International Journal of Science and Research (IJSR)
ISSN (Online): 2319-7064
Index Copernicus Value (2015): 78.96 | Impact Factor (2015): 6.391

Derivation of Cycle Index Formulas for Dihedral Group Acting on Unordered Triples

Geoffrey Muthoka

Pure and Applied Sciences Department, Kirinyaga University, P.O. Box 143-10300, Kerugoya

Abstract: ***The cycle index of dihedral group $D_n$acting on the set X of the vertices of a regular n-gon was studied by Harary and Palmer in 1973 (See [1]). In this paper we derive the cycle index formulas of the dihedral group(D_n) acting on unordered triples from the set $X = \{1,2,\ldots,n\}$. In each case the actions of the cyclic part and the reflection part are studied separately for both an even value of n and an odd value of n.***

Keywords: Cycle index, Cycle type, Monomial

1. Introduction

The concept of the cycle index was discovered by Polya (See [2]) and he gave it its present name. He used the cycle index to count graphs and chemical compounds via the Polya's Enumeration Theorem. More current cycle index formulas include the cycle index of the reduced ordered triples groups $S_n^{[3]}$ (See [3]) which was further extended by Kamuti and Njuguna to cycle index of the reduced ordered r-group $S_n^{[r]}$ (See [4]). The Cycle Index of Internal Direct Product Groups was done in 2012 (See [5]).

2. Definitions and Preliminaries

Definition 1
The cycle index of the action of G on X is the polynomial (say over the rational field$Q$) in $t_1, t_2, \ldots, t_n$given by; $Z(G) = Z_{G,X}(t_1, t_2, \ldots t_n) = \frac{1}{|G|}\sum_{g\in G}\{mon(g)\}$.

Definition 2
A cycle type of a permutation is the data of how many cycles of each length are present in the cycle decomposition of the permutation.

Definition 3
A monomial is a product of powers of variables with nonnegative integer exponents possibly with repetitions.

Preliminary result 1
Suppose $g\epsilon D_n$with $mon(g) = t_1^{\alpha_1}\ldots t_n^{\alpha_n}$ and g' is the corresponding permutation in$D_n^{(3)}$. To obtain an expression for $Z\left(D_n^{(3)}\right)$we need to find $mon(g')$ for every $g\epsilon D_n$.

We have to consider the following cases:

(i) Contributions whereby all the three points lying in a triple come from a common cycle of length (k≥ 3) divisible by 3, (say $k = 3m$).
(ii) Contributions whereby all the three points lying in a triple come from a common cycle of g of length $(k \geq 4)$ not divisible by 3.
(iii) Contributions whereby a pair of points in a triple comes from a common cycles of g and the remaining point comes from a different cycle.
(iv) Contributions whereby all the three points of a triple come from different cycles of g.

We now consider the cases listed above:

i) Contributions whereby all the three points lying in a triple come from a common cycle of length (k≥ 3) divisible by 3, (say $k = 3m$).

In this case we have the contribution;

$$t_{3m}^{\alpha_{3m}} \longrightarrow \left(b_m b_{3m}^{\frac{3m(m-1)}{2}}\right)^{\alpha_{3m}} \quad (2.1)$$

ii) Contributions whereby all the three points lying in a triple come from a common cycle of g of length $(k \geq 4)$ not divisible by 3.

In this case we have the contribution;

$$t_m^{\alpha_m} \longrightarrow b_m^{\frac{(m-1)(m-2)\alpha_m}{6}} \quad (2.2)$$

iii) Contributions where by a pair of points in a triple comes from a common cycles of g and the remaining point comes from a different cycle.

(a) If a pair of points in a triple comes from a common cycles of even length (say 2m) and the remaining point comes from a cycle of length q then we have;

If $q \neq 2m$ then,

$$t_{2m}^{\alpha_{2m}} t_q^{\alpha_q} \longrightarrow \left(b_{[2m,q]}^{(m-1)(2m,q)} b_{[m,q]}^{(m,q)}\right)^{\alpha_q \alpha_{2m}} \quad (2.3)$$

if$q = 2m$ then,

$$t_{2m}^{\alpha_{2m}} \longrightarrow b_{2m}^{2m(2m-1)\binom{\alpha_{2m}}{2}} \quad (2.4)$$

(b) If a pair of points in a triple comes from a common cycles of odd length (say 2m+1) and the remaining point comes from a cycle of length r then we have;

If $r \neq 2m+1$

$$t_{2m+1}^{\alpha_{2m+1}} t_r^{\alpha_r} \longrightarrow b_{[2m+1,r]}^{m(2m+1,r)\alpha_r \alpha_{2m+1}} \quad (2.5)$$

If $r = 2m+1$

$$t_{2m+1}^{\alpha_{2m+1}} \longrightarrow b_{2m+1}^{2m(2m+1)\binom{\alpha_{2m+1}}{2}} \quad (2.6)$$

iv) Contributions where by all the three points of a triple come from different cycles of g (say d, e and f) then we have:

If $d \neq e \neq f$

$$t_d^{\alpha_d} t_e^{\alpha_e} t_f^{\alpha_f} \longrightarrow b_{[d,e,f]}^{\frac{def}{[d,e,f]}\alpha_d \alpha_e \alpha_f} \quad (2.7)$$

If $d = e \neq f$

$$t_d^{\alpha_d} t_f^{\alpha_f} \longrightarrow b_{[d,f]}^{d(df)\binom{\alpha_d}{2}\alpha_f} \quad (2.8)$$

If $d = e = f = m$

$$t_m^{\alpha_m} \longrightarrow b_m^{m^2\binom{\alpha_m}{3}} \quad (2.9)$$

For the proofs and details of the above results see [1].

Preliminary result 2

The cycle index formulas of dihedral group D_n acting on the set X of the vertices of a regular n-gon are given by:

$$Z_{D_n,X} = \frac{1}{2n}\left[\sum_{d|n}\phi(d)t_d^{\frac{n}{d}} + \frac{n}{2}t_1^2 t_2^{\frac{n-2}{2}} + \frac{n}{2}t_2^{\frac{n}{2}}\right] \quad 2.10(a)$$

if n is even and

$$Z_{D_n,X} = \frac{1}{2n}\left[\sum_{d|n}\phi(d)t_d^{\frac{n}{d}} + nt_1 t_2^{\frac{n-1}{2}}\right] \quad 2.10(b)$$

if n is odd. Where ϕ is the Euler's phi formula.

The proof to these important results can be found in several books and articles (e.gSee[6], [7] and [1])

3. Cycle index of D_n acting on unordered triples

We first consider the cyclic part.

We note that from 2.10(a)and 2.10(b)the cycle index of the cyclic part of D_n acting on the set X of the n vertices of a regular n-gon for both n even and odd is given by;

$$Z_{C_n} = \frac{1}{n}\sum_{d|n}\phi(d)t_d^{\frac{n}{d}}$$

We now consider all the possible cases that can be made from $t_d^{\frac{n}{d}}$.

If 3|d then we have the following;

(a) If three elements lying in a triple comes from the same cycle then from 4.1.1 we have;

$$t_d^{\frac{n}{d}} \longrightarrow \left(b_{\frac{d}{3}} b_d^{\frac{d\left(\frac{d}{3}-1\right)}{2}}\right)^{\frac{n}{d}} \quad (3.1)$$

(b) If a pair of elements in a triple comes from a cycle of length d and the other element from a cycle of length d since the lengths of the 2 cycles are equal, then the induced monomials will be the same for the two cases (i.e. if d is odd or even) then from (2.6) we have;

$$t_d^{\frac{n}{d}} \longrightarrow b_d^{d(d-1)\binom{n/d}{2}} \quad (3.2)$$

(c) If all the three elements in a triple come from different cycles of g, then all these cycles must be of length d and hence from (2.9) we have;

$$t_d^{\frac{n}{d}} \longrightarrow b_d^{d^2\binom{n/d}{3}} \quad (3.3)$$

Multiplying (3.1), (3.2) and (3.3) we have;

$$t_d^{\frac{n}{d}} \longrightarrow b_{\frac{d}{3}}^{\frac{n}{d}} b_d^{\frac{n}{2}\left(\frac{d}{3}-1\right)+d(d-1)\binom{\frac{n}{d}}{2}+d^2\binom{n/d}{3}}$$

$$= b_{\frac{d}{3}}^{\frac{n}{d}} b_d^{\frac{n^2(n-3)}{6d}} \quad (3.4)$$

If 3∤d then we have the following;

If the three elements lying in a triple come from the same cycle then from (3.2) we have;

$$t_d^{\frac{n}{d}} \longrightarrow b_d^{\frac{n(d-1)(d-2)}{6d}} \quad (3.5)$$

(a) If a pair of the elements comes from a cycle of length d and the other element comes from another cycle of length d. Since the lengths of the 2 cycles are equal then the induced monomials will be the same for the two cases (i.e. if d is odd or even) then from (2.6) we have;

$$t_d^{\frac{n}{d}} \longrightarrow b_d^{d(d-1)\binom{n/d}{2}} \quad (3.6)$$

(b) If all the three elements in a triple come from different cycles of g, then all these cycles must be of length d and hence from (2.9) we have;

$$t_d^{\frac{n}{d}} \longrightarrow b_d^{d^2\binom{n/d}{3}} \quad (3.7)$$

Multiplying (3.5), (3.6) and (3.7) we have;

$$t_d^{\frac{n}{d}} \longrightarrow b_d^{d^2\binom{n/d}{3}+d(d-1)\binom{\frac{n}{d}}{2}+\frac{n(d-1)(d-2)}{6d}}$$

$$= b_d^{\frac{n(n-1)(n-2)}{6d}} \quad (3.8)$$

Therefore the cycle index of C_n acting on unordered triples is given by;

$$Z_{C_n,X^{(3)}}=\frac{1}{n}\sum_{\substack{d|n\\3|d}}\left[\phi(d)\,b_{\frac{d}{3}}^{\frac{n}{d}}\,b_d^{\frac{n^2(n-3)}{6d}}+\sum_{\substack{d|n\\3\nmid d}}\phi(d)\,b_d^{\frac{n(n-1)(n-2)}{6d}}\right] \quad (3.9)$$

If n is even we now consider the reflection part $t_1^2 t_2^{\frac{n-2}{2}}$ from 2.10(a)

If a pair of the elements lying in a triple comes from a cycle of length 2 and the other element come from another cycle of length 2, then from 2.4 we have;

$$t_2^{\frac{n-2}{2}} \longrightarrow b_2^{2\binom{\frac{n-2}{2}}{2}} = b_2^{\frac{n^2-6n+8}{4}} \quad (3.10)$$

If a pair of the elements lying in a triple comes from a cycle of length 2 and the other element comes from another cycle of length one, then from (2.3) we have;

$$t_1^2 t_2^{\frac{n-2}{2}} \longrightarrow (b_2^0 b_1^1)^{n-2} = b_1^{n-2} \quad (3.11)$$

If all the three elements in a triple come from different cycles of g, with 2 elements coming from cycles of length 1 and the other one from a cycle of length 2, then from (2.8) we have;

$$t_1^2 t_2^{\frac{n-2}{2}} \longrightarrow b_2^{\frac{n-2}{2}} \quad (3.12)$$

If all the three elements in a triple come from different cycles of g, with 2 elements coming from cycles of length two and and the other from a cycle of length one, then from (2.8) we have;

$$t_1^2 t_2^{\frac{n-2}{2}} \longrightarrow b_2^{\frac{n^2-6n+8}{2}} \quad (3.13)$$

If each element in a triple comes from a different cycle of length 2 then from (2.9) we have;

$$t_2^{\frac{n-2}{2}} \longrightarrow b_2^{2^2\binom{\frac{n-2}{2}}{3}} \quad (3.14)$$

Multiplying (3.10), (3.11), (3.12), (3.13) and (3.14) we have;

$$t_1^2 t_2^{\frac{n-2}{2}} \longrightarrow b_1^{n-2} b_2^{2^2\binom{\frac{n-2}{2}}{3}+\frac{n^2-6n+8}{2}+\frac{n-2}{2}+\frac{n^2-6n+8}{4}}$$

$$= b_1^{n-2} b_2^{\frac{(n^2-4)(n-3)}{12}}$$

From 2.10(a), we have $\frac{n}{2}$ monomials of the form $t_1^2 t_2^{\frac{n-2}{2}}$ and hence a total of $\frac{n}{2}$ monomials will be induced giving;

$$\frac{n}{2} t_1^2 t_2^{\frac{n-2}{2}} \longrightarrow \frac{n}{2} b_1^{n-2} b_2^{\frac{(n^2-4)(n-3)}{12}} \quad (3.15)$$

Next we consider the part $t_2^{\frac{n}{2}}$.

If a pair of the elements lying in a triple comes from a cycle of length two and the other element comes from another cycle of length two, then from 2.4) we have;

$$t_2^{\frac{n}{2}} \longrightarrow b_2^{2\binom{n/2}{2}} = b_2^{\frac{n(n-2)}{4}} \quad (3.16)$$

If each element in a triple comes from a different cycle of length 2 then from (2.9) we have;

$$t_2^{\frac{n}{2}} \longrightarrow b_2^{2^2\binom{n/2}{3}} = b_2^{\frac{n^3-6n^2+8n}{12}} \quad (3.17)$$

Multiplying (3.16) and (3.17) we have;

$$t_2^{\frac{n}{2}} \longrightarrow b_2^{\frac{n(n-2)}{4}+\frac{n^3-6n^2+8n}{12}} = b_2^{\frac{n(n-1)(n-2)}{12}}$$

From 2.10(a), we have $\frac{n}{2}$ monomials of the form $t_2^{\frac{n}{2}}$ and hence $\frac{n}{2}$ monomials have to be induced giving;

$$\frac{n}{2} t_2^{\frac{n}{2}} \longrightarrow \frac{n}{2} b_2^{\frac{n(n-1)(n-2)}{12}} \quad (3.18)$$

To get the cycle index of D_n acting on $X^{(3)}$ when n is even we sum(3.9), (3.15) and (3.18) we have the cycle inedx below when n is even.

$$Z_{D_n,X^{(3)}} = \frac{1}{2n}\left[\sum_{\substack{d|n\\3|d}}\phi(d)\,b_{\frac{d}{3}}^{\frac{n}{d}}\,b_d^{\frac{n^2(n-3)}{6d}}+\sum_{\substack{d|n\\3\nmid d}}\phi(d)\,b_d^{\frac{n(n-1)(n-2)}{6d}}+\frac{n}{2}b_1^{n-2}b_2^{\frac{(n^2-4)(n-3)}{12}}+\frac{n}{2}b_2^{\frac{n(n-1)(n-2)}{12}}\right] \quad (3.19)$$

4. For an odd value of n

Since the cyclic part is the same for both even and odd values of n (from2.10(a) and 2.10(b)) we next consider the reflection part when n is odd $t_1 t_2^{\frac{n-1}{2}}$.

If a pair of the elements lying in a triple comes from a cycle of length two and the other element comes from another cycle of length one, then from (2.3) we have;

$$t_1 t_2^{\frac{n-1}{2}} \longrightarrow (b_2^0 b_1^1)^{\frac{n-1}{2}} = b_1^{\frac{n-1}{2}} \quad (4.1)$$

If a pair of the elements lying in a triple comes from a cycle of length two and the other element comes from another cycle of length two, then from (2.4) we have;

$$t_2^{\frac{n-1}{2}} \longrightarrow b_2^{2\binom{\frac{n-1}{2}}{2}} = b_2^{\frac{n^2+3-4n}{4}} \quad (4.2)$$

If all the three elements in a triple come from different cycles of g, with two elements coming from cycles of length two and and the other from a cycle of length one, then from (2.8) we have;

$$t_1 t_2^{\frac{n-1}{2}} \longrightarrow b_2^{\frac{n^2+3-4n}{4}} \quad (4.3)$$

If all the elements in a triple come from different cycles of length two then from (2.9) we have;

$$t_2^{\frac{n-1}{2}} \longrightarrow b_2^{2^2\binom{\frac{n-1}{2}}{3}} = b_2^{\frac{n^3+9n^2+23n-15}{12}} \quad (4.4)$$

Multiplying(4.1), (4.2), (4.3) and (4.4) we have;

$$t_1 t_2^{\frac{n-1}{2}} \longrightarrow b_1^{\frac{n-1}{2}} b_2^{\frac{n^3+9n^2+23n-15}{12}+\frac{n^2+3-4n}{4}+\frac{n^2+3-4n}{4}}$$

$$= b_1^{\frac{n-1}{2}} b_2^{\frac{(n^2-1)(n-3)}{12}}.$$

From 2.10(b) we have *n* monomials of the form $t_1 t_2^{\frac{n-1}{2}}$ and hence *n* monomials have to be induced giving;

$$n t_1 t_2^{\frac{n-1}{2}} \longrightarrow n b_1^{\frac{n-1}{2}} b_2^{\frac{(n^2-1)(n-3)}{12}} \qquad (4.5)$$

To get the cycle index of D_n acting on $X^{(3)}$ when n is odd we sum(3.9) and (4.5)

$$Z_{D_n,X^{(3)}} = \frac{1}{2n}\left[\sum_{\substack{d|n \\ 3|d}} \phi(d)\, b_{\frac{d}{3}}^{\frac{n}{d}} b_d^{\frac{n^2(n-3)}{6d}} + \sum_{\substack{d|n \\ 3\nmid d}} \phi(d)\, b_d^{\frac{n(n-1)(n-2)}{6d}} + n b_1^{\frac{n-1}{2}} b_2^{\frac{(n^2-1)(n-3)}{12}}\right] \quad (4.6)$$

Example 1

Let $n = 6$, then the dihedral group D_6 of degree 6 acting on unordered triples of the set $X = \{1,2,3,4,5,6\}$. Then;

$|D_6| = 12, \quad d = 1,2,3,6$

$\phi(1) = 1, \phi(2) = 1, \phi(3) = 2$ and

$\phi(6) = 2$

We note that in this case n is even, 3 and 6 are divisible by 3 and 1 and 2 are not divisible by 3and hence from (3.19) we have;

$$Z_{D_6,X^{(3)}} = \frac{1}{12}\left[b_1^{20} + 4\, b_2^{10} + 2b_1^2 b_3^6 + 2b_3 b_6^3 + 3\, b_1^4 b_2^8\right]$$

Example 2

Let $n = 5$, then thedihedral group D_5 of degree 5 acting on unordered triples of the set$X = \{1,2,3,4,5\}$. Then;

$|D_5| = 10, \quad d = 1,5$

$\phi(1) = 1$ and $\phi(5) = 4$. Since n is odd and d is not divisible by 2 for both cases and hence from (4.6) we have;

$$Z_{D_5,X^{(3)}} = \left[\frac{1}{10} b_1^{10} + 4\, b_5^2 + 5\, b_1^2 b_2^4\,.\right]$$

References

[1] Harary, F. & Palmer E. 1973. Graphical Enumeration Academic press, New York.

[2] Polya G. 1937 KombinatorischeAnzahlbestimmungenfürGruppen, Graphen Und ChemischeVerbindungen, Acta Math. 68, 145-254.

[3] Kamuti I. N. &Obong'o, J. O. 2002. The derivation of cycle index of $S_n^{[3]}$. QuaestionesMathematicae 25: 437-444.

[4] Kamuti, I. N. &Njuguna, L. N. 2004. On the cycle index of the reduced ordered r-group. East African journal of physical Sciences; 5(2); 99-108.

[5] Kamuti I. N. 2012 Cycle Index of Internal Direct Product Groups. Vol. 7 30(1491-1494).

[6] Harary, F. 1969. Graph Theory. Addison-Wesley Publishing Company, New York.

[7] Krishnamurthy, V. 1985. Combinatorics: Theory and application. Affiliated East-West press private limited, New Delhi.