International Journal of Science and Research (IJSR)
ISSN (Online): 2319-7064
Index Copernicus Value (2015): 78.96 | Impact Factor (2015): 6.391

End-ψ-Primary Submodules

Nuhad S. AL-Mothafar[1], Adwia J. Abdil–Khalik[2]

[1]Dept. of Mathematics, College of Science, University of Baghdad

[2]Dept. ofMathematics, College of Science, Al-Mustansiriya University

Abstract: ***Let R be a commutative ring with identity and M an unitary R-module. Let $\delta(M)$ be the set of all submodules of M, and ψ: $\delta(M) \longrightarrow \delta(M) \cup \{\phi\}$ be a function. We say that a proper submodule P of M is end-ψ-primary if for each $\alpha \in End_R(M)$ and $x \in M$, if $\alpha(x) \in P$, then either $x \in P + \psi(P)$ or $\alpha^n(M) \subseteq P + \psi(P)$ for some $n \in Z_+$. Some of the properties of this concept will be investigated. Some characterizations of end-ψ-primarysubmodules will be given.***

Keywords: Primesubmodules, Primary submodules, end-ψ-primary submodules, end-ψ-primesubmodules.

1. Introduction

Throughout this paper,R is a commutative ring with identity and M is an unitary R-module.For an ideal I of R and a submodule N of M let$\sqrt{I}$ denote the radical of I, and [N:M] $=\{r \in R: rM \subseteq N\}$ which is clearly an ideal of R.Prime ideals play an essential role in ring theory. One of the natural generalizations of prime ideals which have attracted the interest of several authors in the last two decades is the notion of prime submodule. A proper submodule P of M is called a prime submodule if $r \in R$ and $x \in M$ with $rx \in P$ implies that either $r \in [P:M]$or $x \in P$,[1].A proper submodule P of M is called a primary submodule if $r \in R$ and $x \in M$ with $rx \in P$ implies that either$r^n \in [P:M]$ for some $n \in Z_+$ or $x \in P$,[2].There are several generalizations of the notion of prime submodules, such as Nuhad S. , Adwia J., introduced and studied end-Ψ-prime submodules, where a proper submodule p of M is said to be end-Ψ-prime submodule if for each $\alpha \in End_R(M)$ and $x \in M$, if $\alpha(x) \in P$, then either $x \in P + \psi(P)$ or $\alpha(M) \subseteq P + \psi(P)$,where$\psi$: $\delta(M) \longrightarrow \delta(M) \cup \{\phi\}$ be a function and $\delta(M)$ be the set of all submodules of M, [3].Another generalization of prime submodule is the concept of S-prime submodules, where a proper submodule p of M is said to be S-prime submodule of M if $f(m) \in P$,where $f \in S=End(M)$ and $m \in M$ implies that either $m \in P$ or $f(M) \subseteq P$,[4]. Shireen introduced and studied S-primary submodules, where a proper submodule p of M is said to be S-primary submodule of M if $f(m) \in P$,where $f \in S=End(M)$ and $m \in P$ implies that either $m \in P$ or there exists a positive integer n, such that $f^n(M) \subseteq P$,[4].Nuhad S. , Adwia J., in [5] extended the notion of prime submodule to ψ-primary. Let M be an R-module and$\delta(M)$ be the set of all submodules of M andψ: $\delta(M) \longrightarrow \delta(M) \cup \{\phi\}$ be a function. A proper submodule P of M is said to be ψ-primary if $r \in R$ and $x \in M$, $rx \in P$ implies that either$r^nM \subseteq P+\psi(P)$ for some $n \in Z_+$ or $x \in P+\psi(P)$.In this paper,we define and study the notion of end-ψ-primarysubmodules. Let $\delta(M)$ be the set of all submodules of M andψ: $\delta(M) \longrightarrow \delta(M) \cup \{\phi\}$ be a function. A proper submodule P of M is said to be end-ψ-primary if for each $\alpha \in End_R(M)$ and $x \in M$, if $\alpha(x) \in P$, then either $x \in P + \psi(P)$ or $\alpha^n(M) \subseteq P + \psi(P)$ for some $n \in Z_+$.

2. Basic Properties of end-ψ-PrimarySubmodules

First we give the following definition.

Definition (2:1):
Let M be an R-module and$\delta(M)$ be the set of all submodules of M. Let $\psi:\delta(M) \longrightarrow \delta(M) \cup \{\phi\}$ be a function. A proper submodule N of M is said to be **end-ψ-primary** if for each $\alpha \in End_R(M)$ and $x \in M$, if$\alpha(x) \in N$, then either $x \in N + \psi(N)$ or $\alpha^n(M) \subseteq N + \psi(N)$ for some $n \in Z_+$.

Remarks and Examples (2.2):
(1) It is clear that every end-ψ-prime submodule is end-ψ-primarysubmodule.

(2) Let $M = Z_8$ as Z-module,$N = \{\bar{0}, \bar{4}\}$.Then N is an end-ψ-primarysubmodule of M (since N is end-Ψ-prime submodule of M,[3]) .

(3) It is clear that not every end-ψ-primary submodule is prime submodule , see example in remark(2.2,(2)) .

(4) It is clear that every S-primarysubmodule is end-ψ-primarysubmodule.

The convers is not true as the following example shows .Let $M=Z_8$ as Z- module ,$N=\{\bar{0}, \bar{4}\}$.Then N is not S-primarysubmodule of M (since if $f(\bar{x}) = 2\bar{x}$,$\forall \bar{x} \in Z_8$ where $f: Z_8 \longrightarrow Z_8$ and $f(\bar{2}) = 2.\bar{2} = \bar{4} \in N$. But$\bar{2} \notin N$ and $f(M) =\{\bar{0}, \bar{2}, \bar{4}, \bar{6}\} \not\subseteq N$,hence N is notS- primary submodule of M.But N is end ψ - primary submodule of M .

(5) Let $M = Z_4$as Z-module,$N = \{\bar{0}\}$. Then N is an end-ψ-primarysubmodule of M (since N is S- primarysubmodule of M,[4])

(6) The only end-ψ-primarysubmodule of a simple module is {0}.Therefore$(\bar{0})$in the simple Z-module Z_p (p is prime number) is end-ψ-primarysubmodule.

(7) If$\psi(N) = N$ or $\psi(N) = 0$,then every end-ψ-primarysubmodule is S-primarysubmodule and hence is primarysubmodule, [4].

(8) Let $M = Z_{12}$ as Z-module, then $N = \{\bar{0}, \bar{6}\}$is not end-ψ-primary of M .Since if $f : Z_{12} \longrightarrow Z_{12}$, where$f(\bar{m}) = 2\bar{m}$for all$\bar{m} \in Z_{12}$and let$\psi:\delta(M) \longrightarrow \delta(M) \cup \{\phi\}$ such that $\psi(N) = N + \{\bar{0}, \bar{6}\}$,$\forall$ $N \subseteq M$ .Since$f(\bar{3}) = \bar{6} \in N$,but $\bar{3} \notin N + \psi(N) = \{\bar{0}, \bar{6}\}$and $f(Z_{12}) = 2Z_{12} \nsubseteq N + \psi(N) = \{\bar{0}, \bar{6}\}$.Therefore $N = \{\bar{0}, \bar{6}\}$ is not end-ψ-primarysubmodule of Z_{12}.

(9) It is clear that every S-prime submodule is end-ψ-primary submodule.

Proof: Since every S-prime submodule is an S- primary submodule of M, [4]. Therefore N is end-ψ-primary submodule by (2.2,(4))..

Proposition (2.3):
Every end-ψ-primary submodule of an R-module M is a ψ-primary submodule of M.

Proof: Let N be an end-ψ-primary submodule of an R-module M and suppose that for some $r \in R$ and $m \in M$ such that $rm \in N$. Assume that $m \notin N + \psi(N)$, we must prove that $r^n M \subseteq N + \psi(N)$ for some $n \in Z_+$. Define $h: M \to M$ by $h(x) = rx$ for all $x \in M$. *Clearly,* $h \in End(M)$. Now, $h(m) = rm \in N$. But N is end-ψ-primary and$m \notin N + \psi(N)$, so$h^n(M) \subseteq N + \psi(N)$ for some $n \in Z_+$ .This implies$r^n M \subseteq N + \psi(N)$ for some $n \in Z_+$ and hence N is a ψ-primary submodule.

Recall that an R-module M is called **scalar** if for every $f \in End(M)$, $\exists\, r \in R$, $r \neq 0$ such that $f(m) = rm$ for all $r \in R$, [6].

The following proposition shows that(scalar R-module) is a sufficient condition for ψ-primarysubmodule to b end-ψ-primarysubmodule.

Proposition (2.4):
Let M be a scalar R-module, and N is aψ-primary submodule of M. Then N is an end-ψ-primarysubmodule of M.

Proof: Let $f \in End(M)$, $m \in M$ such that $f(m) \in N$. Since M is scalar, $\exists\, r \in R$, $r \neq 0$ such that $f(x) = rx$ for all $x \in M$. Hence $f(m) = rm \in N$. But N is ψ-primary , so either $m \in N + \psi(N)$ or $r^n M \subseteq N + \psi(N)$ for some $n \in Z_+$. Thus either $m \in N + \psi(N)$ or $f^n(M) \subseteq N + \psi(N)$ for some $n \in Z_+$. Therefore N is end-ψ-primarysubmodule.

Recall that a nonzero module M is called a **multiplication** module if for eachsubmodule N of M there exists an ideal I of R such that N = IM, [7]

Corollary (2.5):
Let N a ψ-primarysubmodule of a finitely generated multiplication R-module M.Then N is an end-ψ-primarysubmodule of M

Proof: By [6,corollary 1.1.11]Every finitely generated multiplication R-module M is a scalar module and so by proposition (2.4) we get the result.

The following proposition give another sufficient condition for ψ-primary submodule to be end-ψ-primary submodule.

Proposition (2.6):
If N is aψ-primarysubmoduleof a cyclic R-module M and $[N + \psi(N): M]$ is a semi prime ideal of R, then N is end-ψ-primarysubmodule of M.

Proof: Let N be aψ-primary submodule of a cyclic R-module M and $f(m) \in N$, for some $m \in M$and $f \in End(M)$. Assume that $m \notin N + \psi(N)$. Let $x \in M$, and since M is a cyclic, then $M = \langle a \rangle$. $x = ra$, $m = r_1 a$, for some $r, r_1 \in R$. Now, $f(m) = f(r_1 a) = r_1 f(a) \in N$. But N is ψ-primary, then either $f(a) \in N + \psi(N)$ or $r_1^n \in [N + \psi(N) : M]$ for some $n \in Z_+$ and hence $r_1 \in \sqrt{[N + \psi(N) : M]} = [N + \psi(N) : M]$ since $[N + \psi(N) : M]$ is a semi prime ideal of R. But $r_1 \notin [N + \psi(N):M]$, for $m = r_1 a \notin N + \psi(N)$. Hence $f(a) \in N + \psi(N)$ and therefore $f(x) = f(ra) = r f(a) \in N + \psi(N)$, thus $f(M) \subseteq N + \psi(N)$ and so N is an end-ψ-primesubmodule of M .Therefore N is end-ψ-primary submodule of M.

A proper submodule N of M is said to be **S- semi prim**e if for each $\alpha \in End_R(M)$ and $x \in M$, if $\alpha^2(x) \in N$, then $\alpha(x) \in N$, [4].

Proposition (2.7):
If N is an end-ψ-primary submodule of an R-module M and $N + \psi(N)$is anS-semi prime submodule of M, then N is an end-ψ-primesubmodule of M.

Proof: Let N be an end-ψ-primary submodule of an R-module M and $f(m) \in N$, for some $m \in M$ and $f \in End(M)$. Assume that $m \notin N + \psi(N)$, we must prove that $f(M) \subseteq N + \psi(N)$. Since N is an end-ψ-primary of M and $m \notin N + \psi(N)$, then there exists a positive integer nsuch that $f^n(M) \subseteq N + \psi(N)$. But $N + \psi(N)$ is an S-semi prime submodule of M,hence $f(M) \in N + \psi(N)$ by [4]. Therefore N is an end-ψ-prime submodule of M.

Recall that a submodule N of an R –module M is called **S-relatively divisible** denoted by **S-RD** if $rM \cap N = rN$ for each $r \in R$, $f(M) \cap N = f(N)$ for all $f \in End(M)$, [8] .

By using this concept,we can give the following result.

Corollary (2.8):
If N is an end-ψ-primary submodule of an R-module M and every submodule is S-RD submodule of M, then N is an end-ψ-prime submodule of M.
Proof: By [4,porosition 2.2.17, P.70], each proper submodule of M is an S-semi prime submodule of M. Hence by previous proposition (2.7) we get the result.

Recall that a nonzero module M is called **quasi-Dedekind** if $Hom(M/N, M) = 0$ for all nonzero submodule N of M. Equivalently, M is **quasi-Dedekind** if for any $f \in End(M)$, $f \neq 0$, then $\ker f = 0$, (i.e. f is 1-1), [9].

Proposition (2.9):
Let M be a quasi–Dedekind R-module.Then every proper S-RD submodule of M is end-ψ-primarysubmodule of R-module M .

Proof: By [3, Prop. 2.9],every proper S-RD submodule of M is an end- ψ-prime. Hence every proper submodule of an R-module M is end ψ-primarysubmodule by.(2.2,(1)) .

A nonzero R-module M is said to be **monoform** if, for each N⊆M and for each f∈Hom (N,M), f≠ 0, then ker$f$ = 0, [10].

By using this concept ,we can give the following result.

Corollary (2.10):
Let M be a monoformR-module .Then every proper S-RD submodule of M is end-ψ-primary submodule of R-module M .

Proof: By [10], every propermonoform R-module is an quasi–Dedekind R-module . Hence by previous proposition (2.9) we get the result.

A nonzero R-module M is said to be **S- secondary module,** if for each α∈End$_R$(M), then either α(M) = M or there exists a positive integer n, such that α^n (M) = 0, [11].

By using this concept ,we can give the following result.

Proposition (2.11):
If N is a propersubmodule of an **S- secondary** R-module M. If N is an S-semi prime submodule of M, then N is an end-ψ-primarysubmodule of M.

Proof: Let f∈ End(M), m ∈M such that f (m) ∈ N. Since M is an S-secondary R- module, then either f(M) = M or there exists a positive integer n, such that f^n (M) = 0,
Now, if f^n(M) = 0∈N, then f^n(M)⊆ N + ψ(N) . If f(M) = M and m ∈Mthere exists y ∈ M, such that m = f(y), f(m) = f2(y) ∈ N, but N is an S-semi prime submodule of M, thus m = f(y) ∈ N + ψ(N).Therefore N is end-ψ-primary submodule .

3. More about End-ψ-primarySubmodules

In this section, several fundamental properties of end-ψ-primarysubmodule are given.

Proposition (3.1):
Let M be an R-module, N < M, I ≤ R. If P is an end-ψ-primary submodule of M such that IN⊆ P, then N ⊆ P + ψ(P)orI ⊆$\sqrt{[P\ +\ \psi(P):M]}$.

Proof: Suppose IN⊆ P, where I is an ideal of R and N, P are two submodules. SupposeN ⊈ P + ψ(P), then there exists x ∈ N and x ∉ P + ψ(P). It is clear for each a∈ I, thusax∈ P. Define f : M ⟶M by f (m) = a m for all m ∈M, it is clear that f∈End$_R$(M) and f (x) = a x ∈ IN ⊆ P. But P is an end-ψ-primarysubmodule of M and x ∉ P + ψ(P). Hencef^n (M) = a^nM ⊆ P + ψ(P), so a^n∈[P + ψ(P):M] for some n ∈ Z_+. Thus, a∈$\sqrt{[P\ +\ \psi(P):M]}$ and hence I ⊆$\sqrt{[P\ +\ \psi(P):M]}$.

Recall that a submodule N of an R-module M is said to be **fully invariant** if f (N) ⊆ N, for each R-endomorphism f of M. And an R – module M is said to be **fully invariant** module if for each submodule N of M is fully invariant ,[12].

By using this concept ,we can give the following result.

Proposition (3.2):
Let M be afully invariant an R-module, let ϕ∈ End(M). If N is an end-ψ-primarysubmodulesof an R-module M and End(M) is a commutative ring, such that ϕ(M) ⊈ N and ψ(ϕ^{-1}(N)) = ϕ^{-1}(ψ(N)) , then ϕ^{-1}(N) is also end-ψ-primarysubmodule of M.

Proof: First, we must show that ϕ^{-1}(N) is a proper submodule of M. Suppose that ϕ^{-1}(N) = M, then ϕ(M) ⊆ N, which a contradiction to the assumption.

Now, let f (m) ∈ϕ^{-1}(N), where f∈ End(M) and m ∈ M. Then ϕ (f (m)) ∈ N. Since End(M) is a commutative ring , then f (ϕ (m)) ∈ N.But N is an end-ψ-primary of M so either ϕ(m) ∈ N + ψ(N) or f^n (M) ⊆ N + ψ(N) for some n ∈ Z_+ . If ϕ(m) ∈ N + ψ(N), then ϕ^{-1}(ϕ(m)) = m ∈ϕ^{-1}(N) + ϕ^{-1}(ψ(N)) = ϕ^{-1}(N) + ψ(ϕ^{-1}(N)). If f^n(M) ⊆ N + ψ(N), thenf^n (M) ⊆ϕ^{-1}(N) + ϕ^{-1}(ψ(N)) = ϕ^{-1}(N) + ψ(ϕ^{-1}(N)) (since M is a fully invariant, so ϕ (N + ψ(N)) ⊆ N + ψ(N), implies N + ψ(N) ⊆ϕ^{-1}(N + ψ(N)). Then ϕ^{-1}(N) is an end-ψ-primary submodule of M.

Corollary (3.3):
Let M be a multiplication R-module, let ϕ∈ End(M). If N is an end-ψ-primary submodules of an R-module M and End(M) is a commutative ring, such that ϕ(M) ⊈ N and ψ(ϕ^{-1}(N)) = ϕ^{-1} (ψ(N)) , then ϕ^{-1}(N) is also end-ψ-primary submodule of M.

Proof: Since M is a multiplication R-module, so every submodule of M is fully invariant by [14]. Hence by previous proposition (3.2) we get the result.

Recall that a submodule N of an R-module M is said to be **fullystable** if f (N) ⊆ N, for each f∈Hom(N,M).
And an R–module M is said to be **fully stable** module if for each submodule N of M is fully stable,[13].

Corollary (3.4):
Let M be a fully stable R-module, let ϕ∈ End(M). If N is an end-ψ-primary submodules of an R-module M, such that ϕ(M) ⊈ N and ψ(ϕ^{-1}(N)) = ϕ^{-1} (ψ(N)) , then ϕ^{-1}(N) is also end-ψ-primary submodule of M.

Proof: Since M is a fully stable R-module, so every submodule of M is fully invariant by [13]. Also End(M) is a commutative ring by [13]. Hence by previous proposition (3.2) we get the result.

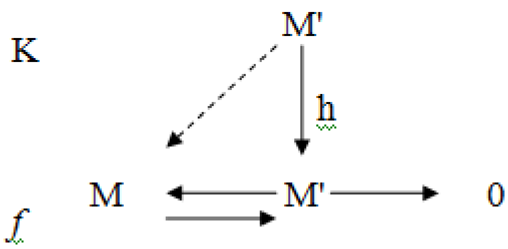
Recall that a submodule N of an R-module M is said to be **satisfy Baer Criterion** if for each R-homomorphism f:N→ M, there exists an element r ∈ Rsuch that f(n) = rn for each n in N, [13].

Corollary (3.5):
Let M be an R-module which satisfies Baer Criterion, let $\phi \in$ End(M). If N is an end-ψ-primary submodules of an R-module M, such that ϕ(M) $\not\subseteq$ N and ψ(ϕ^{-1}(N)) = ϕ^{-1}(ψ(N)), then ϕ^{-1}(N) is also end-ψ-primary submodule of M.
Proof: Since every module which satisfies Baer Criterion is fully stable by [13]. Hence by previous corollary (3.4) we get the result.

Recall that an R-module M is called **A-projective** (where A is an R-module) if for every X < A and every homoorphismϕ: M $\longrightarrow$A/ X can be lifted to a homomorphism ψ: M $\longrightarrow$M, [15]. If M is **A-projective** for each R-module A,thenM is called **projective.**

Theorem (3.6):
Let f: M $\longrightarrow$M' be an epimorphism and let N < M such that ker$f \leq$ N and End(M) is a commutative ring . If N is an end-ψ-primary submodule of a module M, then f (N) is an end-ψ'-primarysubmodule of a module of M', where M' is M-projective module and ψ'(f (N)) = f (ψ (N)).

Proof: First, we must show that f(N) is a proper submodule of a module M'. Suppose$f$(N) = M'. But f is an epimorphism, thus f (N) = f (M) and hence M = N + ker$f$. This implies that M = N. A contradiction.
Now, let h(m')$\in f$ (N), whereh $\in$ End(M') and m' $\in$ M' and suppose that m' $\notin f$ (N) + ψ'(f (N)), we have to show that h^n(M') $\subseteq f$ (N) + ψ'(f (N)) for some n $\in Z_+$. Since f is an epimorphism and m' $\in$ M', then there exists m $\in$ M, such that f (m) = m' $\notin f$ (N) + ψ'(f (N)), thus m $\notin$ N + f^{-1}(ψ'(f (N))) = N + ψ(N). Consider the following diagram:



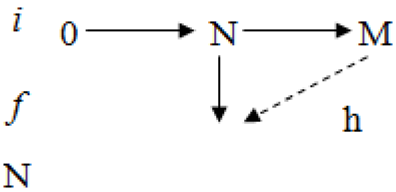
since M' is M-projective module, then there exists a homoorphism k: M' $\longrightarrow$ M, such that $f \circ$k = h. Clearly, k$\circ f \in$ End(M). Note that f (k$\circ f$ (m)) = ($f \circ$k)(f (m)) = h(m') $\in f$ (N) and since ker$f \subseteq$ N, we get (k$\circ f$)(m) $\in$ N. But N is an end-ψ-primarysubmopdule of M and m $\notin$ N + ψ(N). Therefore there exist a positive integer n, such that (k$\circ f$)^n(M) $\subseteq$ N + ψ(N) and since End(M) is a commutative ring hence *(f ∘ k)*^{n}(M)) = h^n(M) $\subseteq$N + ψ(N). Thus *f*(h^n*(M)* $\subseteq$ *f (N) + f (ψ(N)) and hence* h^n*(f (M))*$\subseteq f$ (N) + f (ψ(N)), which implies that h^n(M') $\subseteq f$ (N) + ψ'(f (N)).

Corollary (3.7):
Let M be an R-module, let K < N < M and N is an end-ψ-primary of M and End (M/K) is a commutative ring.Then N/K is end-ψ'-primary in M/K, provided thatM/K is M-projective.

Recall that an R-module M is **A-injective** (where A is an R-module) if for every X ≤ M, any homomorphism ϕ: X $\longrightarrow$ M can be extended to a homomorphism ψ: A $\longrightarrow$ X[15].If M is **M-injective** M is called **quasi-injective**[16].

Proposition (3.8):
Let K be an end-ψ-primary of an R-module M and let N < M which is M-injective and ψ(K) $\subseteq$ K. Then either N $\subseteq$ K or K ∩ N is an end-ψ-primary in N.

Proof: Suppose that N $\not\subseteq$ K, then K ∩ N is a proper submodule in N. Let f (x) $\in$K ∩ N, where $f \in$ End(N) and x $\in$ N. Suppose x $\notin$ (K ∩ N) + ψ'(K ∩ N), where ψ': δ(N) $\longrightarrow$δ(N)∪{ϕ} be a function, then x $\notin$ K. We must show that f (N) ≤ (K ∩ N) + ψ'(K ∩ N).
Now, consider the following diagram:



Where i is the inclusion map.
Since N is M-injective, then there exists h: M $\longrightarrow$ N, such that h∘i = f, clearly h $\in$ End(M). But f (X) = (h∘i)(x) = h(x) $\in$ K. Since K is an end-ψ-prime submodule of M and x $\notin$ K + ψ(K), this implies that h(M) $\subseteq$ K + ψ(K). Also, f (N) = (h∘i)(N) = h(N) $\subseteq$ N (since f (N) $\subseteq$K ∩ N) and f (N) $\subseteq$ h(N) $\subseteq$ h(M) $\subseteq$ K +ψ(K). Therefore, f (N) $\subseteq$ N ∩ (K + ψ(K)) = N ∩ K $\subseteq$ N ∩ K + ψ'(N ∩ K).

Corollary (3.9):
Let K be an end-ψ-primarysubmodule of a quasi-injective R-module M, and let N < M. Then either N ≤ K or K ∩ N is an end-ψ-primary in N.

Proposition (3.10):
Let M be an R-module and let K < N < M and K is fully invariant. If N/K is an end-ψ'-primarysubmodule of M/K andψ'(N/K) = (N + ψ(N)) /K , then N is an end-ψ-primarysubmodule of M.

Proof: Suppose that$f$ (m) $\in$ N, where $f \in$ End(M) and m $\in$ M. We must show that either m$\in$ N + ψ(N) orf^n (M) $\subseteq$ N + ψ(N) for some n $\in$ Z+. Define f *: M/K$\longrightarrow$M/K by f *(x + K) = f (x) + K, ∀ x$\in$M. To prove f * is well define, let x + K = y + K where x, y $\in$ M, then x – y $\in$ K and hence f (x – y) $\in f$ (K) $\subseteq$ K, since K is fully invariant. This implies that f (x) – f (y) $\in$ K. Thus f (x) + K = f (y) + K.. Now,f *(m + K) = f (m) + K $\in$ N/K. But N/K is an end-ψ''-primary of M/K, so either m+K$\in$ N/K + ψ'(N/K) =N/ K + (N + ψ(N)) /K = (N + ψ(N)) /K and thus m $\in$ N + ψ(N) or f^{*n} (M/ K) $\subseteq$ N/K + ψ'(N/K) for some n $\in$ Z+. Thus f^{*n} (m + K) $\subseteq$ N/K + ψ'(N/K) which implies that f^n (m) + K $\subseteq$ N/ K + (N + ψ(N)) /K = (N + ψ(N)) /K, thus f^n (m) $\subseteq$ N + ψ(N) and since m is an arbitrary element of M, thus f^n (M) $\subseteq$ N + ψ(N) for some n $\in$ Z+.

Paper ID: 1061705 DOI: 10.21275/1061705

References

[1] Lu, C.P.,(1981),**Prime Submodules of Modules**,Commutative Mathematics ,University Spatula,33,16-69.
[2] Lu, C.P., (1989), **M- radicals of Submodules in Modules**, Mathematics,Japon,34,211-219.
[3] Nuhad S. Al- Mothafar andAdwia J. Abdul- Al- Khalik,(2016), **End-ψ-prime Submodules,**In Al-Haitham Jour. For Pure and Appl, Sci., 29(2).
[4] Shireen ,O.D.,(2010),**S-prime Submodules and Some Related Concepts**, M.Sc. Thesis,College of science ,University of Baghdad.
[5] Nuhad S. Al- Mothafar andAdwia J. Abdul- Al- Khalik,(2017), **ψ-primarySubmodules,** International Jour. Of Sci. and Res., Vol.6(4).
[6] Shihab, B.N.,(2004), **Scalar Reflexive Modules**, Ph.D. Thesis, Univ.of Baghdad.
[7] El-Bast Z. A. and Simth P. F. (1988), **Multiplication Modules** , Comm. In Algebra, 16, 755-779.
[8] Faith, C.,(1973), Rings, **Modules and Categories**, I, Springer, Berlin, Iteidelberg , New York.
[9] Mijbass,A.S.,(1997),**Quasi-Dedekind Modules and Quasi-InvertibleSubmodules**, Ph.D. Thesis, Univ. of Baghdad.
[10] J. M. Zelmanowitz, (1986), **Representation of Rings With Faithful Polyform Module,** Comm. In Algebra, 14 (6), 1141-1169.
[11] Gungoroglu ,C., (2000),**Strongly Prime Ideals in CS-Rings** ,Turk .J.Math., 24,233-238.
[12] Faith, C.,(1973), **Rings, Modules and Categories**, I, Springer , Berlin, Iteidelberg , New York .
[13] Abbas, M.S., (1990),**On Fully Stable Modules**, Ph. D. Thesis, University of Baghdad.
[14] A. G. Naoum, (1994), **On the Ring of Endomorphisms of Multiplication Modules,** Periodica Math., Hungarica, 29, 277-284.
[15] Azumaya, G., Mbuntum, F. and Varadarajan, K.J., (1975), **On Projective and M-Injective Modules,** 95, 9-16.
[16] G. F. Brikenmeeier ,(1976), **On the Cancellation of Quasi- Injective Modules,** Comm. In Algebra, 4, 101-109.

Paper ID: 1061705 DOI: 10.21275/1061705