International Journal of Science and Research (IJSR)
ISSN (Online): 2319-7064
Index Copernicus Value (2015): 78.96 | Impact Factor (2015): 6.391

Chromatic Edge Stability Number of a Graph

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Abstract: ***Let G be a graph with chromatic number $\chi(G) = k$. The chromatic edge stability number $es_\chi(G)$ is the minimum number of edges whose removal results in a graph G_1 with $\chi(G_1) = \chi(G)-1$. The chromatic bondage number $\rho(G)$ is the minimum number of edges between two color classes in a k coloring of G, where the minimum is taken over all k-colorings of G. We present several interesting results and unsolved problems on Chromatic edge stability number and Chromatic bondage number. In this paper we introduce two fundamental parameters which involve independent domination and chromatic number.***

Keywords: chromatic number, chromatic bondage number, Chromatic edge stability number

1. Introduction

Let G be a graph with chromatic number $\chi(G) = k$. The chromatic edge stability number $es_\chi(G)$ is the minimum number of edges whose removal results in a graph G_1 with $\chi(G_1) = \chi(G) - 1$. The chromatic bondage number $\rho(G)$ is the minimum number of edges between two color classes in a k-coloring of G, where the minimum is taken over all k-colorings of G. We present several interesting results and unsolved problems on Chromatic edge stability number and Chromatic bondage number.

For any graph theoretic parameter the effect of removal of an edge on the parameter is of practical importance. Another related problem is the determination of the minimum number of edges whose removal alters the value of the parameter. For example, the minimum number of edges of G whose removal increases the domination number of G was studied by Bauer et al., [6] who called it as the edge stability number of G. Fink et al., [12] studied the same concept and used the term bondage number. We observe that the chromatic number of a graph can be reduced by removing a set of edges from the graph. Hence, analogous to the concept of bondage number for domination, it is natural to study the minimum number of edges whose removal reduces the chromatic number of the graph, which we call the chromatic edge stability number. In this paper , we initiate a study of these two parameters.

2. Main Results

Definition 1.1. *The chromatic edge stability number of a graph G is defined to be the minimum number of edges of G whose removal results in a graph G_1 with $\chi(G_1) = \chi(G)-1$ and is denoted by $es_\chi(G)$.*

Example 1.2.
(i) $es_\chi(G) = m$ if and only if G is bipartite.
(ii) $es_\chi(K_n) = 1$ for all $n \geq 2$.•
(iii) $es_\chi(C_n) = 1$ if n is odd.

If $\{V_1,V_2,...,V_k\}$ is a k-coloring of a k-chromatic graph G and if G_1 is the graph obtained from G by removing all the edges between two color classes V_i and V_j for some $i \neq j$, then $\chi(G_1) = k - 1$. This observation motivates the following definition.

Definition 1.3. *The minimum number of edges between two color classes in a k-coloring of a k-chromatic graph G, where the minimum is taken over all k-colorings of G, is called the chromatic bondage number of G and is denoted by $\rho(G)$.*

Obviously, for any graph G, $es_\chi(G) \leq \rho(G)$. Further, for all the graphs given in Example 1.2, $es_\chi = \rho$. In the following lemma we give another family of graphs with this property.

Lemma 1.4. *For the complete tripartite graph $G = K_{\lambda,\lambda,\lambda}$, $es_\chi(G) = \lambda^2$.*

Proof. Since $\rho(G) = \lambda^2$, we have $es_\chi(G) \leq \lambda^2$. Further G is decomposable into λ^2 triangles and hence $es_\chi(G) \geq \lambda^2$. Thus $es_\chi(G) = \lambda^2$.

For the complete k-partite graph $G = K_{\lambda_1, \lambda_2, \lambda_3,..., \lambda_k}$, we have $\rho(G) = \min\{\lambda_i \lambda_j : 1 \leq i < j \leq k\}$. In this connection we pose the following conjecture.

Conjecture 1.5. *For the complete k-partite graph $G = K_{\lambda_1, \lambda_2, \lambda_3,..., \lambda_k}$, $es_\chi(G) = \rho(G)$.*

More generally, we have the following problem.

Problem 1.6. *Characterize graphs G for which $es_\chi(G) = \rho(G)$.*

The following theorem shows that the difference between ρ and es_χ can be made arbitrarily large.

Theorem 1.7. *Given two positive integers k and r with $k \geq 3$, there exists a graph G such that $\chi(G) = k$ and $\rho - es_\chi > r$.*

Proof. Let G_1 be a copy of K_k. For each edge $e = xy$ of G_1, we attach $r + 1$ copies of K_k, say $G_{1,1}, G_{i,2},..., G_{i,r+1}$ such that $V(G_{i,j}) \cap V(G_1) = \{x,y\}$, where $1 \leq j \leq r + 1$. Let G be the resultant graph. Clearly $\chi(G) = k$. If H is the subgraph of G obtained by removing the $\binom{k}{2}$ edges of G_1, we have $\chi(H) = k - 1$.Further, if H_1 is any subgraph of G obtained by removing a set of m edges, where $m < \binom{k}{2}$, then H_1 contains a copy of K_k and hence $\chi(H_1) = k$. Thus $es_\chi(G) = \binom{k}{2}$. Now, let V_1 and V_2 be two color classes in a k-coloring of G. Let e_s be the unique edge of G_1 whose ends are in V_1 and V_2. Now, each $G_{i,j}$, where $1 \le i \le \binom{k}{2}$, $i \ne s$ and $1 \le j \le r+1$, contains exactly one edge with ends in V_1 and V_2. Hence $\rho = [\binom{k}{2} - 1](r+1) + 1$. Hence $\rho - es_\chi = r[\binom{k}{2} - 1] > r$.

Remark 1.8. *Obviously, $es_\chi(G) \le \rho(G) \le \frac{m}{\binom{k}{2}}$. If $es_\chi(G) = \frac{m}{\binom{k}{2}}$ then $\rho(G) = \frac{m}{\binom{k}{2}}$ and hence the number of edges between any two color classes in any k-coloring of G is $\frac{m}{\binom{k}{2}}$. However, the converse is not true. For example, for the uniquely 3-colorable graph G given in Figure 1.1, the number of edges between any two color classes in the 3-coloring of G is $3 = \frac{m}{\binom{k}{2}}$. However, $es_\chi(G) = 2$.*

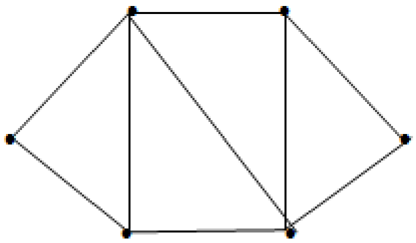

Figure 1.1

In the following theorem, we characterize k-chromatic graphs with $\Delta = n - 1$ for which $\rho = \frac{m}{\binom{k}{2}}$.

Theorem 1.9. *Let G be a k-chromatic graph with $\Delta = n-1$. Then $\rho = \frac{m}{\binom{k}{2}}$ if and only if G has exactly one cut vertex and has ρ blocks each isomorphic to K_k.*

Proof. Let u be a vertex of G with deg $u = n - 1$. Suppose $\rho = \frac{m}{\binom{k}{2}}$. Let $\{V_1, V_2, \ldots, V_k\}$ be any k-coloring of G with $V_1 = \{u\}$. Now, the number of edges between any pair of color classes (V_i, V_j) is ρ and hence $|V_i| = \rho$, for all i, where $2 \le i \le k$ and the subgraph induced by the ρ edges between V_i and V_j, where $2 \le i < j \le k$, is isomorphic to ρK_2. Let $V_i = \{v_{i1}, v_{i2}, \ldots, v_{i\rho}\}$ and $E(V_i \cup V_j) = \{v_{i1}v_{j1}, v_{i2}v_{j2}, \ldots, v_{i\rho}v_{j\rho}\}$, $2 \le i < j \le k$. Now, for each $i = 1,2,\ldots,\rho$, the subgraph of G induced by $\{u, v_{2i}, v_{3i}, \ldots, v_{ki}\}$ is isomorphic to K_k. Hence G is the graph with exactly one cut-vertex and ρ blocks each isomorphic to K_k.

The converse is obvious.

Theorem 1.10. *For any r-regular graph G on n vertices, $\rho(G) \le (n-r)^2$. Further equality holds if and only if $n - r$ divides n and G is isomorphic to the complete k-partite graph $K_{\lambda,\lambda,\ldots,\lambda}$ where $\lambda = n - r$ and $k = \frac{n}{n-r}$.*

Proof. Since G is r-regular, the number of vertices in a color class in any k-coloring of G is at most $n - r$ and hence $\rho(G) \le (n-r)^2$.

Suppose $\rho(G) = (n-r)^2$. Let $\{V_1, V_2, \ldots, V_k\}$ be a k-coloring of G. Since G is r-regular, $|V_i| \le n - r$ for all $i = 1,2,\ldots,k$. Suppose $|V_i| < n - r$ for some i. Then the number of edges between V_i and V_j for all $j \ne i$ is less than $(n-r)^2$, which is a contradiction. Hence $|V_i| = n - r$ for all $i = 1,2,\ldots,k$. Since the number of edges between V_i and V_j for all $i \ne j$, is $(n-r)^2$, it follows that G is a complete k-partite graph in which each part contains exactly $n - r$ vertices. Hence $k(n-r) = n$, so that $n - r$ divides n. The converse is obvious.

Theorem 1.11. *Let G be a k-chromatic graph having a vertex u with deg $u = n - 1$. Then $\rho = r$, where r is the minimum number of vertices in a color class other than $\{u\}$ in a k-coloring of G and the minimum is taken over all k-colorings of G.*

Proof. Since $\{u\}$ is a color class in any k-coloring of G, we have $\rho \le r$. Suppose $\{V_1, V_2, \ldots, V_{k-1}, \{u\}\}$ is a k-coloring of G such that the number of edges between V_1 and V_2 is less than r. Since $|V_1| \ge r$ and $|V_2| \ge r$, there exists a set $S \subseteq V_1$ such that $|S| \ge |V_1| - r + 1$ and $V_2 \cup S$ is independent. Hence $\{V_1 \setminus S, V_2 \cup S, V_3, \ldots, V_{k-1}, \{u\}\}$ is a k-coloring of G with $|V_1 \setminus S| < r$, which is a contradiction to the minimality of r. Thus $\rho = r$.

Corollary 1.12. *For any k-chromatic graph G with $\Delta = n - 1$, we have $\rho(G) \le \beta_0(G)$, where $\beta_0(G)$ is the independence number of G.*

Proof. Since $\rho = r$ and $r \le \beta_0$, the result follows.

Remark 1.13. *The bound given in the above corollary is sharp. For example, for the graph G given in Figure 1.2, $\rho(G) = \beta_0(G) = 2$.*

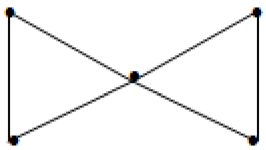

Figure 1.2

Theorem1.14. *Let G be a k-chromatic graph having a vertex u with deg $u = n - 1$. Then $\rho = \beta_0$ if and only if for any k-coloring $\{V_1, V_2, \ldots, V_{k-1}, \{u\}\}$ of G, $|V_i| = \beta_0$ for all $i = 1,2,\ldots,k-1$ and G_{ij} has a perfect matching, where G_{ij} is the induced subgraph $\langle V_i \cup V_j \rangle$, $1 \le i < j \le k-1$.*

Proof. Suppose $\rho = \beta_0$. Let $\{V_1, V_2, \ldots, V_{k-1}, \{u\}\}$ be any k-coloring of G. Since the number of edges between the color classes V_i and $\{u\}$ is $|V_i|$, it follows that $|V_i| \ge \beta_0$. Further since each V_i is an independent set in G, we have $|V_i| = \beta_0$.

Theorem 1.15. *Let G be a bipartite graph with bipartition(X,Y). Then G contains a matching that saturates every vertex in X if and only if $|N(S)| \ge |S|$ for all $S \subseteq X$*

Now, consider $G_{ij} = \langle V_i \cup V_j \rangle$, $i < j$. Let $S \subseteq Vi$. If $|N(S)| < |S|$, then $X_1 = (V_j - N(S)) \cup S$ is an independent set and $|X| > \beta_0$, which is a contradiction. Thus $|N(S)| \ge |S|$ for all $S \subseteq V_i$ and hence it follows from Theorem 1.15, that G_{ij} has a perfect matching.

Conversely, if for any k-coloring $\{V_1, V_2, \ldots, V_{k-1}, \{u\}\}$, $|V_i| = \beta_0$ and G_{ij} has a perfect matching, then it follows that $|E(G_{ij})| \ge \beta_0$ and $|E(\langle V_i \cup \{u\} \rangle)| = \beta_0$ and hence $\rho = \beta_0$.

Corollary 1.16. *Let G be a k-chromatic graph with* $\Delta = n - 1$ *and* $\rho(G) = \beta_0(G)$. *Then* $n = (k-1)\beta_0 + 1$. *Also* $|E(G)| \le \frac{(k-1)\beta_0[(k-2)\beta_0+2]}{2}$ *and equality holds if and only if G is isomorphic to the complete k-partite graph* $K_{1,\beta_0,\beta_0,\ldots,\beta_0}$.

Corollary1.17. *Let G be a 2-chromatic graph with* $\Delta = n - 1$. *Then* $\rho(G) = \beta_0(G)$ *if and only if G is isomorphic to the star* $K_{1,n-1}$.

Corollary 1.18. Let G be a 3-chromatic graph with $\Delta = n - 1$ *and* $\rho(G) = \beta_0(G)$. *Then* $3\beta_0 \le |E(G)| \le \beta_0(\beta_0 + 1)$. *Further* $|E(G)| = 3\beta_0$ *if and only if G has exactly one cut vertex u and has* β_0 *blocks each isomorphic to* K_3. *Also* $|E(G)| = \beta_0(\beta_0 + 1)$ *if and only if G is isomorphic to* K_{1,β_0,β_0}.

Proof. Let $\{V_1,V_2,\{u\}\}$ be a 3-coloring of G. Since $|V_1| = |V_2| = \beta_0$ and $\langle V_1 \cup V_2\rangle$ has a perfect matching, it follows that $3\beta_0 \le |E(G)| \le \beta_0(\beta_0 + 1)$. Now, $|E(G)| = 3\beta_0$. if and only if $\langle V_1 \cup V_2\rangle = \beta_0 K_2$ and hence u is the unique cut vertex of G and G has β_0 blocks each isomorphic to K_3. Further $|E(G)| = \beta_0(\beta_0 + 1)$ if and only if $\langle V_1 \cup V_2\rangle = K_{\beta_0,\beta_0}$ and hence the result follows.
We now proceed to investigate properties of graphs G for which $es_\chi(G) = 1$.

Theorem 1.19. *Let G be a k-chromatic graph. Then* $es_\chi(G) = 1$ *if and only if there exists a k-coloring* $\{V_1,V_2,\ldots,V_k\}$ *of G such that* $|V_1| = 1$ *and there is exactly one edge between* V_1 *and* V_2.

Proof. Suppose $es_\chi(G) = 1$. Let $e = uv$ be an edge of G such that $\chi(G - e) = k - 1$. Let $\{V_1,V_2,\ldots,V_{k-1}\}$ be a $(k-1)$-coloring of $G - e$. Clearly u and v belong to the same color class, say V_1. Now, $\{\{u\},V_1-\{u\},V_2,\ldots,V_{k-1}\}$ is a k-coloring of G and e = uv is the only edge between $\{u\}$ and $V_1-\{u\}$. The converse is obvious.

Theorem 1.20. *Let G be a k-chromatic graph with* $es_\chi(G) = 1$. *Then there exist two adjacent vertices u and v such that* $\deg u + \deg v \ge 2(k-1)$.

Proof. It follows from Theorem 1.19 that there exists a k-coloring $\{\{u\}, V_1,V_2,\ldots,V_{k-1}\}$ of G such that there is exactly one edge $e = uv$ between $\{u\}$ and V_1. Since u is adjacent to at least one vertex of V_i, for all $i \ge 1$, we have $\deg u \ge k - 1$. We now claim that $\deg v \ge k - 1$. Suppose $\deg v < k - 1$. Then there exists a color class, say V_2, such that v is not adjacent to any vertex of V_2. Now, $\{(V_1-\{v\})\cup\{u\},V_2\cup\{v\},V_3,\ldots,V_{k-1}\}$ is a $(k-1)$-coloring of G, which is a contradiction. Hence $\deg v \ge k - 1$ so that $\deg u + \deg v \ge 2(k-1)$.

Theorem1.21.*In the k-coloring of a uniquely k- colorable graph, the subgraph induced by the union of any two color classes is connected.*

Theorem 1.22. *Let G be a uniquely k-colorable graph. Then* $es_\chi(G) = 1$ *if and only if the k-coloring of G contains two singleton color classes.*

Proof. Let $\{V_1,V_2,\ldots,V_k\}$ be the k-coloring of G. Let $es_\chi(G) = 1$. Then by Theorem 1.19 there exist color classes say V_1 and V_2 such that $|V_1| = 1$ and there is exactly one edge between V_1 and V_2. By Theorem 1.21, we have $\langle V_1 \cup V_2\rangle$ is connected and hence $|V_2| = 1$. The converse is obvious.

Problem 1.23. *Characterize graphs G for which* $es_\chi(G) = 1$.
The following theorem gives Nordhaus-Gaddum type result for the parameter $es_\chi(G)$.

Theorem 1.24. *For any graph G, we have*
(i) $2 \le es_\chi(G) + es_\chi(\bar{G}) \le \binom{n}{2}$ and
(ii) $1 \le es_\chi(G)\cdot es_\chi(\bar{G}) \le m[\binom{n}{2} - m]$.

Further, the following are equivalent.
(a) $es_\chi(G) + es_\chi(\bar{G}) = \binom{n}{2}$
(b) $es_\chi(G)\cdot es_\chi(\bar{G}) = m[\binom{n}{2} - m]$.
(c) G is one of the graphs $P_3,\bar{P}_3,P_4,C_4$ *and* $\bar{C}_4$.

Proof. Since for any graph G, $1 \le es_\chi(G) \le m$, the inequalities follow. Now, $es_\chi(G) + es_\chi(\bar{G}) = \binom{n}{2}$ if and only if $es_\chi(G) = |E(G)| = m$ and $es_\chi(G) = |E(\bar{G})| = \binom{n}{2} - m$.Hence G and $\bar{G}$ are both bipartite, so that (a), (b) and (c) are equivalent.

3. Conclusion

In this paper , we initiate a study of two parameters chromatic edge stability number es_χ and chromatic bondage number ρ and investigate properties of graphs G for which $es_\chi(G) = 1$

4. Future Scope

Further we can develop this paper and find the relation between chromatic edge stability number es_χ and chromatic bondage number ρ for all graphs.

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