

End- ψ -Primary Submodules

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Abstract: Let R be a commutative ring with identity and M an unitary R -module. Let $\delta(M)$ be the set of all submodules of M , and $\psi: \delta(M) \rightarrow \delta(M) \cup \{\phi\}$ be a function. We say that a proper submodule P of M is end- ψ -primary if for each $\alpha \in \text{End}_R(M)$ and $x \in M$, if $\alpha(x) \in P$, then either $x \in P + \psi(P)$ or $\alpha^n(M) \subseteq P + \psi(P)$ for some $n \in \mathbb{Z}_+$. Some of the properties of this concept will be investigated. Some characterizations of end- ψ -primary submodules will be given.

Keywords: Primesubmodules, Primary submodules, end- ψ -primary submodules, end- ψ -primesubmodules.

1. Introduction

Throughout this paper, R is a commutative ring with identity and M is an unitary R -module. For an ideal I of R and a submodule N of M let \sqrt{I} denote the radical of I , and $[N:M] = \{r \in R: rM \subseteq N\}$ which is clearly an ideal of R . Prime ideals play an essential role in ring theory. One of the natural generalizations of prime ideals which have attracted the interest of several authors in the last two decades is the notion of prime submodule. A proper submodule P of M is called a prime submodule if $r \in R$ and $x \in M$ with $rx \in P$ implies that either $r \in [P:M]$ or $x \in P$, [1]. A proper submodule P of M is called a primary submodule if $r \in R$ and $x \in M$ with $rx \in P$ implies that either $r^n \in [P:M]$ for some $n \in \mathbb{Z}_+$ or $x \in P$, [2]. There are several generalizations of the notion of prime submodules, such as Nuhad S. , Adwia J., introduced and studied end- ψ -prime submodules, where a proper submodule p of M is said to be end- ψ -prime submodule if for each $\alpha \in \text{End}_R(M)$ and $x \in M$, if $\alpha(x) \in P$, then either $x \in P + \psi(P)$ or $\alpha^n(M) \subseteq P + \psi(P)$, where $\psi: \delta(M) \rightarrow \delta(M) \cup \{\phi\}$ be a function and $\delta(M)$ be the set of all submodules of M , [3]. Another generalization of prime submodule is the concept of S -prime submodules, where a proper submodule p of M is said to be S -prime submodule of M if $f(m) \in P$, where $f \in S = \text{End}(M)$ and $m \in M$ implies that either $m \in P$ or $f(M) \subseteq P$, [4]. Shireen introduced and studied S -primary submodules, where a proper submodule p of M is said to be S -primary submodule of M if $f(m) \in P$, where $f \in S = \text{End}(M)$ and $m \in P$ implies that either $m \in P$ or there exists a positive integer n , such that $f^n(M) \subseteq P$, [4]. Nuhad S. , Adwia J., in [5] extended the notion of prime submodule to ψ -primary. Let M be an R -module and $\delta(M)$ be the set of all submodules of M and $\psi: \delta(M) \rightarrow \delta(M) \cup \{\phi\}$ be a function. A proper submodule P of M is said to be ψ -primary if $r \in R$ and $x \in M$, $rx \in P$ implies that either $r^n M \subseteq P + \psi(P)$ for some $n \in \mathbb{Z}_+$ or $x \in P + \psi(P)$. In this paper, we define and study the notion of end- ψ -primary submodules. Let $\delta(M)$ be the set of all submodules of M and $\psi: \delta(M) \rightarrow \delta(M) \cup \{\phi\}$ be a function. A proper submodule P of M is said to be end- ψ -primary if for each $\alpha \in \text{End}_R(M)$ and $x \in M$, if $\alpha(x) \in P$, then either $x \in P + \psi(P)$ or $\alpha^n(M) \subseteq P + \psi(P)$ for some $n \in \mathbb{Z}_+$.

2. Basic Properties of end- ψ -Primary Submodules

First we give the following definition.

Definition (2.1):

Let M be an R -module and $\delta(M)$ be the set of all submodules of M . Let $\psi: \delta(M) \rightarrow \delta(M) \cup \{\phi\}$ be a function. A proper submodule N of M is said to be **end- ψ -primary** if for each $\alpha \in \text{End}_R(M)$ and $x \in M$, if $\alpha(x) \in N$, then either $x \in N + \psi(N)$ or $\alpha^n(M) \subseteq N + \psi(N)$ for some $n \in \mathbb{Z}_+$.

Remarks and Examples (2.2):

- (1) It is clear that every end- ψ -prime submodule is end- ψ -primary submodule.
- (2) Let $M = \mathbb{Z}_8$ as \mathbb{Z} -module, $N = \{\bar{0}, \bar{4}\}$. Then N is an end- ψ -primary submodule of M (since N is end- ψ -prime submodule of M , [3]).
- (3) It is clear that not every end- ψ -primary submodule is prime submodule, see example in remark(2.2,(2)).
- (4) It is clear that every S -primary submodule is end- ψ -primary submodule.

The convers is not true as the following example shows. Let $M = \mathbb{Z}_8$ as \mathbb{Z} -module, $N = \{\bar{0}, \bar{4}\}$. Then N is not S -primary submodule of M (since if $f(\bar{x}) = 2\bar{x}, \forall \bar{x} \in \mathbb{Z}_8$ where $f: \mathbb{Z}_8 \rightarrow \mathbb{Z}_8$ and $f(\bar{2}) = 2\bar{2} = \bar{4} \in N$. But $\bar{2} \notin N$ and $f(M) = \{\bar{0}, \bar{2}, \bar{4}, \bar{6}\} \not\subseteq N$, hence N is not S -primary submodule of M . But N is end ψ -primary submodule of M .

- (5) Let $M = \mathbb{Z}_4$ as \mathbb{Z} -module, $N = \{\bar{0}\}$. Then N is an end- ψ -primary submodule of M (since N is S -primary submodule of M , [4]).
- (6) The only end- ψ -primary submodule of a simple module is $\{0\}$. Therefore $(\bar{0})$ in the simple \mathbb{Z} -module \mathbb{Z}_p (p is prime number) is end- ψ -primary submodule.
- (7) If $\psi(N) = N$ or $\psi(N) = 0$, then every end- ψ -primary submodule is S -primary submodule and hence is primary submodule, [4].

(8) Let $M = Z_{12}$ as Z -module, then $N = \{\bar{0}, \bar{6}\}$ is not end- ψ -primary of M . Since if $f : Z_{12} \rightarrow Z_{12}$, where $f(\bar{m}) = 2\bar{m}$ for all $\bar{m} \in Z_{12}$ and let $\psi : \delta(M) \rightarrow \delta(M) \cup \{\phi\}$ such that $\psi(N) = N + \{\bar{0}, \bar{6}\}, \forall N \subseteq M$. Since $f(\bar{3}) = \bar{6} \in N$, but $\bar{3} \notin N + \psi(N) = \{\bar{0}, \bar{6}\}$ and $f(Z_{12}) = 2Z_{12} \not\subseteq N + \psi(N) = \{\bar{0}, \bar{6}\}$. Therefore $N = \{\bar{0}, \bar{6}\}$ is not end- ψ -primary submodule of Z_{12} .

(9) It is clear that every S -prime submodule is end- ψ -primary submodule.

Proof: Since every S -prime submodule is an S -primary submodule of M , [4]. Therefore N is end- ψ -primary submodule by (2.2,(4)).

Proposition (2.3):

Every end- ψ -primary submodule of an R -module M is a ψ -primary submodule of M .

Proof: Let N be an end- ψ -primary submodule of an R -module M and suppose that for some $r \in R$ and $m \in M$ such that $rm \in N$. Assume that $m \notin N + \psi(N)$, we must prove that $r^n M \subseteq N + \psi(N)$ for some $n \in Z_+$. Define $h : M \rightarrow M$ by $h(x) = rx$ for all $x \in M$. Clearly, $h \in \text{End}(M)$. Now, $h(m) = rm \in N$. But N is end- ψ -primary and $m \notin N + \psi(N)$, so $h^n(M) \subseteq N + \psi(N)$ for some $n \in Z_+$. This implies $r^n M \subseteq N + \psi(N)$ for some $n \in Z_+$ and hence N is a ψ -primary submodule.

Recall that an R -module M is called **scalar** if for every $f \in \text{End}(M), \exists r \in R, r \neq 0$ such that $f(m) = rm$ for all $r \in R$, [6].

The following proposition shows that (scalar R -module) is a sufficient condition for ψ -primary submodule to be end- ψ -primary submodule.

Proposition (2.4):

Let M be a scalar R -module, and N is a ψ -primary submodule of M . Then N is an end- ψ -primary submodule of M .

Proof: Let $f \in \text{End}(M), m \in M$ such that $f(m) \in N$. Since M is scalar, $\exists r \in R, r \neq 0$ such that $f(x) = rx$ for all $x \in M$. Hence $f(m) = rm \in N$. But N is ψ -primary, so either $m \in N + \psi(N)$ or $r^n M \subseteq N + \psi(N)$ for some $n \in Z_+$. Thus either $m \in N + \psi(N)$ or $f^n(M) \subseteq N + \psi(N)$ for some $n \in Z_+$. Therefore N is end- ψ -primary submodule.

Recall that a nonzero module M is called a **multiplication** module if for each submodule N of M there exists an ideal I of R such that $N = IM$, [7]

Corollary (2.5):

Let N a ψ -primary submodule of a finitely generated multiplication R -module M . Then N is an end- ψ -primary submodule of M

Proof: By [6, corollary 1.1.11] Every finitely generated multiplication R -module M is a scalar module and so by proposition (2.4) we get the result.

The following proposition give another sufficient condition for ψ -primary submodule to be end- ψ -primary submodule.

Proposition (2.6):

If N is a ψ -primary submodule of a cyclic R -module M and $[N + \psi(N) : M]$ is a semi prime ideal of R , then N is end- ψ -primary submodule of M .

Proof: Let N be a ψ -primary submodule of a cyclic R -module M and $f(m) \in N$, for some $m \in M$ and $f \in \text{End}(M)$. Assume that $m \notin N + \psi(N)$. Let $x \in M$, and since M is a cyclic, then $M = \langle a \rangle, x = ra, m = r_1 a$, for some $r, r_1 \in R$. Now, $f(m) = f(r_1 a) = r_1 f(a) \in N$. But N is ψ -primary, then either $f(a) \in N + \psi(N)$ or $r_1^n \in [N + \psi(N) : M]$ for some $n \in Z_+$ and hence $r_1 \in \sqrt{[N + \psi(N) : M]} = [N + \psi(N) : M]$ since $[N + \psi(N) : M]$ is a semi prime ideal of R . But $r_1 \notin [N + \psi(N) : M]$, for $m = r_1 a \notin N + \psi(N)$. Hence $f(a) \in N + \psi(N)$ and therefore $f(x) = f(ra) = r f(a) \in N + \psi(N)$, thus $f(M) \subseteq N + \psi(N)$ and so N is an end- ψ -primary submodule of M . Therefore N is end- ψ -primary submodule of M .

A proper submodule N of M is said to be **S-semi prime** if for each $\alpha \in \text{End}_R(M)$ and $x \in M$, if $\alpha^2(x) \in N$, then $\alpha(x) \in N$, [4].

Proposition (2.7):

If N is an end- ψ -primary submodule of an R -module M and $N + \psi(N)$ is an S -semi prime submodule of M , then N is an end- ψ -primary submodule of M .

Proof: Let N be an end- ψ -primary submodule of an R -module M and $f(m) \in N$, for some $m \in M$ and $f \in \text{End}(M)$. Assume that $m \notin N + \psi(N)$, we must prove that $f(M) \subseteq N + \psi(N)$. Since N is an end- ψ -primary of M and $m \notin N + \psi(N)$, then there exists a positive integer n such that $f^n(M) \subseteq N + \psi(N)$. But $N + \psi(N)$ is an S -semi prime submodule of M , hence $f(M) \in N + \psi(N)$ by [4]. Therefore N is an end- ψ -primary submodule of M .

Recall that a submodule N of an R -module M is called **S-relatively divisible** denoted by **S-RD** if $rM \cap N = rN$ for each $r \in R, f(M) \cap N = f(N)$ for all $f \in \text{End}(M)$, [8].

By using this concept, we can give the following result.

Corollary (2.8):

If N is an end- ψ -primary submodule of an R -module M and every submodule is S -RD submodule of M , then N is an end- ψ -primary submodule of M .

Proof: By [4, proposition 2.2.17, P.70], each proper submodule of M is an S -semi prime submodule of M . Hence by previous proposition (2.7) we get the result.

Recall that a nonzero module M is called **quasi-Dedekind** if $\text{Hom}(M/N, M) = 0$ for all nonzero submodule N of M . Equivalently, M is **quasi-Dedekind** if for any $f \in \text{End}(M), f \neq 0$, then $\ker f = 0$, (i.e. f is 1-1), [9].

Proposition (2.9):

Let M be a quasi-Dedekind R -module. Then every proper S -RD submodule of M is end- ψ -primary submodule of R -module M .

Proof: By [3, Prop. 2.9], every proper S-RD submodule of M is an end- ψ -prime. Hence every proper submodule of an R-module M is end- ψ -primary submodule by (2.2, (1)).

A nonzero R-module M is said to be **monoform** if, for each $N \subseteq M$ and for each $f \in \text{Hom}(N, M)$, $f \neq 0$, then $\ker f = 0$, [10].

By using this concept, we can give the following result.

Corollary (2.10):

Let M be a monoform R-module. Then every proper S-RD submodule of M is end- ψ -primary submodule of R-module M .

Proof: By [10], every proper monoform R-module is a quasi-Dedekind R-module. Hence by previous proposition (2.9) we get the result.

A nonzero R-module M is said to be **S- secondary module**, if for each $\alpha \in \text{End}_R(M)$, then either $\alpha(M) = M$ or there exists a positive integer n , such that $\alpha^n(M) = 0$, [11].

By using this concept, we can give the following result.

Proposition (2.11):

If N is a proper submodule of an **S- secondary** R-module M . If N is an S-semi prime submodule of M , then N is an end- ψ -primary submodule of M .

Proof: Let $f \in \text{End}(M)$, $m \in M$ such that $f(m) \in N$. Since M is an S-secondary R-module, then either $f(M) = M$ or there exists a positive integer n , such that $f^n(M) = 0$. Now, if $f^n(M) = 0 \in N$, then $f^n(M) \subseteq N + \psi(N)$. If $f(M) = M$ and $m \in M$ there exists $y \in M$, such that $m = f(y)$, $f(m) = f^2(y) \in N$, but N is an S-semi prime submodule of M , thus $m = f(y) \in N + \psi(N)$. Therefore N is end- ψ -primary submodule.

3. More about End- ψ -primary Submodules

In this section, several fundamental properties of end- ψ -primary submodule are given.

Proposition (3.1):

Let M be an R-module, $N < M$, $I \leq R$. If P is an end- ψ -primary submodule of M such that $IN \subseteq P$, then $N \subseteq P + \psi(P)$ or $I \subseteq \sqrt{[P + \psi(P): M]}$.

Proof: Suppose $IN \subseteq P$, where I is an ideal of R and N, P are two submodules. Suppose $N \not\subseteq P + \psi(P)$, then there exists $x \in N$ and $x \notin P + \psi(P)$. It is clear for each $a \in I$, thus $ax \in P$. Define $f: M \rightarrow M$ by $f(m) = a m$ for all $m \in M$, it is clear that $f \in \text{End}_R(M)$ and $f(x) = a x \in IN \subseteq P$. But P is an end- ψ -primary submodule of M and $x \notin P + \psi(P)$. Hence $f^n(M) = a^n M \subseteq P + \psi(P)$, so $a^n \in [P + \psi(P): M]$ for some $n \in \mathbb{Z}_+$. Thus, $a \in \sqrt{[P + \psi(P): M]}$ and hence $I \subseteq \sqrt{[P + \psi(P): M]}$.

Recall that a submodule N of an R-module M is said to be **fully invariant** if $f(N) \subseteq N$, for each R-endomorphism f of M . And an R-module M is said to be **fully invariant**

module if for each submodule N of M is fully invariant, [12].

By using this concept, we can give the following result.

Proposition (3.2):

Let M be a fully invariant R-module, let $\phi \in \text{End}(M)$. If N is an end- ψ -primary submodule of an R-module M and $\text{End}(M)$ is a commutative ring, such that $\phi(M) \not\subseteq N$ and $\psi(\phi^{-1}(N)) = \phi^{-1}(\psi(N))$, then $\phi^{-1}(N)$ is also end- ψ -primary submodule of M .

Proof: First, we must show that $\phi^{-1}(N)$ is a proper submodule of M . Suppose that $\phi^{-1}(N) = M$, then $\phi(M) \subseteq N$, which is a contradiction to the assumption.

Now, let $f(m) \in \phi^{-1}(N)$, where $f \in \text{End}(M)$ and $m \in M$. Then $\phi(f(m)) \in N$. Since $\text{End}(M)$ is a commutative ring, then $\phi(f(m)) \in N$. But N is an end- ψ -primary of M so either $\phi(m) \in N + \psi(N)$ or $f^n(M) \subseteq N + \psi(N)$ for some $n \in \mathbb{Z}_+$. If $\phi(m) \in N + \psi(N)$, then $\phi^{-1}(\phi(m)) = m \in \phi^{-1}(N) + \phi^{-1}(\psi(N)) = \phi^{-1}(N) + \psi(\phi^{-1}(N))$. If $f^n(M) \subseteq N + \psi(N)$, then $f^n(M) \subseteq \phi^{-1}(N) + \phi^{-1}(\psi(N)) = \phi^{-1}(N) + \psi(\phi^{-1}(N))$ (since M is a fully invariant, so $\phi(N + \psi(N)) \subseteq N + \psi(N)$, implies $N + \psi(N) \subseteq \phi^{-1}(N + \psi(N))$). Then $\phi^{-1}(N)$ is an end- ψ -primary submodule of M .

Corollary (3.3):

Let M be a multiplication R-module, let $\phi \in \text{End}(M)$. If N is an end- ψ -primary submodule of an R-module M and $\text{End}(M)$ is a commutative ring, such that $\phi(M) \not\subseteq N$ and $\psi(\phi^{-1}(N)) = \phi^{-1}(\psi(N))$, then $\phi^{-1}(N)$ is also end- ψ -primary submodule of M .

Proof: Since M is a multiplication R-module, so every submodule of M is fully invariant by [14]. Hence by previous proposition (3.2) we get the result.

Recall that a submodule N of an R-module M is said to be **fully stable** if $f(N) \subseteq N$, for each $f \in \text{Hom}(N, M)$.

And an R-module M is said to be **fully stable** module if for each submodule N of M is fully stable, [13].

Corollary (3.4):

Let M be a fully stable R-module, let $\phi \in \text{End}(M)$. If N is an end- ψ -primary submodule of an R-module M , such that $\phi(M) \not\subseteq N$ and $\psi(\phi^{-1}(N)) = \phi^{-1}(\psi(N))$, then $\phi^{-1}(N)$ is also end- ψ -primary submodule of M .

Proof: Since M is a fully stable R-module, so every submodule of M is fully invariant by [13]. Also $\text{End}(M)$ is a commutative ring by [13]. Hence by previous proposition (3.2) we get the result.

Recall that a submodule N of an R-module M is said to be **satisfy Baer Criterion** if for each R-homomorphism $f: N \rightarrow M$, there exists an element $r \in R$ such that $f(n) = rn$ for each n in N , [13].

Corollary (3.5):

Let M be an R -module which satisfies Baer Criterion, let $\phi \in \text{End}(M)$. If N is an end- ψ -primary submodules of an R -module M , such that $\phi(M) \not\subseteq N$ and $\psi(\phi^{-1}(N)) = \phi^{-1}(\psi(N))$, then $\phi^{-1}(N)$ is also end- ψ -primary submodule of M .

Proof: Since every module which satisfies Baer Criterion is fully stable by [13]. Hence by previous corollary (3.4) we get the result.

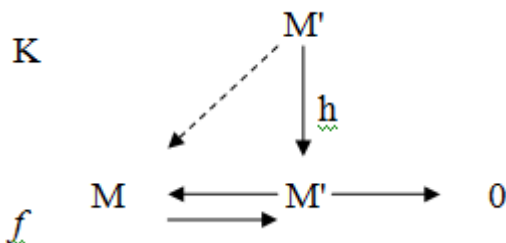
Recall that an R -module M is called **A-projective** (where A is an R -module) if for every $X < A$ and every homomorphism $\phi: M \rightarrow A/X$ can be lifted to a homomorphism $\psi: M \rightarrow M$, [15]. If M is **A-projective** for each R -module A , then M is called **projective**.

Theorem (3.6):

Let $f: M \rightarrow M'$ be an epimorphism and let $N < M$ such that $\ker f \leq N$ and $\text{End}(M)$ is a commutative ring. If N is an end- ψ -primary submodule of a module M , then $f(N)$ is an end- ψ' -primary submodule of a module of M' , where M' is M -projective module and $\psi'(f(N)) = f(\psi(N))$.

Proof: First, we must show that $f(N)$ is a proper submodule of a module M' . Suppose $f(N) = M'$. But f is an epimorphism, thus $f(N) = f(M)$ and hence $M = N + \ker f$. This implies that $M = N$. A contradiction.

Now, let $h(m') \in f(N)$, where $h \in \text{End}(M')$ and $m' \in M'$ and suppose that $m' \notin f(N) + \psi'(f(N))$, we have to show that $h^n(M') \subseteq f(N) + \psi'(f(N))$ for some $n \in \mathbb{Z}_+$. Since f is an epimorphism and $m' \in M'$, then there exists $m \in M$, such that $f(m) = m' \notin f(N) + \psi'(f(N))$, thus $m \notin N + f^{-1}(\psi'(f(N))) = N + \psi(N)$. Consider the following diagram:



since M' is M -projective module, then there exists a homomorphism $k: M' \rightarrow M$, such that $f \circ k = h$. Clearly, $k \circ f \in \text{End}(M)$. Note that $f(k \circ f(m)) = (f \circ k)(f(m)) = h(m') \in f(N)$ and since $\ker f \subseteq N$, we get $(k \circ f)(m) \in N$. But N is an end- ψ -primary submodule of M and $m \notin N + \psi(N)$.

Therefore there exist a positive integer n , such that $(k \circ f)^n(M) \subseteq N + \psi(N)$ and since $\text{End}(M)$ is a commutative ring hence $(f \circ k)^n(M) = h^n(M) \subseteq N + \psi(N)$. Thus $f(h^n(M)) \subseteq f(N) + f(\psi(N))$ and hence $h^n(f(M)) \subseteq f(N) + \psi'(f(N))$, which implies that $h^n(M') \subseteq f(N) + \psi'(f(N))$.

Corollary (3.7):

Let M be an R -module, let $K < N < M$ and N is an end- ψ -primary of M and $\text{End}(M/K)$ is a commutative ring. Then N/K is end- ψ' -primary in M/K , provided that M/K is M -projective.

Recall that an R -module M is **A-injective** (where A is an R -module) if for every $X \leq M$, any homomorphism $\phi: X \rightarrow$

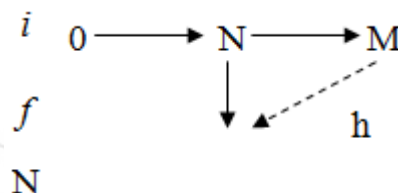
M can be extended to a homomorphism $\psi: A \rightarrow X$ [15]. If M is **M-injective** M is called **quasi-injective**[16].

Proposition (3.8):

Let K be an end- ψ -primary of an R -module M and let $N < M$ which is M -injective and $\psi(K) \subseteq K$. Then either $N \subseteq K$ or $K \cap N$ is an end- ψ -primary in N .

Proof: Suppose that $N \not\subseteq K$, then $K \cap N$ is a proper submodule in N . Let $f(x) \in K \cap N$, where $f \in \text{End}(N)$ and $x \in N$. Suppose $x \notin (K \cap N) + \psi'(K \cap N)$, where $\psi': \delta(N) \rightarrow \delta(N) \cup \{\phi\}$ be a function, then $x \notin K$. We must show that $f(N) \subseteq (K \cap N) + \psi'(K \cap N)$.

Now, consider the following diagram:



Where i is the inclusion map.

Since N is M -injective, then there exists $h: M \rightarrow N$, such that $h \circ i = f$, clearly $h \in \text{End}(M)$. But $f(X) = (h \circ i)(x) = h(x) \in K$. Since K is an end- ψ -prime submodule of M and $x \notin K + \psi(K)$, this implies that $h(M) \subseteq K + \psi(K)$. Also, $f(N) = (h \circ i)(N) = h(N) \subseteq N$ (since $f(N) \subseteq K \cap N$) and $f(N) \subseteq h(N) \subseteq h(M) \subseteq K + \psi(K)$. Therefore, $f(N) \subseteq N \cap (K + \psi(K)) = N \cap K \subseteq N \cap K + \psi'(N \cap K)$.

Corollary (3.9):

Let K be an end- ψ -primary submodule of a quasi-injective R -module M , and let $N < M$. Then either $N \leq K$ or $K \cap N$ is an end- ψ -primary in N .

Proposition (3.10):

Let M be an R -module and let $K < N < M$ and K is fully invariant. If N/K is an end- ψ' -primary submodule of M/K and $\psi'(N/K) = (N + \psi(N)) / K$, then N is an end- ψ -primary submodule of M .

Proof: Suppose that $f(m) \in N$, where $f \in \text{End}(M)$ and $m \in M$. We must show that either $m \in N + \psi(N)$ or $f^n(M) \subseteq N + \psi(N)$ for some $n \in \mathbb{Z}_+$. Define $f^*: M/K \rightarrow M/K$ by $f^*(x + K) = f(x) + K, \forall x \in M$. To prove f^* is well define, let $x + K = y + K$ where $x, y \in M$, then $x - y \in K$ and hence $f(x - y) \in f(K) \subseteq K$, since K is fully invariant. This implies that $f(x) - f(y) \in K$. Thus $f(x) + K = f(y) + K$. Now, $f^*(m + K) = f(m) + K \in N/K$. But N/K is an end- ψ' -primary of M/K , so either $m + K \in N/K + \psi'(N/K) = N/K + (N + \psi(N)) / K = (N + \psi(N)) / K$

and thus $m \in N + \psi(N)$ or $f^{*n}(M/K) \subseteq N/K + \psi'(N/K)$ for some $n \in \mathbb{Z}_+$. Thus $f^{*n}(m + K) \subseteq N/K + \psi'(N/K)$ which implies that $f^n(m) + K \subseteq N/K + (N + \psi(N)) / K = (N + \psi(N)) / K$, thus $f^n(m) \in N + \psi(N)$ and since m is an arbitrary element of M , thus $f^n(M) \subseteq N + \psi(N)$ for some $n \in \mathbb{Z}_+$.

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