Fuzzy Fixed Point Theorems for Fuzzy Mappings Via fuzzy β-Admissible

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Abstract: The aim of this paper is introduced the notion of fuzzy β-admissible, fuzzy β-Jaggi contractive and study some results of fuzzy fixed point for fuzzy mapping via fuzzy β-admissible in Hilbert spaces.

Keywords: fuzzy mapping, fuzzy fixed point, β-admissible mapping and Hilbert space

1. Introduction

The concept of fuzzy set was introduced by L. Zadeh [3] in 1965. After that a lot of work has been done regarding fuzzy set and fuzzy mappings. The concept of fuzzy mapping was first introduced by Heilpern [4]. In 2001, Estruch and Vidal [5] proved a fuzzy fixed point theorem for fuzzy contractive mappings. On the other hand, the concept of an α-admissible mapping was introduced by Samet et al. [2]. Recently, Mohammadi et al. [1] introduced the concept of β-admissible for multivalued mappings. In this paper, introduced fuzzy β-admissible mapping, fuzzy β-Jaggi contractive mapping and study some results of fuzzy fixed point theorems for fuzzy mapping via fuzzy β-admissible in Hilbert space.

2. Preliminaries

In this section, we recall some basic definitions and preliminaries that will be needed in this paper.

Definition 2.1[3]: Let H be a Hilbert space and F(H) be a collection of all fuzzy sets in H. Let A∈ F(H) and α ∈[0, 1] the α - level set of A, denoted by A_α is defined by

\[ A_α = \{ x : A(x) ≥ α \} \]

where α∈[0,1].

Definition 2.2[4]: A fuzzy set A is said to be an approximate quantity if and only if A_0 is compact and convex for each α ∈[0,1], and sup_{α∈[0,1]} A(x) = 1. When A is an approximate quantity and A(x)≠0 for some x∈H, A is identified with an approximate of x_0.

The collection of all fuzzy sets in H is denoted by F(H) and W(H) is the sub collection of all approximate quantities.

Definition 2.3[4]: Let A, B ∈ W(H) and α ∈[0,1]. Then

1) \[ P_α(A,B) = \inf_{x∈A_α, \ y∈B_α} \| x - y \| \]
2) \[ D_α(A,B) = \text{dis}(A_α, B_α) \]
3) \[ D(A, B) = \sup_{α} D_α(A,B) \]
4) \[ P(A, B) = \sup_{α} P_α(A,B) \]

It is to be noted that for any ‘α’, \( P_α \) is a non decreasing as well as continuous function.

Definition 2.4[3]: Let A, B ∈ W(H). An approximate quantity A is said to be more accurate than B (denoted by A< B) if and only if A(x) ≤ B(x), \( \forall \ x \in H \).

Definition 2.5[4]: A mapping T from the set H into W(H) is said to be fuzzy mapping.

Definition 2.6[4]: The point x ∈ H is called fixed point for the fuzzy mapping T if \( \{ x \} \subseteq T(x) \). If x_α ∈ T(x) is called fuzzy fixed point of T. We shall use the following lemmas due to Heilpern.

Lemma 2.7[4]: \( P_α(x,B) ≤ \| x - y \| + P_α(y,B) \), \( \forall x, y \in H \).

Lemma 2.8[4]: If \( x_0 \in A \), then \( P_α(x_0,B) ≤ D_α(A,B) \), \( \forall B \in W(H) \).

Lemma 2.9[4]: Let A ∈ W(H) and x_0 ∈ H, if \( x_0 \in A \) then \( P_α(x_0, A) = 0 \), for each \( α ∈[0,1] \).

Lemma 2.10[4]: Let H be a Hilbert space and T fuzzy mapping from H into W(H) and x_0 ∈ H, then there exist \( x_1 \in H \) such that \( \{ x_1 \} \subseteq T(x_0) \).

3. Fuzzy Fixed Point Theorem

In this section, we introduce the concept of fuzzy β-admissible for fuzzy mapping and some results of fuzzy fixed point theorems.

Definition 3.1[2]: Let H be a Hilbert space \( \beta : H × H → [0, \infty) \), \( \alpha ∈ [0,1] \) and \( T:H→ W(H) \). A mapping T is said to be fuzzy β-admissible if for each \( x, y \in H \) and \( y \in [T(x)]_α \), with \( \beta(x,y) ≥ 1 \), we have \( \beta(y,z) ≥ 1 \) for all \( z \in [Ty]_α \).

Definition 3.2[2]: Let H be a Hilbert space \( \beta : H × H → [0, \infty) \), \( \alpha ∈ [0,1] \) and \( T:H→ W(H) \). A mapping T is said to be fuzzy β-admissible if for each \( x, y \in H \) with \( \beta(x,y) ≥ 1 \), we have \( \beta([Tx]_α, [Ty]_α) ≥ 1 \)

Where \( \beta([Tx]_α, [Ty]_α) = \inf \{ \beta(a,b) \} \), \( a \in [Tx]_α \) and \( b \in Ty_α \).

Remark 3.3: If T is fuzzy β-admissible, then T is also fuzzy β-admissible mapping.
Definition 3.4: Let H be a Hilbert space, $\beta : H \times H \to [0, \infty)$. A fuzzy mapping $T:H \to W(H)$ is called fuzzy $\beta$-Jaggi contractive, if there exists two functions $\beta : H \times H \to [0, \infty)$ and $\Psi : [0, \infty) \to [0, \infty)$, where $\Psi$ is non-decreasing and $\sum_{n=1}^{\infty} \Psi^n(t) < \infty$ for each $t > 0$ and $\Psi^n$ is n-th iteration of $\Psi$ such that $\beta(x,y)D^2(Tx,Ty) \leq \Psi(M(x,y)) + LN(x,y)$, $x, y \in H$, where $L \geq 0, 0 < \alpha \in [0,1], \beta > 0,$

$$M(x,y) = \frac{a_1^n \psi(a, \beta_n, x, y) \nabla(y, \psi(n, x, y))}{\nabla - \nabla_n} + \beta \nabla \nabla_n$$

Then, there exists $x \in H$ such that $x_n$ is a fuzzy fixed point of $T$.

Theorem 3.4: Let H be a Hilbert space and T be a fuzzy $\beta$-Jaggi contractive mapping satisfies the following conditions:

1) $T$ is fuzzy $\beta$-admissible
2) There exists $x_0 \in H$ and $x_1 \in [T x_0]$ such that $\beta(x_0, x_1) \geq 1$
3) If $\{x_n\}$ is a sequence in $H$ such that $\beta(x_n, x_{n+1}) \geq 1$ and $x_n \to u$ as $n \to \infty$, then $\beta(x_n, u) \geq 1$
4) $\Psi$ is continuous

Then, there exists $x \in H$ such that $x_n$ is a fuzzy fixed point of $T$.

Proof: Let $x_0 \in H$ and $x_1 \in [T x_0]$ by condition (2) $(x_0, x_1) \geq 1$.

Since $[T x_0]$ is non-empty compact subset of $H$, there exists $x_2 \in [T x_0]$ such that $\|x_1 - x_2\|^2 = D^2(Tx_0, Tx_2)$

So $\|x_1 - x_2\|^2 \leq D^2(Tx_0, Tx_2)$

Since $\beta(x_0, x_1) \geq 1$, then $\|x_1 - x_2\|^2 \leq D^2(Tx_0, Tx_1)$

$\beta(x_0, x_1)D^2(Tx_0, Tx_1) \leq \Psi(M(x_0, x_1)) + L \min\{P_0^2(x_0, Tx_0), P_0^2(x_1, Tx_1), P_0^2(x_0, x_1)\}$

By the same argument, for $x_2 \in H$ we have $[Tx_2]$, which is a non-empty compact subset of $H$ and there exists $x_3 \in [Tx_2]$, such that $\|x_2 - x_3\|^2 \leq P_0^2(x_2, Tx_2) \leq D^2(Tx_1, Tx_2)$

For $x_3 \in H$ and $x_4 \in [Tx_2]$, with $\beta(x_3, x_4) \geq 1$, by definition of fuzzy $\beta$-admissible, we get $\beta(x_3, x_4) \geq 1$

Then

$\|x_3 - x_4\|^2 \leq D^2(Tx_1, Tx_2)$

$\beta(x_3, x_4)D^2(Tx_1, Tx_2) \leq \Psi(M(x_0, x_1)) + L \min\{P_0^2(x_0, Tx_0), P_0^2(x_1, Tx_1), P_0^2(x_0, x_1)\}$

By induction, we can construct a sequence $\{x_n\}$ in $H$ such that for each $n \in N, x_n \in [Tx_{n-1}]$ with $\beta(x_{n-1}, x_n) \geq 1$ and

$\|x_n - x_{n+1}\|^2 \leq \Psi(M(x_{n-1}, x_n))$.

$M(x_{n-1}, x_n) = \frac{a_1^n \psi(a, \beta_n, x, y) \nabla(y, \psi(n, x, y))}{\nabla - \nabla_n} + \beta \nabla \nabla_n \leq \Psi(M(x_{n-1}, x_n)) + L \min\{P_0^2(x_{n-1}, Tx_{n-1}), P_0^2(x_n, x_{n+1})\}$

$\Psi^{n+1}(t) \leq \Psi^n(t) + L \min\{P_0^2(x_0, Tx_0), P_0^2(x_1, Tx_1), P_0^2(x_0, x_1)\}$

$\Psi^{n+2}(t) \leq \Psi^n(t) + L \min\{P_0^2(x_0, Tx_0), P_0^2(x_1, Tx_1), P_0^2(x_0, x_1)\}$

Next, we will show that $\{x_n\}$ is a Cauchy sequence in $H$.

Since continuous function $\Psi$, there exists $\epsilon > 0$ and integer $h=h(\epsilon)$ such that $\sum_{n=h}^{\infty} \Psi^n(t) < \epsilon$.

Let $m, n > h$. Using the triangular inequality, previous relation, we have $\|x_n - x_m\|^2 \leq \sum_{k=m}^{n-1} \Psi^k(t) \leq n \geq h \Psi \|x_n - x_m\|^2 < \epsilon$.

This implies that $\{x_n\}$ is a Cauchy sequence in $H$. By completeness of $H$, there exists $x \in H$ such that $x_n \to x$ as $n \to \infty$.

Finally, we show that $P_0^2(x, Tx) = 0$. By condition (3), we have $\beta(x_n, x) \geq 1$, for all $n \in N$

Now we have $P_0^2(x, Tx) \leq \|x_n - x\|^2 + \beta \Psi \|x_n - x\|^2 + D^2(Tx_n, Tx)$

Letting $n \to \infty$, it follows that $P_0^2(x, Tx) \leq \Psi(0) = 0 \to P_0^2(x, Tx) = 0$.

Hence by Lemma 2.8, $x_n \in Tx$. This complete the proof.

Theorem 3.5: Let H be a Hilbert space and T be a fuzzy $\beta$-Jaggi contractive mapping satisfies the following conditions:

1) $T$ is fuzzy $\beta$-admissible
2) There exists $x_0 \in H$ and $x_1 \in [Tx_0]$ such that $\beta(x_0, x_1) \geq 1$
3) If $\{x_n\}$ is a sequence in $H$ such that $\beta(x_n, x_{n+1}) \geq 1$ and $x_n \to u$ as $n \to \infty$, then $\beta(x_n, u) \geq 1$
4) $\Psi$ is continuous

Then, there exists $x \in H$ such that $x_n$ is a fuzzy fixed point of $T$.

Proof: Trivial

In Theorem 3.4 and 3.5, we take $\Psi(t) = \theta t$, where $\theta \in (0, 1)$, then we have the following corollary which is a fuzzy extension of fixed point theorem.

Corollary 3.6: Let H be a Hilbert space and T be a fuzzy mapping. Suppose that there exists $\beta : H \times H \to [0, \infty)$ such that $\beta(x, y)D^2(Tx, Ty) \leq \theta M(x, y)$ for all $x, y \in H$, where $L \geq 0$, $0 < \alpha \in [0,1], \beta > 0$.

$M(x, y) = \frac{a_1^n \psi(a, \beta_n, x, y) \nabla(y, \psi(n, x, y))}{\nabla - \nabla_n} + \beta \nabla \nabla_n \leq \Psi(M(x_{n-1}, x_{n-1}))$
And 
\[ N(x,y) = \min\{P_\alpha^2(x,Tx), P_\alpha^2(y,Ty), P_\alpha^2(x,Ty), P_\alpha^2(y,Tx)\} \].

Satisfies the following conditions:
1) $T$ is fuzzy $\beta$ − admissible (or fuzzy $\beta^\ast$ − admissible)
2) There exists $x_0 \in H$ and $x_1 \in [Tx_0]_\alpha$ such that
$\beta(x_0,x_1) \geq 1$
3) If \{$x_n$\} is a sequence in $H$ such that $\beta(x_n,x_{n+1}) \geq 1$
and $x_n \rightharpoonup u$ as $n \to \infty$, then $\beta(x_n,u) \geq 1$

Then, there exists $x \in H$ such that $x_\alpha$ is a fuzzy fixed point of $T$.

**Proof:** Trivial

**References**