Fuzzy Critical Path Method Using Ranking of Fuzzy Numbers

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Abstract: ***In this paper Algorithm is presented to find critical path in fuzzy environment. In Trapezoidal fuzzy number is given to access the critical path from the initial node to terminal node by using ranking of fuzzy number.***

Keywords: Fuzzy set, trapezoidal fuzzy number, critical path method, ranking function

1. Introduction

Critical path method is a network-based method designed for planning and managing of complicated projects in real world applications. The main purpose of critical path method is to evaluating project performance and to identifying the critical activities on the critical path so that the available resources could be utilized on these activities in the project network in order to reduce project completion time. With the help of the critical path, the decision maker can adopt a better strategy of optimizing the time and the available resources to ensure the earlier completion and the quality of the project.

Dubois et al [1] extended the fuzzy arithmetic operations to compute the latest starting time of each activity in a project network. Hapke et al. [2] used fuzzy arithmetic operations to compute the latest starting time of each activity in a project network. To find critical path in a fuzzy project network Yaoet al. [3] used signed distance ranking of fuzzy numbers. Chen et al. [4] used defuzzification method to find possible critical paths in a fuzzy network. Dubois et al [5] assigns a different level of importance to each activity on a critical path for a randomly chosen set of activities.

To deal with completion time management and the critical degrees of all activities for a project network. C. T. Chen and S. F. Huang, applied fuzzy method for measuring criticality in project network. Ravi Shankar et al [6] proposed an analytical method for finding critical path in a fuzzy project network.

The critical path is the one from the start of the project to finish of project where the slack times are all zeros. The purpose of the Critical path method(CPM) is to identify critical activities on the critical path so that resources may be concentrated on these activities in order to reduce project length time.

Besides, CPM has proved very valuable in evaluating project performance and identifying bottlenecks. Thus, CPM is a vital tool for the planning and control of complex projects. A project network is defined as a set of activities that must be performed according to precedence constraints stating which activities must start after the completion of specified other activities [11]. Such a project network can be represented as a directed graph. A path through a project network is one of the routes from the starting node to the ending node. The length of a path is the sum of the durations of the activities on the path. The project duration equals the length of the longest path through the project network. The longest path is called the critical path in the network.

In this paper, we use a new Ranking formula for trapezoidal fuzzy number and apply to the expected duration for each activity in the fuzzy project network to find the critical path.

2. Fuzzy Set Theory

In this section, we briefly review the theory of fuzzy sets from [7-10]. In Fig.1, we see a graph of a crisp set and a fuzzy set. The fuzzy set A can look very different depending on the chosen membership function. Using this function, it is possible to assign a membership degree to each of the element in the universe of discourse X. Elements of the set could but are not required to be numbers as long as a degree of membership can be deduced from them. It is important to note the fact that membership grades are not probabilities. One important difference is that the summation of probabilities on a finite universal set must equal 1, while there is no requirement for membership grades.

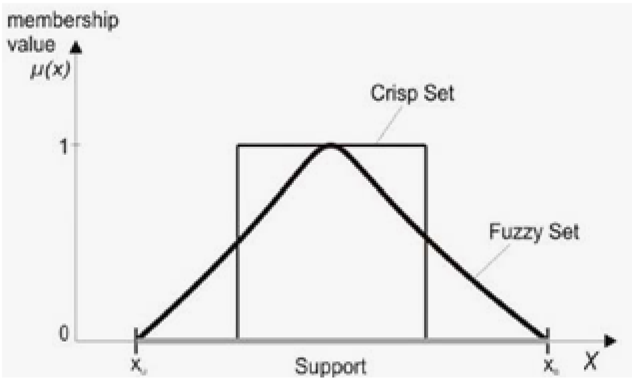

Figure 1: Crisp set and Fuzzy set

Definition 1: A fuzzy number with membership function in the form

$$\mu_{\tilde{A}}(x) = \begin{cases} \frac{x-a}{b-a}, & a \le x < b, \\ \frac{c-x}{c-b}, & b \le x \le c, \\ 0, & otherwise \end{cases}$$

is called a triangular fuzzy number $\tilde{A}$ =(a,b,c,d)

Let X be the universe of discourse, X = {x1,x2,…xn }. A fuzzy set A of X can be represented by

$$A = \mu_A(x_1) / x_1 + \mu_A(x_2) / x_2 + \ldots + \mu_A(x_n) / x_n \qquad (1)$$

where μ_A is the membership function of the fuzzy set A and $\mu A(x_1)$ indicates the grade of membership of xi in the fuzzy set A, where $\mu_A(x_i) \in [0,1]$.

A fuzzy number is a fuzzy set which is both convex and normal. A fuzzy set A of the universe of discourse X is convex if and only if for all x_1, x_2 in X,

$$\mu_A(\lambda x_1 + (1-\lambda)x_2) \ge \text{Min}\,(\mu A(x_1), \mu A(x_2)) \qquad (2)$$

where $\lambda \in [0,1]$. A fuzzy set of the universe of discourse X is called a normal fuzzy set if $\exists\, x_i \in X$, $\mu_A(x_i) = 1$. A trapezoidal fuzzy number A of the universe ofdiscourse X can be characterized by a trapezoidal membership function parameterized by a quadruple (a ,b,c,d) as shown in Fig 2, where a , b, c and d are real values.

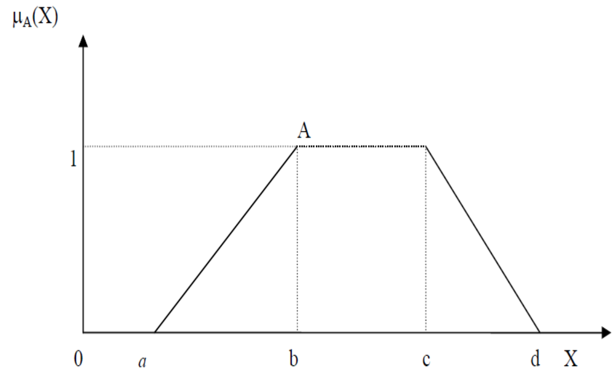

Figure 2: Membership function curve of trapezoidal fuzzy number A

From Fig.2, we can see that if *a* = b and c = d , then A is called a crisp interval; if *a* = b = c = d , then A is a crisp value.

2.1 Arithmetic operations on Trapezoidal fuzzy numbers

Let A_1 and A_2 be two trapezoidal fuzzy numbers parameterized by the quadruple (a_1,b_1,c_1,d_1) and (a_2,b_2,c_2,d_2), respectively. The simplified fuzzy number arithmetic operations between the trapezoidal fuzzy numbers A1 and A2 are as follows:

Fuzzy numbers addition $\oplus$:
$(a_1, b_1, c_1, d_1) \oplus (a_2, b_2, c_2, d_2)$
$= (a_1 + a_2, b_1+b_2, c_1+c_2, d_1+d_2)$. (3)

Fuzzy numbers subtraction Θ :
$(a_1, b_1, c_1, d_1)\ \Theta\ (a_2, b_2, c_2, d_2)$
$= (a_1 - d_2, b_1 - c_2, c_1 - b_2, d_1 - a_2)$. (4)

For example: Let A_1 and A_2 be two trapezoidal fuzzy numbers, where
A1= (14, 18 ,20, 22) and A2 = (4,5,6,7) .
Then,
A1 $\oplus$A2 = (14, 18, 20, 22) $\oplus$ (4,5,6,7)
= (18,23,26,29)
A1 ΘA2 = (14, 18 ,20,22) Θ (4,5,6,7)
= (7, 12, 15,18)

3. Centroid of Ranking Method

In this section we take the trapezoidal fuzzy numbers whose endpoints are represented by

$\tilde{A}$= (a,b,c,d; ω) [12]. The expression of the the membership functions corresponding to TrFNs is given by

$$\mu\tilde{A}(x) = \frac{w(x-a)}{b-a}; \quad a \le x \le b$$
$$\mu\tilde{A}(x) = w\,; \qquad b \le x \le c$$
$$\mu\tilde{A}(x) = \frac{w(d-x)}{d-c}; \quad c \le x \le d$$

When T_rFN has been taken

$$\int \mu_{\tilde{A}}(x)dx = \int_a^b \omega\left(\frac{x-a}{x-b}\right)dx + \int_b^c \omega dx + \int_c^d \omega\left(\frac{d-x}{d-c}\right)dx$$

$$= \frac{\omega}{(b-a)}\left(\frac{(x-a)2}{2}\right)\Big|_a^b + \omega(c-b) + \frac{\omega}{d-c}((d-x)^2 /-2\,)\Big|_c^d$$

$$= \frac{\omega}{2}(d + cb - a)$$

$\int x.\mu_{\tilde{A}}(x)dx=$

$$\int_a^b xw\left(\frac{x-a}{b-a}\right)dx + \int_b^c xw dx + \int_c^d xw\left(\frac{d-x}{d-c}\right)dx$$

$$= \frac{w}{b-a}\left[\left(\frac{(x-a)3}{3} + a\left(\frac{(x-a)2}{2}\right)\right]\Big|_a^b + w\left(\frac{c^2\ -b^2}{2}\right) + \frac{w}{(d-c)}\left[\frac{(d-x)^3}{3} - \frac{(d-x)^2}{2}\right]_c^d$$

$$= \frac{w(b-a)}{6}(2b+a) + w\left(\frac{c^2\ - b^2}{2}\right) + \frac{w(d-c)}{6}(d+2c)$$

$$= \frac{w}{6}(d^2 + c^2 - b^2 - a^2 - ab + cd)$$

$$R(\tilde{A}) = \frac{\int x.\mu_{\tilde{A}}(x)dx}{\int \mu_{\tilde{A}}(x)dx}$$

$$= \frac{1}{3}\left(\frac{d^2\ + c^2 - b^2 - a^2 - ab + cd}{d + c - b - a}\right)$$

$$= \frac{1}{3}\left[d + c + b + a - \left(\frac{cd - ab}{d+c-b-a}\right)\right]$$

$$R(\tilde{A}) = \frac{1}{3}\left[a + b + c + d - \left(\frac{cd - ab}{(c+d)-(a+b)}\right)\right] \qquad (5)$$

4. Fuzzy Critical Path Analysis

A Fuzzy project network is an acyclic digraph, where the vertices represent events and directed edges represent the activities to be performed in a project we denote this fuzzy project network by $\tilde{N} = (\tilde{V}, \tilde{A}, \tilde{T})$

Let $\tilde{v} = \{\tilde{v}_1, \tilde{v}_2 \tilde{v}_3 \ldots \cdot \tilde{v}_n\}$ be the set of fuzzy vertices (events), where $\tilde{v}_1$ and $\tilde{v}_n$ are the tail and head events of the project, and each $\tilde{v}_i$ belongs to some path from $\tilde{v}_1$ to $\tilde{v}_n$.

Let Let Ã= ($\tilde{v}$ X $\tilde{v}$)be the set of directed edges Ã={ $\tilde{a}_{ij}$ =($\tilde{v}_i, \tilde{v}_j$)/ for $\tilde{v}_i, \tilde{v}_j \in \tilde{v}$ }that represents the activities to be performed in the project. Activity $\tilde{a}_{ij}$ is then represented by one, and only one, arrow with a tail event $\tilde{v}_i$ and a head event $\tilde{v}_j$. For each activity $\tilde{a}_{ij}$, a fuzzy number $\tilde{t}_{ij} \in \tilde{T}$ is defined as the fuzzy time required for the completion of $\tilde{a}_{ij}$. A critical path is a longest path from the initial event $\tilde{v}_1$ to the terminal event to $\tilde{v}_n$ of the project network, and an activity $\tilde{a}_{ij}$ a on a critical path is called a critical activity.

4.1 Notations

$\tilde{t}_{ij}$ The fuzzy activity time of activity $\tilde{a}_{ij}$

$\tilde{E}s_j$: The earliest fuzzy time of event $\tilde{v}_j$

$\tilde{L}s_i$: The latest fuzzy time of event $\tilde{v}_i$

$\tilde{T}_{ij}$: The total float of fuzzy activity $\tilde{a}_{ij}$

P_i: The i-th path of the fuzzy project network.

P: The set of all paths in a fuzzy project network.

CPM (P_k): The fuzzy completion time of path P_k in a fuzzy project network.

Property 1 If $\tilde{a}_{ij}$ =($\tilde{v}_i, \tilde{v}_j$), $\tilde{a}_{mn}$ = ($\tilde{v}_m, \tilde{v}_n$) are two fuzzy activities, activity $\tilde{a}_{ij}$ is a predecessor of activity $\tilde{a}_{mn}$ iff there is a chain from event j to event m in project network.

Property 2

If $\tilde{a}_{ij}$ =($\tilde{v}_i, \tilde{v}_j$), $\tilde{a}_{mn}$ = ($\tilde{v}_m, \tilde{v}_n$) (are two fuzzy activities, activity $\tilde{a}_{ij}$ is an immediate predecessor of activity $\tilde{a}_{mn}$ iff either j = m, or there exists a chain from event j to event m in the project network consisting of dummy activities only.

Property 3

$$\textbf{CPM(Pk)} = \sum_{1 \le i < j \le n\ i,j \in P_k} \tilde{T}_{ij}, P_k \in P$$

Definition 4.1 Assume that there exists a path PC in a fuzzy project network such that CPM (PC) = min {CPM (Pi) | Pi ∈ P}, then the path PC is a fuzzy critical path.

Theorem 1 Assume that the fuzzy activity times of all activities in a project network are trapezoidal fuzzy numbers, then there exists fuzzy critical path in the project network

4.2 Fuzzy Critical Path Algorithm

Step 1: Construct network diagram according to Fulkerson rule

Step 2: Calculate Earliest starting time according to forward pass calculation

i.e., $E_j = Max_i \{E_i + \tilde{D}_{ij}\}$, i = no of preceding nodes

Step 3: Calculate Earliest finishing time EFT = EST + NT

Step 4: Calculate Latest Starting time according to backward pass calculation.

i.e., $L_i = Min_j \{L_i - \tilde{D}_{ij}\}$, j = number of succeeding nodes.

Step 5: Calculate the Latest Finishing time LFT = LST – NT

Step 6: Calculate Total Floating time TFT = LFT – EFT

Step 7: If TFT = 0 those activities are called critical activities

5. Description of the Model

Trapezoidal fuzzy number are converted into expected (normal time) by ranking method. These expected time treated as the time between the nodes and fuzzy critical path is calculated by using conventional method.

Numerical Example

Suppose that there is a project network with the set of fuzzy events $\tilde{v}$ = {1, 2, 3, 4, 5}, the fuzzy activity time for each activity is shown in Table 1. All the durations are in hours.

Table 1: Activity duration of each activity in a fuzzy project network

Activity	Activity duration
1-2	(10,15,15,20)
1-3	(30,40,40,50)
2-3	(30,40,40,50)
1-4	(15,20,25,30)
2-5	(60,100,150,180)
3-5	(60,100,150,180)
4-5	(60,100,150,180)

Consider the example mentioned above using the ranking method find the duration time of each activity.

Table 2: Activity duration of each activity and Expected duration in a fuzzy project network

Activity	Activity duration	Expected Duration
1-2	(10,15,15,20)	15
1-3	(30,40,40,50)	40
2-3	(30,40,40,50)	40
1-4	(15,20,25,30)	22.5
2-5	(60,100,150,180)	122.16
3-5	(60,100,150,180)	122.16
4-5	(60,100,150,180)	122.16

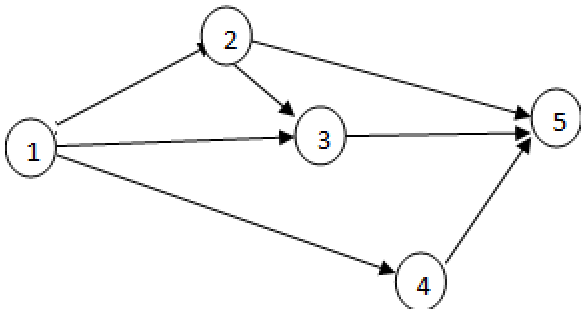

Figure 3: Fuzzy project network –I

The revised form is as follows

Activity	1-2	1-3	2-3	1-4	2-5	3-5	4-5
Duration (hours)	15	40	40	22.5	122.16	122.16	122.16

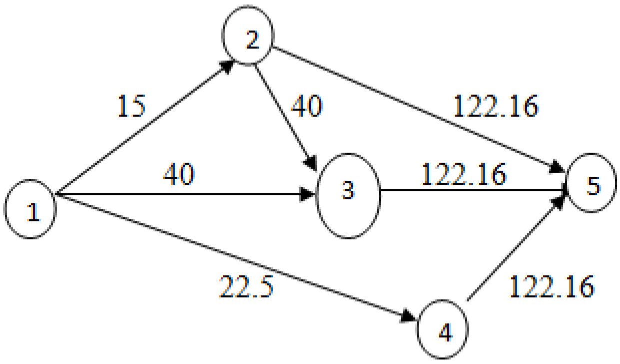

Fig 4: Project network-II

Now apply algorithm we obtain Critical path as 1-2-3-5
Critical path for fuzzy project network is 1-2-3-5

6. Conclusion

A new ranking method for finding critical path in a fuzzy project network has been proposed. We have used ranking formula for trapezoidal fuzzy number and applied to the duration time for each activity in the fuzzy project network to find the critical path. The comparison reveals that the method proposed in this paper is more effective in determining the activity criticalities and finding the critical path.

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