Chromatic Edge Stability Number of a Graph

A. Muthukamatchi

Department of Mathematics, R.D. Government Arts College, Sivagangai-630 561, Tamil Nadu, India

Abstract: Let \( G \) be a graph with chromatic number \( \chi(G) = k \). The chromatic edge stability number \( es_k(G) \) is the minimum number of edges whose removal results in a graph \( G_1 \), with \( \chi(G_1) = \chi(G) - 1 \). The chromatic bondage number \( \rho(G) \) is the minimum number of edges between two color classes in a \( k \)-coloring of \( G \), where the minimum is taken over all \( k \)-colorings of \( G \). We present several interesting results and unsolved problems on Chromatic edge stability number and Chromatic bondage number. In this paper we introduce two fundamental parameters which involve independent domination and chromatic number.

Keywords: chromatic number, chromatic bondage number, Chromatic edge stability number

1. Introduction

Let \( G \) be a graph with chromatic number \( \chi(G) = k \). The chromatic edge stability number \( es_k(G) \) is the minimum number of edges whose removal results in a graph \( G_1 \) with \( \chi(G_1) = \chi(G) - 1 \). The chromatic bondage number \( \rho(G) \) is the minimum number of edges between two color classes in a \( k \)-coloring of \( G \), where the minimum is taken over all \( k \)-colorings of \( G \). We present several interesting results and unsolved problems on Chromatic edge stability number and Chromatic bondage number.

For any graph theoretic parameter the effect of removal of an edge on the parameter is of practical importance. Another related problem is the determination of the minimum number of edges whose removal alters the value of the parameter. For example, the minimum number of edges of \( G \) whose removal increases the domination number of \( G \) was studied by Bauer et al., [6] who called it as the edge stability number of \( G \). Fink et al., [12] studied the same concept and used the term bondage number. We observe that the chromatic number of a graph can be reduced by removing a set of edges from the graph. Hence, analogous to the concept of bondage number for domination, it is natural to study the minimum number of edges whose removal reduces the chromatic number of the graph, which we call the chromatic edge stability number. In this paper, we initiate a study of these two parameters.

2. Main Results

Definition 1.1. The chromatic edge stability number of a graph \( G \) is defined to be the minimum number of edges of \( G \) whose removal results in a graph \( G_1 \) with \( \chi(G_1) = \chi(G) - 1 \) and is denoted by \( es_k(G) \).

Example 1.2.
(i) \( es_2(G) \) is \( m \) if and only if \( G \) is bipartite.
(ii) \( es_2(K_n) = 1 \) for all \( n \geq 2 \).
(iii) \( es_2(C_n) = 1 \) if \( n \) is odd.

If \([V_1, V_2, \ldots, V_k]\) is a \( k \)-coloring of a \( k \)-chromatic graph \( G \) and if \( G_1 \) is the graph obtained from \( G \) by removing all the edges between two color classes \( V_i \) and \( V_j \) for some \( i \neq j \), then \( \chi(G_1) = k - 1 \). This observation motivates the following definition.

Definition 1.3. The minimum number of edges between two color classes in a \( k \)-coloring of a \( k \)-chromatic graph \( G \), where the minimum is taken over all \( k \)-colorings of \( G \), is called the chromatic bondage number of \( G \) and is denoted by \( \rho(G) \).

Obviously, for any graph \( G \), \( es_k(G) \leq \rho(G) \). Further, for all the graphs given in Example 1.2, \( es_k \neq \rho \). In the following lemma we give another family of graphs with this property.

Lemma 1.4. For the complete tripartite graph \( G = K_{n, n, n} \), \( es_k(G) = \lambda^2 \).

Proof. Since \( \rho(G) = \lambda^2 \), we have \( es_k(G) \leq \lambda^2 \). Further \( G \) is decomposable into \( \lambda^2 \) triangles and hence \( es_k(G) \geq \lambda^2 \). Thus \( es_k(G) = \lambda^2 \).

For the complete \( k \)-partite graph \( G = K_{\lambda_1, \lambda_2, \ldots, \lambda_k} \), we have \( \rho(G) = \min \{ \lambda_i \lambda_j : 1 \leq i < j \leq k \} \). In this connection we pose the following conjecture.

Conjecture 1.5. For the complete \( k \)-partite graph \( G = K_{\lambda_1, \lambda_2, \ldots, \lambda_k} \), \( es_k(G) = \rho(G) \).

More generally, we have the following problem.

Problem 1.6. Characterize graphs \( G \) for which \( es_k(G) = \rho(G) \).

The following theorem shows that the difference between \( \rho \) and \( es_k \) can be made arbitrarily large.

Theorem 1.7. Given two positive integers \( k \) and \( r \) with \( k \geq 3 \), there exists a graph \( G \) such that \( \chi(G) = k \) and \( \rho - es_k > r \).

Proof. Let \( G_1 \) be a copy of \( K_k \). For each edge \( e = xy \) of \( G_1 \), we attach \( r + 1 \) copies of \( K_k \), say \( G_{i1}, G_{i2}, \ldots, G_{ir+1} \) such that \( V(G_{ij}) \cap V(G_{ij+1}) = \{ x, y \} \), where \( 1 \leq j \leq r + 1 \). Let \( G \) be the resultant graph. Clearly \( \chi(G) = k \). If \( H \) is the subgraph of \( G \) obtained by removing the \( \binom{k}{2} \) edges of \( G_1 \), we have \( \chi(H) = k - 1 \). Further, if \( H_1 \) is any subgraph of \( G \) obtained by removing a set of \( m \) edges, where \( m < \binom{k}{2} \), then \( H_1 \) contains a copy of \( K_k \) and hence \( \chi(H_1) = k \). Thus \( es_k(G) = \binom{k}{2} \). Now, let \( V_1 \) and \( V_2 \) be two color classes in a \( k \)-coloring of \( G \). Let \( e \) be the...
unique edge of $G_1$ whose ends are in $V_1$ and $V_2$. Now, each $G_{ij}$, where $1 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor$, $i \neq s$ and $1 \leq j \leq r + 1$, contains exactly one edge with ends in $V_j$ and $V_2$. Hence $\rho = \lceil \binom{k}{2} \rceil - 1 + 1$. Hence $\rho - e_s = r\lceil \binom{k}{2} \rceil > r$.

**Remark 1.8.** Obviously, $e_s(G) \leq \rho(G) \leq \binom{m}{2}$. If $e_s(G) = \binom{m}{2}$ then $\rho(G) = \binom{m}{2}$ and hence the number of edges between any two color classes in any $k$-coloring of $G$ is $\binom{m}{2}$. However, the converse is not true. For example, for the uniquely 3-colorable graph $G$ given in Figure 1.1, the number of edges between any two color classes in the 3-coloring of $G$ is $\binom{3}{2}$. However, $e_s(G) = 2$.

**Figure 1.1**

In the following theorem, we characterize k-chromatic graphs with $\Delta = n - 1$ for which $\rho = \binom{m}{2}$.

**Theorem 1.9.** Let $G$ be a k-chromatic graph with $\Delta = n - 1$. Then $\rho = \binom{m}{2}$ if and only if $G$ has exactly one cut vertex and has $\rho$ blocks each isomorphic to $K_2$.

**Proof.** Let $u$ be a vertex of $G$ with deg $u = n - 1$. Suppose $\rho = \binom{m}{2}$. Let $\{V_1,V_2,...,V_i\}$ be any k-coloring of $G$ with $V_i = \{u\}$. Now, the number of edges between any pair of color classes $(V_i,V_j)$ is $\rho$ and hence $|V_i| = \rho$, for all $i$, where $2 \leq i \leq k$ and the subgraph induced by the $\rho$ edges between $V_i$ and $V_j$, where $2 \leq i < j \leq k$, is isomorphic to $\rho K_2$. Let $V_i = \{v_{i1},v_{i2},...,v_{ip}\}$ and $E(V_i,V_j) = \{v_{i1}v_{j1},v_{i2}v_{j2},...,v_{ip}v_{jp}\}$, $2 \leq i < j \leq k$. Now, for each $i = 1,2,...,p$, the subgraph of $G$ induced by $\{u,v_{i1},v_{i2},...,v_{ip}\}$ is isomorphic to $K_2$. Hence $G$ is the graph with exactly one cut-vertex and $\rho$ blocks each isomorphic to $K_2$.

The converse is obvious.

**Theorem 1.10.** For any r-regular graph $G$ on $n$ vertices, $\rho(G) \leq (n - r)^2$. Further equality holds if and only if $n - r$ divides $n$ and $G$ is isomorphic to the complete k-partite graph $K_{\lambda_1,...,\lambda_k}$ where $\lambda = n - r$ and $k = \frac{n}{n - r}$.

**Proof.** Since $G$ is r-regular, the number of vertices in a color class in any k-coloring of $G$ is at most $n - r$ and hence $\rho(G) \leq (n - r)^2$.

Suppose $\rho(G) = (n - r)^2$. Let $\{V_1,V_2,...,V_i\}$ be a k-coloring of $G$. Since $G$ is r-regular, $|V_i| \leq n - r$ for all $i = 1,2,...,k$. Suppose $|V_i| < n - r$ for some $i$. Then the number of edges between $V_i$ and $V_j$ for all $j \neq i$ is less than $(n - r)^2$, which is a contradiction. Hence $|V_i| = n - r$ for all $i = 1,2,...,k$. Since the number of edges between $V_i$ and $V_j$ for all $i \neq j$, is $(n - r)^2$, it follows that $G$ is a complete k-partite graph in which each part contains exactly $n - r$ vertices. Hence $k(n - r) = n$, so that $n - r$ divides $n$. The converse is obvious.

**Theorem 1.11.** Let $G$ be a k-chromatic graph having a vertex $u$ with deg $u = n - 1$. Then $\rho = r$, where $r$ is the minimum number of vertices in a color class other than $\{u\}$ in a k-coloring of $G$ and the minimum is taken over all k-colorings of $G$.

**Proof.** Since $\{u\}$ is a color class in any k-coloring of $G$, we have $\rho \leq r$. Suppose $\{V_1,V_2,...,V_{n-1}\}$ is a k-coloring of $G$ such that the number of edges between $V_1$ and $V_2$ is less than $r$. Since $|V_1| \geq r$ and $|V_2| \geq r$, there exists a set $S \subseteq V_1$ such that $|S| \geq |V_1| - r + 1$ and $V_2 \cup S$ is independent. Hence $\{V_1 \cup S, V_2 \cup S, V_3, ..., V_{n-1}\}$ is a k-coloring of $G$ with $|V_1 \setminus S| < r$, which is a contradiction to the minimality of $r$. Thus $\rho = r$.

**Corollary 1.12.** For any k-chromatic graph $G$ with $\Delta = n - 1$, we have $\rho(G) \leq \beta_0(G)$, where $\beta_0(G)$ is the independence number of $G$.

**Proof.** Since $\rho = r$ and $r \leq \beta_0$, the result follows.

**Remark 1.13.** The bound given in the above corollary is sharp. For example, for the graph $G$ given in Figure 1.2, $\rho(G) = \beta_0(G) = 2$.

**Theorem 1.14.** Let $G$ be a k-chromatic graph having a vertex $u$ with deg $u = n - 1$. Then $\rho = \beta_0$ if and only if for any k-coloring $\{V_1,V_2,...,V_{n-1}\}$ of $G$, $|V_i| = \beta_0$ for all $i = 1,2,...,k - 1$ and $G_{ij}$ has a perfect matching, where $G_{ij}$ is the induced subgraph $(V_i \cup V_j)$, $1 \leq i < j \leq k - 1$.

**Proof.** Suppose $\rho = \beta_0$. Let $\{V_1,V_2,...,V_{n-1}\}$ be any k-coloring of $G$. Since the number of edges between the color classes $V_i$ and $\{u\}$ is $|V_i|$, it follows that $|V_i| \geq \beta_0$. Further since each $V_i$ is an independent set in $G$, we have $|V_i| = \beta_0$.

**Theorem 1.15.** Let $G$ be a bipartite graph with bipartition $(X,Y)$. Then $G$ contains a matching that saturates every vertex in $X$ if and only if $|N(S)| \geq |S|$ for all $S \subseteq X$.

Now, consider $G_{ij} = (V_i \cup V_j)$, $1 \leq i < j$. Let $S \subseteq V_i$. If $|N(S)| \leq |S|$, then $X_i \cup (V_j \setminus N(S)) \cup S$ is an independent set and $|X_i| > \beta_0$, which is a contradiction. Thus $|N(S)| \geq |S|$ for all $S \subseteq V_i$ and hence it follows from Theorem 1.15, that $G_{ij}$ has a perfect matching.

Conversely, if for any k-coloring $\{V_1,V_2,...,V_{n-1}\}$, $|V_i| = \beta_0$ and $G_{ij}$ has a perfect matching, then it follows that $|E(G_{ij})| \geq \beta_0$ and $|E\left((V_i \cup V_j)\right)| = \beta_0$ and hence $\rho = \beta_0$. 

**Volume 6 Issue 5, May 2017**

www.ijsr.net

Licensed Under Creative Commons Attribution CC BY
Corollary 1.16. Let $G$ be a $k$-chromatic graph with $\Delta = n - 1$ and $\rho(G) = \beta_0(G)$. Then $n = (k - 1)\beta_0 + 1$. Also $|E(G)| \leq \frac{(k-1)\beta_0(k-1)\beta_0+2}{2}$ and equality holds if and only if $G$ is isomorphic to the complete $k$-partite graph $K_{1,\beta_0,\beta_0,...,\beta_0}$.

Corollary 1.17. Let $G$ be a 2-chromatic graph with $\Delta = n - 1$. Then $\rho(G) = \beta_0(G)$ if and only if $G$ is isomorphic to the star $K_{1,n-1}$.

Corollary 1.18. Let $G$ be a 3-chromatic graph with $\Delta = n - 1$ and $\rho(G) = \beta_0(G)$. Then $3\beta_0 \leq |E(G)| \leq \beta_0(\beta_0 + 1)$. Further $|E(G)| = 3\beta_0$ if and only if $G$ has exactly one cut vertex $u$ and has $\beta_0$ blocks each isomorphic to $K_2$. Also $|E(G)| = \beta_0(\beta_0 + 1)$ if and only if $G$ is isomorphic to $K_{1,\beta_0,\beta_0}$.

Proof. Let $\{V_1,V_2,[u]\}$ be a 3-coloring of $G$. Since $|V_1| = |V_2| = \beta_0$ and $(V_1 \cup V_2)$ has a perfect matching, it follows that $3\beta_0 \leq |E(G)| \leq \beta_0(\beta_0 + 1)$. Now, $|E(G)| = 3\beta_0$ if and only if $(V_1 \cup V_2) = \beta_0K_2$ and hence $u$ is the unique cut vertex of $G$ and $G$ has $\beta_0$ blocks each isomorphic to $K_2$. Further $|E(G)| = \beta_0(\beta_0 + 1)$ if and only if $(V_1 \cup V_2) = \beta_0K_{1,\beta_0,\beta_0}$ and hence the result follows.

We now proceed to investigate properties of graphs $G$ for which $es_3(G) = 1$.

Theorem 1.19. Let $G$ be a $k$-chromatic graph. Then $es_3(G) = 1$ if and only if there is no $k$-coloring $\{V_1,V_2,...,V_k\}$ of $G$ such that $|V_1| = 1$ and there is exactly one edge between $V_1$ and $V_2$.

Proof. Suppose $es_3(G) = 1$. Let $e = uv$ be an edge of $G$ such that $\chi(G - e) = k - 1$. Let $\{V_1,V_2,...,V_k-1\}$ be a $(k-1)$-coloring of $G - e$. Clearly $u$ and $v$ belong to the same color class, say $V_i$. Now, $\{\{u\},V_1-\{u\},V_2,...,V_{k-1}\}$ is a $k$-coloring of $G$ and $e = uv$ is the only edge between $\{u\}$ and $V_1-\{u\}$. The converse is obvious.

Theorem 1.20. Let $G$ be a $k$-chromatic graph with $es_3(G) = 1$. Then there exist two adjacent vertices $u$ and $v$ such that $deg(u) + deg(v) \geq 2( k - 1)$.

Proof. It follows from Theorem 1.19 that there exists a $k$-coloring $\{\{u\},V_1,V_2,...,V_{k-1}\}$ of $G$ such that there is exactly one edge $e = uv$ between $\{u\}$ and $V_i$. Since $u$ is adjacent to at least one vertex of $V_i$, for all $i \geq 1$, we have $deg(u) \geq k - 1$. We now claim that $deg(v) \geq k - 1$. Suppose $deg(v) < k - 1$. Then there exists a color class, say $V_j$, such that $v$ is not adjacent to any vertex of $V_j$. Now, $\{(V_1-\{v\})\cup\{u\},V_j\cup\{v\},V_2,...,V_{k-1}\}$ is a $(k - 1)$-coloring of $G$, which is a contradiction. Hence $deg(v) \geq k - 1$ so that $deg(u) + deg(v) \geq 2( k - 1)$.

Theorem 1.21. In the $k$-coloring of a uniquely $k$-colorable graph, the subgraph induced by the union of any two color classes is connected.

Theorem 1.22. Let $G$ be a uniquely $k$-colorable graph. Then $es_3(G) = 1$ if and only if the $k$-coloring of $G$ contains two singleton color classes.

Proof. Let $\{V_1,V_2,...,V_k\}$ be the $k$-coloring of $G$. Let $es_3(G) = 1$. Then by Theorem 1.19 there exist color classes $V_1$ and $V_2$ such that $|V_1| = 1$ and there is exactly one edge between $V_1$ and $V_2$. By Theorem 1.21, we have $(V_1 \cup V_2)$ is connected and hence $|V_2| = 1$. The converse is obvious.

Problem 1.23. Characterize graphs $G$ for which $es_3(G) = 1$.

The following theorem gives Nordhaus-Gaddum type result for the parameter $es_3(G)$.

Theorem 1.24. For any graph $G$, we have

(i) $2 \leq es_3(G) + es_3(\overline{G}) \leq \left(\frac{n}{2}\right)^2$

(ii) $1 \leq es_3(G) - es_3(\overline{G}) \leq m\left(\frac{n}{2}\right) - m$.

Further, the following are equivalent.

(a) $es_3(G) + es_3(\overline{G}) = \left(\frac{n}{2}\right)^2$

(b) $es_3(G) - es_3(\overline{G}) = m\left(\frac{n}{2}\right) - m$

(c) $G$ is one of the graphs $P_n, \overline{P_n}, P_4C_4$ and $\overline{C_6}$.

Proof. Since for any graph $G$, $1 \leq es_3(G) \leq m$, the inequalities follow. Now, $es_3(G) + es_3(\overline{G}) = \left(\frac{n}{2}\right)^2$ if and only if $es_3(G) = |E(G)| = m$ and $es_3(\overline{G}) = |E(\overline{G})| = \left(\frac{n}{2}\right)^2 - m$. Hence $G$ and $\overline{G}$ are both bipartite, so that (a), (b) and (c) are equivalent.

3. Conclusion

In this paper, we initiate a study of two parameters chromatic edge stability number $es_e$ and chromatic bondage number $\rho$ and investigate properties of graphs $G$ for which $es_3(G) = 1$.

4. Future Scope

Further we can develop this paper and find the relation between chromatic edge stability number $es_e$ and chromatic bondage number $\rho$ for all graphs.

References


Author Profile

A. Muthukamatchi received the M.sc., and Ph.D., degrees in Mathematics from Madurai Kamaraj University. He has nearly 23 years of experience in various colleges. Now he is working as Assistant Professor of Mathematics at R. D. Govt. Arts College, Sivaganga, Tamilnadu.