\alpha \) is compact and convex for each \( \alpha \in [0, 1] \) and \( \sup_{x \in \overline{B}} A(x) = 1 \). When \( A \) is an approximate quantity and \( A(x_0) = 1 \) for some \( x_0 \in H \), \( A \) is identified with an approximate of \( x_0 \).

The collection of all fuzzy sets in \( H \) is denoted by \( F(H) \) and \( W(H) \) is the sub collection of all approximate quantities.

Definition 2.2[3]: Let \( A, B \in W(H) \) and \( \alpha \in [0, 1] \). Then

i. \( P_\alpha(A, B) = \inf_{x \in A_\alpha, y \in B_\alpha} \| x - y \| \)

ii. \( D_\alpha(A, B) = \text{dis}(A_\alpha, B_\alpha) \), where “dis” is the Hausdorff distance

iii. \( D(A, B) = \sup_\alpha D_\alpha(A, B) \)

iv. \( P(A, B) = \sup_\alpha P_\alpha(A, B) \).

It is to be noted that for any ‘\( \alpha \)’, \( P_\alpha \) is a non decreasing as well as continuous function.

Definition 2.3[2]: Let \( A, B \in W(H) \). An approximate quantity \( A \) is said to be more accurate than \( B \) (denoted by \( A \preceq B \)) if and only if \( A(x) \leq B(x) \) for all \( x \in H \).

Definition 2.4[3]: A mapping \( T \) from the set \( H \) into \( W(H) \) is said to be fuzzy mapping.

Definition 2.5[3]: The point \( x \in H \) is called a fixed point for the fuzzy mapping \( T \) if and only if \( \{ x \} \subset T(x) \). We shall use the following lemmas due to Heilpern.

Lemma 2.6[3]: \( P_\alpha(x, B) \leq \| x - y \| + P_\alpha(y, B), \forall x, y \in H. \)

Lemma 2.7[3]: If \( \{ x_0 \} \subset A \), then \( P_\alpha(x_0, B) \leq D_\alpha(A, B), \forall B \in W(H). \)

Lemma 2.8[3]: Let \( A \in W(H) \) and \( x_0 \in H \), if \( \{ x_0 \} \subset A \) then \( P_\alpha(x_0, A) = 0 \), for each \( \alpha \in [0, 1] \).

Lemma 2.9[3]: Let \( H \) be a Hilbert space and \( T \) fuzzy mapping from \( H \) into \( W(H) \) and \( x_0 \in H \), then there exist \( x_1 \in H \) such that \( \{ x_1 \} \subset T(x_0). \)

3. Main Results

The proof rely on the parallelogram law in Hilbert spaces.

Theorem 3.1: Let \( C \) be a closed subset of a Hilbert space \( H \). Let \( T, S \) be a fuzzy mapping from \( C \) into \( W(C) \) satisfying :

\[
D^2(T(x), S(y)) \leq \alpha P_\alpha^2(T(x), T(y)) + \beta P_\alpha^2(y, S(y)) + \gamma |x - y|^2 + \delta \min\{P_\alpha^2(x, S(y)), P_\alpha^2(y, T(x))\} + m(x,y) \]

For all \( x, y \in C \) where \( \alpha, \beta, \sigma, \epsilon, \delta \geq 0 \) with \( \alpha + \beta + \epsilon + \gamma + \delta + \sigma < 1 \) and

\[
m(x,y) = \sigma \frac{P_\alpha^2(y, S(y))}{1 + P_\alpha^2(x, S(x))P_\alpha^2(x, T(y))} + \epsilon \frac{P_\alpha^2(x, S(x))}{1 + P_\alpha^2(y, S(x))P_\alpha^2(x, T(y))}
\]

Then \( T \) and \( S \) have a fixed point \( z \in C \) such that \( \{ z \} \subset T(z) \cap S(z) \).

Proof: Let \( x_0 \in C \), we construct the sequence \( \{ x_n \} \subset C \) as follows \( \{ x_n \} \subset T(x_n), \{ x_n \} \subset S(x_n), \ldots , \{ x_{2n+1} \} \subset T(x_{2n}), \{ x_{2n+2} \} \subset S(x_{2n+1}), \ldots \) and \( \| x_{2n+1} - x_{2n+2} \| \leq \| x_{2n+1} - x_{2n+2} \| \leq D_\alpha(T(x_{2n+1}), S(x_{2n+2})) \) for all \( n \in N \).
\[
\|x_{2n+1} - x_n\|^2 \leq D^2(S(x_{2n-1}), T(x_{2n})) \\
\leq \alpha P_\alpha^2(x_{2n-1}, S(x_{2n-1})) \\
+ \beta P_\alpha^2(x_{2n-1}, T(x_{2n})) + \gamma \|x_{2n} - x_{2n-1}\|^2 \\
+ \delta \min \{P_\alpha^2(x_{2n}, S(x_{2n-1})), P_\alpha^2(x_{2n}, T(x_{2n}))\} \\
+ \sigma \left( 1 + P_\alpha^2(x_{2n-1}, S(x_{2n-1})) \right) P_\alpha^2(x_{2n}, S(x_{2n-1})) \\
+ \epsilon \left( 1 + P_\alpha^2(x_{2n-1}, T(x_{2n})) \right) P_\alpha^2(x_{2n}, S(x_{2n-1}))
\]

\[
\leq \alpha \|x_{2n} - x_{2n-1}\|^2 \\
+ \beta \|x_{2n} - x_{2n+1}\|^2 + \gamma \|x_{2n} - x_{2n-1}\|^2 \\
+ \delta \min \{\|x_{2n} - x_{2n-1}\|^2, \|x_{2n+1} - x_{2n-1}\|^2\} \\
+ \sigma \left( 1 + \|x_{2n} - x_{2n+1}\|^2\right) \|x_{2n} - x_{2n-1}\|^2 \\
+ \epsilon \left( 1 + \|x_{2n} - x_{2n+1}\|^2\right) \|x_{2n} - x_{2n-1}\|^2
\]
\[
\leq \|x - x_{2n}\|^2 + \alpha \frac{\alpha P_a^2(z, T(z)) + P_a^2(x_{2n-1}, S(x_{2n-1})) + P_a^2(z, S(x_{2n-1}))}{1 + P_a^2(z, S(x_{2n-1}))} P_a^2(x_{2n-1}, T(z))
+ \beta \left[P_a^2(z, T(z)) + P_a^2(z, S(x_{2n-1}))\right] + \gamma \|x_{2n-1} - z\|^2
\leq \|x - x_{2n}\|^2 + \alpha \frac{\alpha P_a^2(z, T(z)) + \|x_{2n-1} - x_{2n}\|^2 + \|z - x_{2n}\|^2}{1 + \|z - x_{2n}\|^2 P_a^2(x_{2n-1}, T(z))} + \beta \left[P_a^2(z, T(z)) + \|z - x_{2n}\|^2 + \gamma \|x_{2n-1} - z\|^2\right]
\]

Then \((1 - \alpha - \beta) P_a^2(z, T(z)) \leq \alpha \|x_{2n-1} - x_{2n}\|^2

Putting \(q = \frac{\alpha}{1 - \alpha - \beta} < 1\),

Then, we have

\(P_a^2(z, T(z)) \leq q \|x_{2n-1} - x_{2n}\|^2 \leq q^n \|x_{2n-1} - x_{2n}\|^2\)

As \(n \to \infty\), then \(P_a^2(z, T(z)) = 0\)

Hence we get \(z \in T(z)\).

Similarly, \(z \in S(z)\).

References


