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Fuzzy Fixed Point Theorems for Fuzzy Mappings Viafuzzy β-Admissible

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Abstract: ***The aim of this paper is introduced the notion offuzzy β–admissible , fuzzy β –Jaggi contractive and study some results of fuzzy fixed point for fuzzy mapping via fuzzy β –admissible in Hilbert spaces.***

Keywords: fuzzy mapping, fuzzy fixed point , β − admissiblemapping and Hilbert space

1. Introduction

The concept of fuzzy set was introduce by L.Zadeh [3]in 1965.After that a lot of work has been done regarding fuzzy set and fuzzy mappings. The concept of fuzzy mapping was first introduced by Heilpern [4].In 2001 , Estruch and Vidal [5] proved a fuzzy fixed point theorem for fuzzy contractive mappings. On the other hand, the concept of an $\beta-$ admissible mapping was introduced by Samet et al.[2]. Recently, Mohammadi et al. [1] introduced the concept of $\beta-$ admissible for multivalued mappings . In this paper, introduced fuzzy$\beta-$ admissible mapping , fuzzy $\beta-$ Jaggi contractive mappingand study some results of fuzzy fixed point theorems for fuzzy mapping via fuzzy $\beta-$ admissible in Hilbert space.

2. Preliminaries

In this section, we recall some basic definitions and preliminaries that will be needed in this paper.

Definition 2.1[3]: Let H be a Hilbert space and F(H) be a collection of all fuzzy sets in H . Let A∈ F(H) and $\alpha \in [0, 1]$ the $\alpha-$ level set of A, denoted by A_α is defined by

$$A_\alpha = \{x : A(x) \geq \alpha\} \text{ if } \alpha \in [0,1]$$

$$A_0 = \overline{\{x : A(x) > \alpha\}}$$

Where $\overline{B}$ denotes the closure of a set B.

Definition 2.2[4]:A fuzzy set A is said to be an approximate quantity if and only if A_α is compact and convex for each $\alpha \in [0,1]$, and $\sup_{x\in X} A(x) = 1$ .When A is an approximate quantity and$A(x_0)$=1 for some$x_0 \in H$, A is identified with an approximate of x_0 .

The collection of all fuzzy sets in H is denoted by F(H) and W(H) is the sub collection of all approximate quantities.

Definition 2.3[4]: Let A, B ∈ W(H) and $\alpha \in [0,1]$. Then

1) $P_\alpha(A,B) = \inf_{x\in A_\alpha,\ y\in B_\alpha} \|x-y\|$
2) $D_\alpha(A,B) = dis(A_\alpha, B_\alpha)$, where "dis" is the Hausdorff distance
3) $D(A , B) = \sup_\alpha D_\alpha(A,B)$
4) $P(A , B) = \sup_\alpha P_\alpha(A,B)$.

It is to be noted that for any 'α', P_α is a non decreasing as well as continuous function.

Definition 2.4[3]. Let A, B ∈ W(H). An approximate quantity A is said to be more accurate than B (denoted by A⊂ B) if and only if $A(x) \leq B(x)$, $\forall x \in H$.

Definition 2.5[4]:A mapping T from the set H into W(H) is said to be fuzzy mapping.

Definition 2.6[4]: The point x ∈ H is called fixed point for the fuzzy mapping T if {x}⊂Tx. If $x_\alpha \subset Tx$ is called fuzzy fixed point of T . We shall use the following lemmas due to Helipern.

Lemma 2.7[4]: $P_\alpha(x,B) \leq \|x-y\| + P_\alpha(y,B)$, $\forall x, y \in H$.

Lemma 2.8[4]: If $\{x_0\} \subset A$, then $P_\alpha(x_0,B) \leq D_\alpha(A,B)$, $\forall B \in W(H)$.

Lemma 2.9[4]: Let $A \in W(H)$ and $x_0 \in H$, if$\{x_0\} \subset A$ then Pα(x0 ,A)= 0, for each α∈0,1.

Lemma 2.10[4]: Let H be a Hilbert space and T fuzzy mapping from H into W(H) and $x_0 \in H$, then there exist $x_1 \in H$ such that$\{x_1\} \subset Tx_0$.

3. Fuzzy Fixed Point Theorem

In this section, we introduce the concept of fuzzy $\beta-$ admissiblefor fuzzy mapping and some results of fuzzy fixed point theorems.

Definition 3.1[2]: Let H be a Hilbert space ,$\beta: H\times H \to [0,\infty)$, $\alpha \in [0,1]$ and T:H→ W(H). A mapping T is said to be fuzzy $\beta-$ admissible if for each $x \in H$ and $y \in [Tx]_\alpha$, with $\beta(x,y) \geq 1$, we have $\beta(y,z) \geq 1$ for all $z \in [Ty]_\alpha$.

Definition 3.2: Let H be a Hilbert space ,$\beta: H\times H \to [0,\infty)$, $\alpha \in [0,1]$ and T:H→ W(H). A mapping T is said to be fuzzy β^*- admissible if for each x,y∈ H with $\beta(x,y) \geq 1$, we have$\beta([Tx]_\alpha, [Ty]_\alpha) \geq 1$

Where $\beta([Tx]_\alpha, [Ty]_\alpha) = \inf\{\beta(a,b) : a \in [Tx]_\alpha \text{ and } b \in Ty\alpha$.

Remark 3.3: If T is fuzzy β^*- admissible, then Tis also fuzzy $\beta-$ admissible mapping .

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Definition 3.4: Let H be a Hilbert space, $\beta: H \times H \rightarrow [0, \infty)$. A fuzzy mapping $T: H \rightarrow W(H)$ is called fuzzy $\beta-$Jaggi contractive, if there exists two function $\beta: H \times H \rightarrow [0,\infty)$ and $\Psi: [0,\infty) \rightarrow [0,\infty)$, where Ψ is non-decreasing and $\sum_{n=1}^{\infty} \Psi^n(t) < \infty$, *for each* $t > 0$ *and* Ψ^n is $n-$th iteration of Ψ such that $\beta(x,y)D^2(Tx, Ty) \leq \Psi\big(M(x,y)\big) + LN(x,y), \forall\ x\ ,\ y\ \in H$, where $L \geq 0, \alpha \in [0,1]$, $\beta > 0$,

$$M(x,y) = \frac{\alpha P_\alpha^2(x, Tx).\, P_\alpha^2(y, Ty)}{\|x-y\|^2} + \beta\|x-y\|^2 - \alpha P_\alpha^2(x, Tx)$$

And N(x,y) = $\min\{P_\alpha^2(x, Tx), P_\alpha^2(y, Ty), P_\alpha^2(x, Ty), P_\alpha^2(y, Tx)\}$.

Theorem 3.4: Let H be a Hilbert space and T be a fuzzy β-Jaggi contractive mapping satisfies the following conditions:

1) T is fuzzy $\beta-$admissible
2) There exists $x_0 \in H$ and $x_1 \in [T x_0]_\alpha$ such that $\beta(x_0, x_1) \geq 1$
3) If $\{x_n\}$ is a sequence in H such that $\beta(x_n, x_{n+1}) \geq 1$ and $x_n \rightarrow u$ as $n \rightarrow \infty$, then $\beta(x_n, u) \geq 1$
4) Ψ is continuous

Then, there exists $x \in H$ such that x_α is a fuzzy fixed point of T .

Proof: Let $x_0 \in H$ and $x_1 \in [T x_0]_\alpha$ by condition (2) $(x_0, x_1) \geq 1$.

Since $[T x_1]_\alpha$ is non-empty compact subset of H , there exists $x_2 \in [T x_1]_\alpha$, such that $\| x_1 - x_2\|^2 = P_\alpha^2(x_1, T x_1) \leq D^2(T x_0, T x_1)$

So $\| x_1 - x_2\|^2 \leq D^2(T x_0, T x_1)$

Since $(x_0, x_1) \geq 1$, then $\| x_1 - x_2\|^2 \leq D^2(T x_0, T x_1)$

$\leq \beta(x_0, x_1) D^2(T x_0, T x_1) \leq \Psi\big(M(x_0, x_1)\big) + L\min\{P_\alpha^2(x_0, T x_0), P_\alpha^2(x_1, T x_1), P_\alpha^2(x_0, T x_1), P_\alpha^2(x_1, T x_0)\}$

$\leq \Psi\big(M(x_0, x_1)\big) + L\min\{P_\alpha^2(x_0, x_1), P_\alpha^2(x_1, x_2), P_\alpha^2(x_0, x_2), P_\alpha^2(x_1, x_1)\}$

$\| x_1 - x_2\|^2 \leq \Psi\big(M(x_0, x_1)\big)$.

By the same argument, for $x_2 \in H$ we have $[T x_2]_\alpha$ which is non-empty compact subset of H and there exists $x_3 \in [T x_2]_\alpha$, such that

$\| x_2 - x_3\|^2 = P_\alpha^2(x_2, T x_2) \leq D^2(T x_1, T x_2)$

For $x_0 \in H$ and $x_1 \in [T x_0]_\alpha$ with $(x_0, x_1) \geq 1$, by definition of fuzzy $\beta-$admissible , we get $\beta(x_1, x_2) \geq 1$, then

$\| x_2 - x_3\|^2 \leq D^2(T x_1, T x_2) \leq \beta(x_1, x_2) D^2(T x_1, T x_2)$
$\leq \Psi\big(M(x_1, x_2)\big)$
$+ L\min\{P_\alpha^2(x_1, T x_1), P_\alpha^2(x_2, T x_2), P_\alpha^2(x_1, T x_2), P_\alpha^2(x_2, T x_1)\}$
$\leq \Psi\big(M(x_1, x_2)\big)$
$+ L\min\{P_\alpha^2(x_1, x_2), P_\alpha^2(x_2, x_3), P_\alpha^2(x_1, x_3), P_\alpha^2(x_2, x_2)\}$
$\| x_2 - x_3\|^2 \leq \Psi\big(M(x_1, x_2)\big)$.

By induction, we can construct a sequence $\{x_n\}$ in H such that , for each $n \in N$, $x_n \in [T x_n]_\alpha$ with $\beta(x_{n-1}, x_n) \geq 1$ and

$\| x_n - x_{n+1}\|^2 \leq \Psi\big(M(x_{n-1}, x_n)\big)$.

$M(x_{n-1}, x_n) = \frac{\alpha P_\alpha^2(x_{n-1}, T x_{n-1}).\, P_\alpha^2(x_n, T x_n)}{\| x_{n-1} - x_n\|^2} + \beta\| x_{n-1} - x_n\|^2 - \alpha P_\alpha^2(x_{n-1}, T x_{n-1}) \leq \frac{\alpha\| x_{n-1} - x_n\|^2.\, \| x_{n+1} - x_n\|^2}{\| x_{n-1} - x_n\|^2} + \beta\| x_{n-1} - x_n\|^2$

$\leq \alpha\, \| x_{n+1} - x_n\|^2 + \beta\| x_{n-1} - x_n\|^2 - \alpha\, \| x_{n+1} - x_n\|^2$

$M(x_{n-1}, x_n) \leq \beta\| x_{n-1} - x_n\|^2$

We have $\| x_n - x_{n+1}\|^2 \leq \Psi(\beta\| x_{n-1} - x_n\|^2)$ $\leq \Psi(\Psi(\beta\| x_{n-2} - x_{n-1}\|^2))$ …………$\leq \Psi^n(\beta\| x_0 -$ x12 .

Next, we will show that $\{x_n\}$ is a Cauchy sequence in H . Since continuous function Ψ, there exists $\epsilon > 0$ and positive integer $h=h(\epsilon)$ such that $\sum_{n\geq h} \Psi^n(\beta\| x_0 - x_1\|^2) < \epsilon$.

Let $m > n > h$.Using the triangular inequality, previous relation, we have $\| x_n - x_m\|^2 \leq \sum_{k=n}^{m-1} \beta\| x_k - x_{k+1}\|^2 \leq$ n≥hΨnβ x0− x12<ϵ.

This implies that $\{x_n\}$ is a Cauchy sequence in H. By completeness of H, there exists $x \in H$ such that $x_n \rightarrow x$ as $n \rightarrow \infty$.

Finally, we show that $P_\alpha^2(x, Tx) = 0$. By condition (3), we have $\beta(x_n, x) \geq 1$, for all $n \in N$

Now we have $P_\alpha^2(x, Tx) \leq \| x_{n+1} - x_n\|^2 + P_\alpha^2(x_{n+1}, Tx)$
$\leq \| x_{n+1} - x_n\|^2 + D_\alpha^2(Tx_n, Tx)$

$$\leq \| x_{n+1} - x_n\|^2 + \beta(x_n, x)\, D_\alpha^2(Tx_n, Tx)$$

$$\leq \| x_{n+1} - x_n\|^2 + \Psi\left(\frac{\alpha P_\alpha^2(x_n, T x_n).\, P_\alpha^2(x, Tx)}{\| x_n - x\|^2} + \beta\| x_n - x\|^2 - \alpha P_\alpha^2(x_n, T x_n)\right)$$
$$+ L\min\{P_\alpha^2(x_n, T x_n), P_\alpha^2(x, Tx), P_\alpha^2(x_n, Tx), P_\alpha^2(x, T x_n)\}$$

$$\leq \| x_{n+1} - x_n\|^2 + \Psi\left(\frac{\alpha\| x_{n+1} - x_n\|^2.\, P_\alpha^2(x, Tx)}{\| x_n - x\|^2} + \beta\| x_n - x\|^2 - \alpha\| x_{n+1} - x_n\|^2\right)$$
$$+ L\min\{\| x_{n+1} - x_n\|^2, P_\alpha^2(x, Tx), P_\alpha^2(x_n, Tx), P_\alpha^2(x, T x_n)\}$$

Letting $n \rightarrow \infty$, it folloes that

$P_\alpha^2(x, Tx) \leq \Psi(0) = 0 \rightarrow P_\alpha^2(x, Tx) = 0$.

Hence by lemma 2.8 $x_\alpha \subset Tx$ .This complete the proof.

Theorem 3.5: Let H be a Hilbert space and T be a fuzzy β-Jaggi contractive mapping satisfies the following conditions:

1) T is fuzzy β^*-admissible
2) There exists $x_0 \in H$ and $x_1 \in [T x_0]_\alpha$ such that $\beta(x_0, x_1) \geq 1$
3) If $\{x_n\}$ is a sequence in H such that $\beta(x_n, x_{n+1}) \geq 1$ and $x_n \rightarrow u$ as $n \rightarrow \infty$, then $\beta(x_n, u) \geq 1$
4) Ψ is continuous

Then, there exists $x \in H$ such that x_α is a fuzzy fixed point of T .

Proof: Trivial

In Theorem 3.4 and 3.5, we take $\Psi(t) = \theta t$, where $\theta \in (0,1)$, then we have the following corollary which is a fuzzy extension of fixed point theorem .

Corollary 3.6: Let H be a Hilbert space and T be a fuzzy mapping. Suppose that there exists$\beta: H \times H \rightarrow [0, \infty)$such that $\beta(x,y)D^2(Tx, Ty) \leq \theta\big(M(x,y)\big) + LN(x,y), \forall x\ ,\ y \in H$, where $L \geq 0, \alpha \in [0,1]$, $\beta > 0$,

$$M(x,y) = \frac{\alpha P_\alpha^2(x, Tx).\, P_\alpha^2(y, Ty)}{\|x-y\|^2} + \beta\|x-y\|^2 - \alpha P_\alpha^2(x, Tx)$$

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And $N(x,y) = \min\{P_\alpha^2(x, Tx), P_\alpha^2(y, Ty), P_\alpha^2(x, Ty), P_\alpha^2(y, Tx)\}$.

Satisfies the following conditions:

1) T is fuzzy $\beta-$admissible(or fuzzyβ^*-admissible)
2) There exists $x_0 \in H$ and $x_1 \in [T x_0]_\alpha$ such that $\beta(x_0, x_1) \geq 1$
3) If $\{x_n\}$ is a sequence in H such that $\beta(x_n, x_{n+1}) \geq 1$ and $x_n \rightarrow u$ as $n \rightarrow \infty$, then $\beta(x_n, u) \geq 1$

Then, there exists $x \in H$ such that x_α is a fuzzy fixed point of T .

Proof: Trivial

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