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# Precise Solutions of a Viscoelastic Fluid Flow in an Annular Pipe under an Impulsive Pressure with the Fractional Generalized Burgers' Model

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**Abstract:** ***This paper deals with an analytical study of flow of an incompressible generalized Burgers' fluid (GBF) in an annular pipe. We discussed in this problem the flow induced by an impulsive pressure gradient and compare the results with flow due to a constant pressure gradient. Analytic solutions for velocity is earned by using discrete Laplace transform (DLT) of the sequential fractional derivatives (FD) and finite Hankel transform (FHT). The influences of different parameters are analyzed on a velocity distribution characteristics and a comparison between two cases is also presented, and discussed in details. Eventually, the figures are plotted to exhibit these effects.***

**Keywords:** Generalized Burgers' fluid, Constant pressure gradient, impulsive pressure gradient.

## 1. Introduction

Non-Newtonian fluids do not illustrate a linear interconnection between the rate of strain and the stress , and acquired great attention for their diverts applications in technology and industry , like paints , polymers solutions , and heavy oils , the models of these fluids found in various manners with their constitutive equations which differ seriously in their complexity [1], [2].

The actions of all non-Newtonian fluids cannot be recited by an individual model, due to the fact that they own difficult behavior. A numerous basical equations for non-Newtonian fluid models were proposed.for instance, the Burgers' fluid which has a representative relationship between the shear stress and the strain rate that cannot be described , that is why many of models of constitutive equations have been suggested for those fluids [3], [4], [5]. Flow of viscoelastic fluid in an annular pipe of Burgers' model with fractional derivatives has been discussed by Hyder … etc. [6]. The flow of GBF in an annular pipe has also been discussed by Tong … [7].

Many applications of this fluid type can be found in [8], [9], [10], [11]. And the development in the viscoelastic flows theory have been mainly limited to the basical equations and fundamental models [12], [13]. Whereas a large number of fractional calculus implementations have been found in dynamics of fluid, nonlinear control theory, turbulence, and stochastic dynamical system [14], [15], [16]. In an annular pipe, the flow of unsteady rotating non-Newtonian fluid with oldroyd-B fluid model has been studied currently by Tong and Liu [17]. The oldroyd-B fluid flow in an annular pipe with fractional derivative has also been discussed by Tong … etc. [18].Later on , the results of MDH on the unsteady flow in an annular pipe of a viscoelastic fluid with a model of fractional GBF has been studied by Ibraheem and Abdulhadi[2].

In this paper, our target is to study the flow of unsteady viscoelastic fluid in an annular pipe with a model of fractional GBF under impulsive pressure, and compare it with flow under constant pressure. The exact solution for the distribution of velocity is performed by utilizing the FHT and DLT of the fractional sequential derivatives.

## 2. Dominant Equations

The constitutive equations for an incompressible GBF fractional are given by

$$\mathbf{T} = -p\mathbf{I} + \mathbf{S}\,,\ (1+\lambda_1^{\alpha}\tilde{D}_t^{\alpha} + \lambda_2^{\alpha}\tilde{D}_t^{2\alpha})\mathbf{S} = \mu(1+\lambda_3^{\beta}\tilde{D}_t^{\beta})\mathbf{A_1} \quad (1)$$

where $\mathbf{T}$ indicated the cauchy stress, $-p\mathbf{I}$ is the indeterminate spherical stress, $\mathbf{S}$ denoted the extra stress tensor, $\mathbf{A_1} = \mathbf{L} + \mathbf{L}^{T}$ is the first Rivlin- Ericksen tensor with the velocity gradient where $\mathbf{L} = \text{grad}\,\mathbf{V}$, $\mu$ indicated the fluid dynamic viscosity, $\lambda_1$ and $\lambda_3$ ($<\lambda_1$)are the repose and tardiness times respectively, $\lambda_2$ is the Burger's fluid new material parameter, $\alpha$ and $\beta$ are the parameters of the fractional calculus such as $0 \le \alpha \le \beta \le 1$ and $\tilde{D}_t^{p}$ the upper convected fractional derivative describe by

$$\tilde{D}_t^{\alpha}\mathbf{S} = D_t^{\alpha}\mathbf{S} + (\mathbf{V}.\nabla)\mathbf{S} - \mathbf{LS} - \mathbf{SL}^{T} \quad (2)$$

$$\tilde{D}_t^{\beta}\mathbf{A_1} = D_t^{\beta}\mathbf{A_1} + (\mathbf{V}.\nabla)\mathbf{A_1} - \mathbf{LA_1} - \mathbf{A_1L}^{T} \quad (3)$$

in which $D_t^{\alpha}$ and $D_t^{\beta}$ are the fractional differentiation operators of order $\alpha$ and $\beta$ depend on the Riemann-Liouville definition, identify as

$$D_t^{p}[f(t)] = \frac{1}{\Gamma(1-p)}\frac{d}{dt}\int_0^t (t-\tau)^{-p} f(\tau)d\tau \qquad ,0 \le p \le 1$$

$$\text{and } D_t^{2p}\mathbf{S} = D_t^{p}(D_t^{p}\mathbf{S}) \quad (4)$$

here $\Gamma(.)$ is the Gamma function.

The model diminished to the generalized Oldroyd- B (O-B) model when and in addition to that, if the model of ordinary O-B ought to earn. We suppose for the unidirectional flow the following form for both of velocity field and shear stress

$$\mathbf{V}(\mathbf{r},\mathbf{t}) = \omega(r,t)e_y \quad ,\mathbf{S} = \mathbf{S}(r,t) \quad (5)$$

where indicated the unit vector y - along direction. Replacing equation (5) into (1) and observance of the initial situation

$$\mathbf{S}(r,0) = 0 \quad (6)$$

we earned

$$(1+\lambda_1^\alpha D_t^\alpha + \lambda_2^\alpha D_t^{2\alpha})S_{ry} = \mu(1+\lambda_3^\beta D_t^\beta)\partial_r \omega(r,t)$$

$$(1+\lambda_1^\alpha \mathrm{D}_\mathrm{t}^\alpha + \lambda_2^\alpha \mathrm{D}_\mathrm{t}^{2\alpha})\mathrm{S}_\mathrm{yy} - 2\,\mathrm{S}_\mathrm{ry}(\lambda_1^\alpha + \lambda_2^\alpha \mathrm{D}_\mathrm{t}^\alpha)\partial_\mathrm{r}\omega(r,t) = -2\mu\,\lambda_3^\beta(\partial_r \omega(r,t))^2 \quad (7)$$

where $\mathrm{S}_{\mathrm{rr}} = \mathrm{S}_{rx} = \mathrm{S}_{xy} = \mathrm{S}_{xx} = 0$. Furthermore, in the existence of pressure gradient in y- direction, the motion equation supply the following equation of scalar:

$$\rho\frac{d\omega}{dt} = \frac{\partial P}{\partial y} + \frac{1}{r}\frac{\partial}{\partial r}(rS_{ry}) \quad (8)$$

here indicated the fixed density of the fluid. Removing amidst (7) and (8) eqs., we obtained the following differential fractional equation

$$(1+\lambda_1^\alpha \mathrm{D}_\mathrm{t}^\alpha + \lambda_2^\alpha \mathrm{D}_\mathrm{t}^{2\alpha})\frac{\partial\omega}{\partial \mathrm{t}} = -\frac{1}{\rho}(1+\lambda_1^\alpha \mathrm{D}_\mathrm{t}^\alpha + \lambda_2^\alpha \mathrm{D}_\mathrm{t}^{2\alpha})\frac{\mathrm{dP}}{\mathrm{dy}} + v(1+\lambda_3^\beta \mathrm{D}_\mathrm{t}^\beta)(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r})\omega \quad (9)$$

where $v = \frac{\mu}{\rho}$ denoted the kinematic viscosity.

## 3. Flow of Plane Poiseuille

Regard the problem of an incompressible generalized Burgers' fluid flow is firstly at rest between two long coaxial infinitely cylinders of radii $R_0$ and $R_1$ $(> R_0)$. At time $t = 0^+$ the fluid is generated as a result of an impulsive pressure gradient that acts on liquid in y- direction. Pointing to Eq. (9), the coinciding differential fractional partial equation which describe such flow has the following form

$$(1+\lambda_1^\alpha \mathrm{D}_\mathrm{t}^\alpha + \lambda_2^\alpha \mathrm{D}_\mathrm{t}^{2\alpha})\frac{\partial\omega}{\partial \mathrm{t}} = -\mathrm{k}\left(1+\lambda_1^\alpha\frac{t^{-\alpha-1}}{\Gamma(-\alpha)} + \lambda_2^\alpha\frac{t^{-2\alpha-1}}{\Gamma(-2\alpha)}\right) + v(1+\lambda_3^\beta \mathrm{D}_\mathrm{t}^\beta)(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r})\omega \quad (10)$$

where $\mathrm{K}\delta(\mathrm{t}) = \frac{1}{\rho}\frac{dp}{dy}$ indicated the fixed pressure gradient

The related beginning and ending states are as follows

$$\omega(r,0) = \frac{\partial}{\partial t}\omega(r,0) = \frac{\partial^2}{\partial t^2}\omega(r,0) = 0 \qquad ,R_0 \le r \le R_1 \quad (11)$$

$$\omega(R_0,t) = \omega(R_1,t) = 0 \qquad ,t > 0 \quad (12)$$

To earn the accurate analytical resolution of the previous problem (10)- (12), First, we applied the principle of Laplace transform [19] with regard to t, that we acquired

$$\mathrm{s}\,(1+\lambda_1^\alpha s^\alpha + \lambda_2^\alpha s^{2\alpha})\overline{\omega} = -\mathrm{K}(1+\lambda_1^\alpha s^\alpha + \lambda_2^\alpha s^{2\alpha}) + v(1+\lambda_3^\beta s^\beta)(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r})\overline{\omega} \quad (13)$$

$$\overline{\omega}(r,0) = 0$$
$$\overline{\omega}(R_0,s) = \overline{\omega}(R_1,s) = 0 \qquad ,t > 0 \quad (14)$$

where $\overline{\omega}(r,s)$ denoted the image function of $\omega(r,t)$ and $s$ denoted a converting parameter. We use the restricted Hankel transform [19], described like the follows

$$\overline{\omega}_H = \int_{R_0}^{R_1} r\,\overline{\omega}\,B_0(rk_i)dr \qquad , i = 1,2,3,\ldots \quad (15)$$

and its inverse is

$$\overline{\omega} = \frac{\pi^2}{2}\sum_{i=1}^{\infty}\frac{k_i^2\overline{\omega}_H\,B_0(rk_i)J_0^2(R_1k_i)}{J_0^2(R_0k_i) - J_0^2(R_1k_i)} \quad (16)$$

where $k_i$ are the positive roots of equation $B_0(R_1k_i) = 0$ and $B_0(rk_i) = J_0(rk_i)Y_0(R_0k_i) - Y_0(rk_i)J_0(R_0k_i)$

where $J_0(.)$ and $Y_0(.)$ are the functions of Bessel of both the first and second types of order zero.

Yet, using the restricted Hankel transform to (13)-(14) Eqs. with respect to r, that we obtained

$$\overline{\omega}_H = -K\frac{(1+\lambda_1^\alpha s^\alpha + \lambda_2^\alpha s^{2\alpha})}{\mathrm{s}(1+\lambda_1^\alpha s^\alpha + \lambda_2^\alpha s^{2\alpha}) + \nu\,\mathrm{k}_\mathrm{i}^2(1+\lambda_3^\beta s^\beta)} \quad (17)$$

Now, writing Eq. (17) in series form as

$$\overline{\omega}_H = -K(1+\lambda_1^\alpha s^\alpha + \lambda_2^\alpha s^{2\alpha})\sum_{k=0}^{\infty}(-1)^k \sum_{a,b,c\ge 0}^{a+b+c=k} k!\,\frac{(\nu k_i^2)^c(\lambda_1^\alpha)^{-k-1}(\lambda_2^\alpha)^b(\lambda_3^\beta)^c s^\delta}{a!b!c!\left(s^{\alpha+1} + \frac{\nu k_i^2}{\lambda_1^\alpha}\right)^{k+1}} \quad (18)$$

where $\delta = k + 2\alpha b + c(\beta - 1)$. And its separated reverse Laplace transform [19] will take the following form

$$\omega_H = -K\sum_{k=0}^{\infty}(-1)^k\sum_{a,b,c\ge 0}^{a+b+c=k}\frac{(\nu k_i^2)^c(\lambda_1^\alpha)^{-k-1}(\lambda_2^\alpha)^b(\lambda_3^\beta)^c}{a!b!c!}t^{(\alpha+1)k+(\alpha+1-\delta)-1}$$
$$\left\{E_{\alpha+1,\alpha+1-\delta}^k\left(-\frac{vk_i^2}{\lambda_1^\alpha}t^{\alpha+1}\right) + t^\alpha\lambda_1^\alpha E_{\alpha+1,1-\delta}^k\left(-\frac{vk_i^2}{\lambda_1^\alpha}t^{\alpha+1}\right) + t^{2\alpha}\lambda_2^\alpha E_{\alpha+1,1-\alpha-\delta}^k\left(-\frac{vk_i^2}{\lambda_1^\alpha}t^{\alpha+1}\right)\right\} \quad (19)$$

where $E_{\alpha,\beta}^m(z) = \sum_{j=0}^{\infty}\frac{(j+m)!z!}{j!\Gamma(\alpha j + \alpha m + \beta)}$ indicated the generalized Mittag- Leffler function [19] andtoearn Eq. (19), the following feature of reverse Laplace transform is applied [19]

$$L^{-1}\left\{\frac{m!\,s^{\lambda-\mu}}{(s^\lambda \mp c)}\right\} = t^{\lambda m+\mu-1}E_{\lambda,\mu}^m(\pm ct^\lambda) \qquad , \left(\mathrm{Re}(s) > |c|^{\frac{1}{\lambda}}\right) \quad (20)$$

eventually, the reverse restricted Hankel transform obtains the analytic resolution of speed distribution

$$\omega(r,t) = -\frac{K\pi^2}{2}\sum_{i=1}^{\infty}\frac{k_i^2B_0(rk_i)J(R_ik_i)}{J_0^2(R_0k_i) - J_0^2(R_ik_i)}$$
$$\left[\sum_{k=0}^{\infty}(-1)^k\sum_{a,b,c\ge 0}^{a+b+c=k}\frac{(\nu k_i^2)^c(\lambda_1^\alpha)^{-k-1}(\lambda_2^\alpha)^b(\lambda_3^\beta)^c}{a!b!c!d!n!}t^{(\alpha+1)k+(\alpha+1-\delta)-1}\left\{E_{\alpha+1,\alpha+1-\delta}^k\left(-\frac{vk_i^2}{\lambda_1^\alpha}t^{\alpha+1}\right) + t^\alpha\lambda_1^\alpha E_{\alpha+1,1-\delta}^k\left(-\frac{vk_i^2}{\lambda_1^\alpha}t^{\alpha+1}\right) + t^{2\alpha}\lambda_2^\alpha E_{\alpha+1,1-\alpha-\delta}^k\left(-\frac{vk_i^2}{\lambda_1^\alpha}t^{\alpha+1}\right)\right\}\right] \quad (21)$$

### 3.1 The Limiting Status

Making the limits of Eq. number (21), when $\alpha \ne 0$, $\lambda_2 \to 0$ (b=0), we could obtained the distribution of velocity for a generalized Oldroyd- B fluid. Thus the field of velocity reduces to

$$\omega(r,t) = -\frac{K\pi^2}{2}\sum_{i=1}^{\infty}\frac{k_i^2\, B_0(rk_i)J_0^2(R_1k_i)}{J_0^2(R_0k_i) - J_0^2(R_1k_i)}$$
$$\left[\sum_{k=0}^{\infty}(-1)^k \sum_{a+c=k}^{a,c\geq 0}\frac{(\nu k_i^2)^c(\lambda_1^{\alpha})^{-k-1}(\lambda_3^{\beta})^c}{a!\, c!}t^{(\alpha+1)k+(\alpha+1-\delta)-1} \right. \tag{23}$$
$$\left.\left\{E^k_{\alpha+1,\alpha+1-\delta}\left(-\frac{\nu k_i^2}{\lambda_1^{\alpha}}t^{\alpha+1}\right) + \frac{\lambda_1^{\alpha}}{t^{\alpha}}E^k_{\alpha+1,\alpha+1-\delta}\left(-\frac{\nu k_i^2}{\lambda_1^{\alpha}}t^{\alpha+1}\right)\right\}\right]$$

where $\delta = k + c(\beta - 1)$.

## 4. Discussion and Numerical Results

In this topic, the flow in an annular pipe was discussed due to impulsive pressure gradient for the generalized Burger's fluid. The accurate resolution for the field of velocity $u$ is obtained by applying the detached Laplace and restricted Hankel transforms. In addition, few figures were plotted to revealed the behavior of various parameters included in velocity expressions $u$.

A rapprochement between flow due to impulsive pressure gradient (Panel a) and the flow due to constant pressure gradient (Panel b) is likewise made graphically in Figs 1-6.

Fig. 1 was provided the graphical illustrations for the effect of the non- integer fractional parameter $\alpha$ on the fields of velocity. Velocity is decreasing with the increased the $\alpha$ for the flow as a result to the impulsive pressure gradient, whereas quite opposite effect was observed for the flow due to a constant pressure gradient.

Fig. 2 showed that the field of velocity is increased with the increasing the $\beta$ of both cases.

Fig. 3 provided the graphical explanation of the effect of repose parameter $\lambda_1$ on the fields of velocity. The velocity is decreased with the increase of $\lambda_1$ for both cases.

Figs. 4 and 5 were prepared to show the effect of the material parameter $\lambda_2$ and the tardiness parameter $\lambda_3$ on the field of velocity. The field of velocity has similar behavior for both cases the velocity is increase with the increase of $\lambda_2$ and $\lambda_3$.

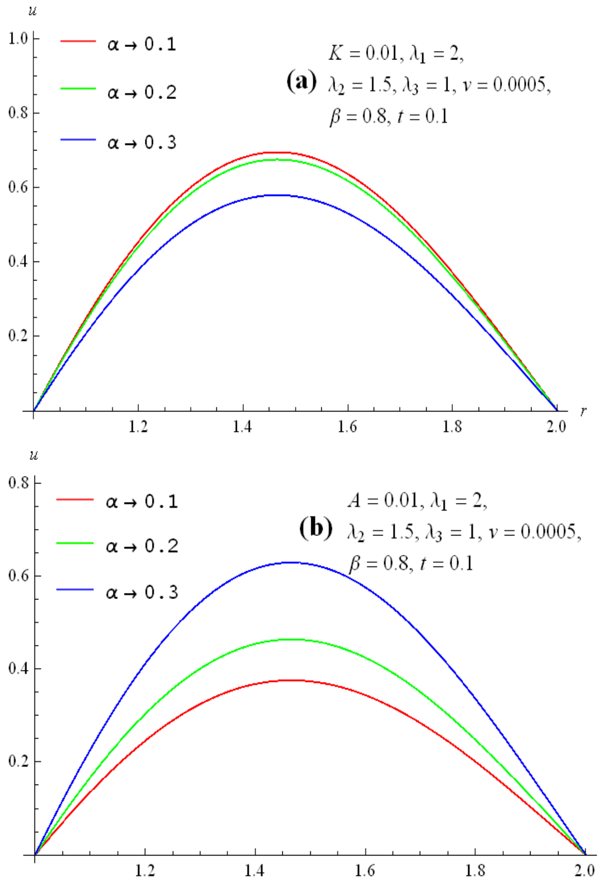


**Figure 1:** Velocity of different value of $\alpha$ while maintaining another parameters constant a) flow due to impulsive pre. grad. b) flow due to constant pre. grad.

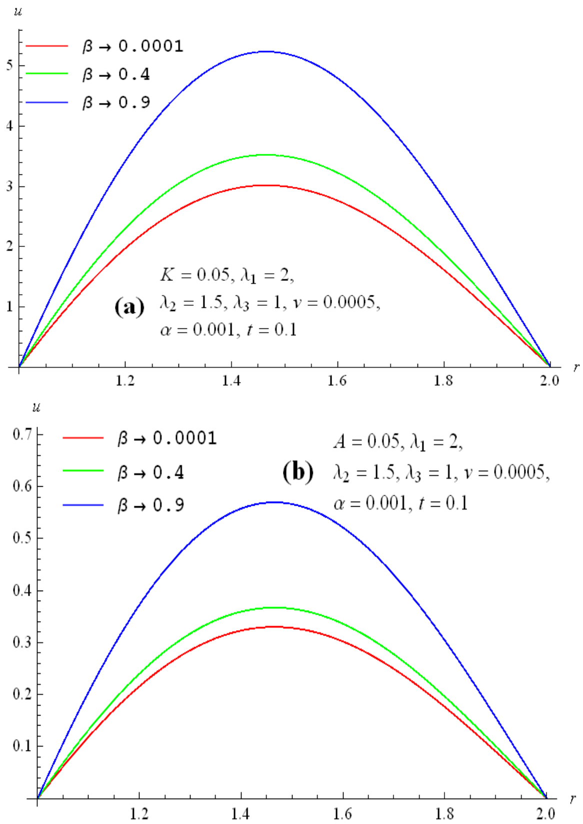


**Figure 2:** Velocity of different value of $\beta$ while maintaining another parameters constant a) flow due to impulsive pre. grad. b) flow due to constant pre. grad.

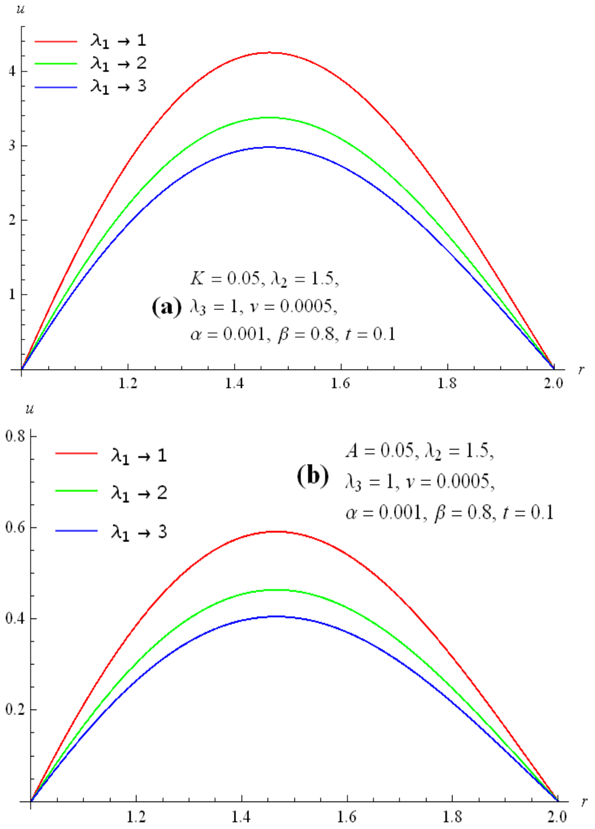


**Figure 3:** Velocity of different value of $\lambda_1$ while maintaining another parameters constant a) flow due to impulsive pre. grad. b) flow due to constant pre. grad.

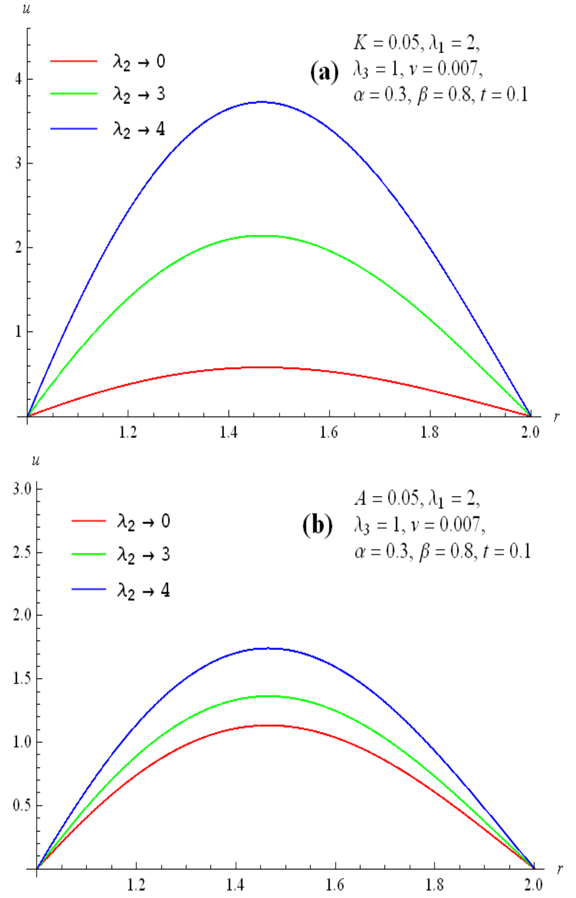


**Figure 4:** Velocity of different value of $\lambda_2$ while maintaining another parameters constant a) flow due to impulsive pre. grad. b) flow due to constant pre. grad.

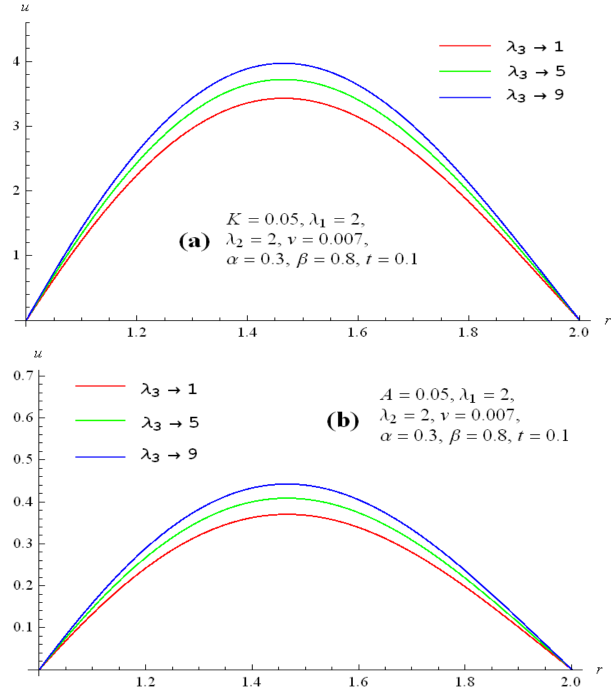


**Figure 5:** Velocity of different value of $\lambda_3$ while maintaining another parameters constant a) flow due to impulsive pre. grad. b) flow due to constant pre. grad.

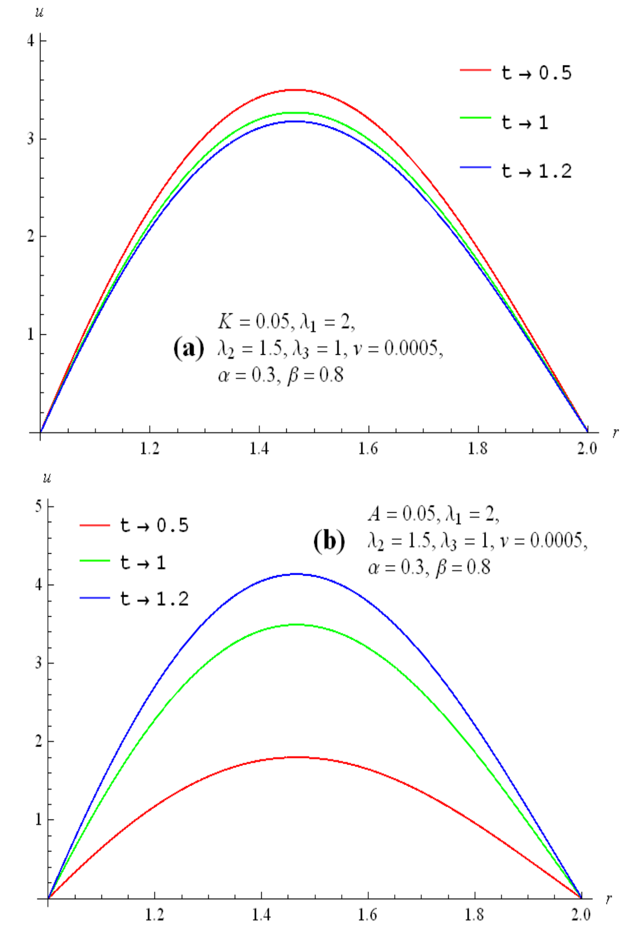


**Figure 6:** Velocity of different value of $t$ while maintaining another parameters constant a) flow due to impulsive pre. grad. b) flow due to constant pre. grad.

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