The Optimal Dividend Problem in the Compound Poisson Model with Covering the Deficit at Ruin

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Abstract: ***In this paper, we consider the optimal dividend problem in the compound Poisson model with covering the deficit at ruin, which is restrained to strategies with restricted densities. As we all known, the shareholders should repay the deficit at ruin. Therefore, we want to maximize the expectation of the difference between the accumulated discounted dividends until ruin and the discounted deficit at ruin, and find out the optimal dividend strategy. Further,when the claim amount distribution is exponential, we obtain explicit solutions of the value function.***

Keywords: The Value Function, HJB equation, Optimal Barrier Strategy

1. Introduction

For continuous-time risk model, Gerber(1969) firstly studied the optimal dividend problem . Until 1997, Asmussen and Taksar researched the Brownian motion with drift model, by the method of HJB equation, they got the optimal dividend policy. In the classical risk model ,the optimal dividend problem got further research. For example, Gerber and Shiu proved that the optimal strategy is a threshold strategy; Kulenko and Schmidli (2008) discussed the optimal strategy problem in the classical risk model with capital injection, they got a more general and more comprehensive results. Dickson and Waters(2004) pointed out that the shareholders should be more responsible to repay the bankruptcy of those deficits. Complying with this view, Gerber, Shiu and Smith (2006) did the further research for the modified model.

Based on the above theory, this paper is dedicated to the following research. Our goal is to maximize the difference value between the cumulative expected discounted dividend and the discounted deficit. Further, we can derive the optimal dividend strategy. When the claim size obey exponential, we calculated value function and find the optimal dividend barrier.

2. Optimal Dividend Payments With Covering the Deficit at Ruin

Given an initial surplus x, the free surplus X_t of the insurance company at time t can be written as

$$X_t = x + ct - \sum_{i=1}^{N_t} Y_i \qquad (1)$$

In this paper, the classical collective risk model is completely determined by the premium rate c, the intensity β and the claim-size distribution function $F(x) = P(Y_i \le x)$. We can describe this model in a rigorous way by defining its filtered probability space $(\Omega,\ \Sigma, (F_t)_{t\ge 0}, P)$. We make the following assumptions on the distribution of sizes and occurrences of the claims:

(1) The first claim cannot occur at time zero, two claims cannot occur at the same time, and the number of claims in any time interval is finite. So $0<\tau_1<\tau_2<\tau_3<\cdots, N_0=0$ and N_t is finite for any t.

(2) The claim sizes are mutually independent and they are also independent of the claim-arrival times.

(3) The claim sizes are identically distributed.

(4) The number of claims in a time interval only depends on the length of the interval, that is

$$P(N_{t_1+\Delta t} - N_{t_1} = k) = P(N_{t_2+\Delta t} - N_{t_2} = k) \text{ for any } t_1, t_2 \ge 0.$$

Assume L_t is the cumulative expected discounted dividends until the time t, and it is an adapted cadlag(left continuous with right limits) stochastic process. So the surplus becomes the controlled process

$$X_t^L = x + ct - \sum_{i=1}^{N_t} Y_i - L_t \qquad (2)$$

We say that a dividend strategy L is admissible if it is non decreasing, predictable with respect to the filtration $\{F_t\}_{t\ge 0}$. We denote by A the set of all admissible strategies. Define the corresponding ruin time as $T = T_x^L := \inf\{t : X_t^L > 0\}$ .Because we consider the optimal dividend policy in an interval $[0,\ T]$,without loss of generality, when $t>T$, X_t stops. So when $t>T$, $X_t^L = x + c(t\wedge T) - \sum_{i=1}^{N_{t\wedge T}} Y_i - L_{t\wedge T}$ .For a strategy $L\in\Pi$, then the corresponding value function is defined as

$$V_L(x) = E_x\left[\int_0^\tau e^{-\delta t} dL_t - ce^{-\delta T} X_T^L\right], x>0$$

This formula shows that the cumulative expected discounted dividends subtract the deficit, that is to say, the net profit of shareholders until the bankruptcy times, where $\delta>0$ .Our goal is to seek for the optimal strategy and maximize the value function. Then the value function is defined as

$$V(x) = \sup_{L\in A} V_L(x)$$

3. Basic Properties of the Value Functions

In this section we study the properties of the value function under the constraint of the dividend policy.

Lemma 3.1 The value function $V(x)$ is monotonically increasing on the interval $(0, \infty)$ and

$$yV(x) - V(t) \geq x - y \text{ for } x\geq y\geq 0 \quad (3)$$

Proof: For $x\geq y\geq 0$, there exits $L^y \in A$ such that $V^{L^y} \geq V(y) - \varepsilon$, for all $\varepsilon>0$. And we define a new strategy L^x, dividend $x-y$ immediately. And take the policy L^y, we can get

$$V(x) \geq V^{L_x}(x) = x - y + V^{L^y}(y) = x - y + V(y) - \varepsilon.$$

Because of the arbitrariness of ε, so

$V(x) - V(y) \geq x - y$, $V(x)$ is obvious monotonically increasing.

Lemma 3.2 For $x>0$, any dividend policy L has a bound $\frac{\lambda\mu}{\delta}$.

Proof: In a worst case, we need to compensate for each claim size. When we consider no dividend, the kth claim size obeys $\Gamma(\lambda, k)$, then

$$E\left[\sum_{k=1}^{\infty} Y_k e^{-\delta T_k}\right] = \mu \sum_{k=1}^{\infty}\left(\frac{\lambda}{\lambda+\delta}\right)^k = \frac{\lambda\mu}{\delta}.$$

Lemma 3.3 For $x\in R^+$, the value function $V(x)$ has a bound $\frac{\mu_0}{\delta}$ and Lipschitz continuous and satisfies

$$\lim_{x\to\infty} V(x) = \frac{\mu_0}{\delta}. \quad (4)$$

Proof. Because $V(x)$ is monotonically increasing, so

$$V(x) = \int_0^\infty e^{-\delta t}\mu_0 dt = \frac{\mu_0}{\delta}.$$

Consider a policy $U_t = \mu_0$,

$$T_x^U = \inf\left\{t: x + (c-u_0)\ t - \sum_{i=1}^{N_t} Y_i < 0\right\}$$

By Lemma 3.2, we can get $e^{-\delta T_x^U} X_{T_x^U}^U \leq e^{-\delta T_x^U}\frac{\lambda\mu}{\delta}$.When $x\to\infty$, T_x^U is unbounded. And $P_x\left[e^{-\delta T_x^U} X_{T_x^U}^U > \varepsilon\right]$ is converge to zero, i.e.

$$V(x) \geq V^U(x) \geq E\left[\int_0^{T_x^U} e^{-\delta t}\mu_0 dt - e^{-\delta T_x^U} X_{T_x^U}^U\right] \to \frac{\mu_0}{\delta}.$$

Let $x\geq y\geq 0$, and $U\in A_y$ is a strategy with the initial surplus y .Define h is the time from reserve y to x, and there is no claim size. Choose a new strategy $\tilde{U}\in A_x$,where A_x is the admissible strategy with initial reserve x. Denote

$$U_t = \begin{cases} 0, & t\leq h or T\leq h \\ \tilde{U}_{t-h}, & T\wedge t > h \end{cases},$$

assume that the first claim size occurs based on density $\lambda e^{-\lambda t}$,then

$$
\begin{aligned}
V(y) &\geq V^{U}(y) \\
&= E\left[\left[E\int_{0}^{T} e^{-\delta t}U_{t}dt - e^{-\delta T}X_{T}^{U}\right]\middle| T_{1}\right] \\
&= P(T_{1}\geq h)E\left(\int_{h}^{T} e^{-\delta t}U_{t}dt - e^{-\delta T}X_{T}^{U} \middle| T_{1}\geq h\right) \\
&- P(T_{1}< h)E\left[e^{-\delta t}X_{T}^{U} \middle| T_{1}< h\right] \\
&\geq e^{-\lambda h}E\left[E\left[\int_{h}^{T} e^{-\delta t}U_{t}dt - e^{-\delta T}X_{T}^{U}\right]\middle| F_{h}\right] \\
&= e^{-(\lambda+\delta)h}E\left[E\left[\int_{0}^{\infty} e^{-\delta t}\tilde{U}_{t}dt - e^{-\delta T}X_{T}^{U}\right]\middle| F_{h}\right] \\
&= e^{-(\lambda+\delta)h}V^{\tilde{U}}(x)
\end{aligned}
$$

We take supremum from A_{x}, so $V(y)\geq e^{-(\lambda+\delta)h}V(x)$.

By the boundedness of $V(x)$, we can get the Lipschitz continuous:

$$
\begin{aligned}
V(x)-V(y) &\leq V(x)\left(1-e^{-(\lambda+\delta)h}\right) \\
&\leq (\lambda+\delta)hV(x) \\
&\leq \frac{\mu_{0}}{\delta}(\lambda+\delta)h
\end{aligned}
$$

Due to $V(x)-V(y)\geq x-y$, so $V(x)$ is Lipschitz continuous.

4. The HJB Equation and The Optimal Strategy

In section, we find heuristically the first-order equations which satisfy the value function of the stability criteria defined above. Now, we use the so-called dynamic programming principle (DPP in short) and establish the HJB equation. That is to say,

$$\sup_{0\leq\mu\leq\mu_{0}}\left\{(c-\mu)V^{'}(x)+\mu-(\lambda+\delta)V(x)+\lambda\int_{0}^{\infty}V(x-y)dF(y)\right\}=0, x\geq 0 \quad (5)$$

in which,

$$\int_{0}^{\infty}V(x-y)dF(y)=\int_{0}^{x}V(x-y)dF(y)+\int_{x}^{\infty}(x-y)dF(y)$$

So the HJB equation becomes

$$\sup_{0\leq\mu\leq\mu_{0}}\left\{(c-\mu)V^{'}(x)+\mu-(\lambda+\delta)V(x)+\lambda\int_{0}^{\infty}V(x-y)dF(y)+\lambda\int_{x}^{\infty}(x-y)dF(y)\right\}=0, x\geq 0 \quad (6)$$

Maximize the left of formula (5) , so it becomes

$$\mu(1-V^{'}(x)) \quad (7)$$

Because the first item of formula (7) is linear, so maximized the left of formula (5) can be written as :

$$U_{x}=\begin{cases}0, V^{'}(x)>1\\ \in[0,\mu_{0}], V^{'}(x)=1\\ \mu_{0}, V^{'}(x)<1\end{cases}$$

If $V(x)$ is concave on $(0, \infty)$, there exits $b:=\inf\left\{x: V^{'}(x)\leq 1\right\}$, such that

$$U_{x}=\begin{cases}0, x<b \Leftrightarrow V^{'}(x)>1\\ \mu_{0}, x\geq b \Leftrightarrow V^{'}(x)\leq 1\end{cases} \quad (8)$$

Let $V(x;b)$ be the cumulative expected discounted dividends subtract the deficit, where x is the initial reserve, define $b := \inf\{x : V'(x) \le 1\}$

$$cV'(x;b)-(\lambda+\delta)V(x;b)+\lambda\int_0^\infty V(x-y;b)dF(y)=0, \quad 0 \le x < b \qquad (9)$$

$$(c-u_0)V'(x:b)+u_0-(\lambda+\delta)V(x:b)+\lambda\int_0^\infty V(x-y:b)dF(y)=0 \quad x \ge b \qquad (10)$$

By discussing, we get the equation (9) and equation (10) are equivalent to the equation (5).

Proposition 4.1 Assume that $V(x;b)$ is concave on $(0, \infty)$, then $V(x;b)$ is continuously differentiable on $(0, \infty)$.

Proof. Because of the concavity of $V(x;b)$, the formula (8) is proper. By using HJB equation and the continuous of $V(x;b)$, we get

$$cV'(x+;b)-(\lambda+\delta)V(x;b)+\lambda\int_0^\infty V(x-y;b)dF(y)$$
$$=cV'(x-;b)-(\lambda+\delta)V(x;b)+\lambda\int_0^\infty V(x-y;b)dF(y)$$

So $V'(x+;b)=V'(x-;b)$. Similarly, we can proof $V(b,\infty)$ is continuous on (b,∞). Now we assume that $b>0$, and

$$(c-\mu_0)V'(b+;b)+\mu_0-(\lambda+\delta)V(b;b)+\lambda\int_0^\infty V(x-y;b)dF(y)=0$$
$$cV'(b-;b)+\mu_0-(\lambda+\delta)V(b;b)+\lambda\int_0^\infty V(x-y;b)dF(y)=0$$

so we can get

$$cV'(b-;b)=(c-\mu_0)V'(b+;b)+\mu_0$$

or

$$c(V'(b-;b)-V'(b+;b))=\mu_0(1-V'(b+;b)).$$

If $\mu_0<c$, we can get $V'(b-;b)=V'(b+;b)=1$ or $V'(b-;b)<1$. Because the later formula is impossible, so $V(x;b)$ is continuous. Otherwise, if $\mu_0>c$, we can get the optimal strategy is barrier strategy. In fact, due to the surplus process is on the interval $[0, b]$, for $\mu_0>c$, the corresponding strategy is admissible. When $x\in[0,b]$, the policy is optimal. When $x=b$, the expected discounted dividend is

$$\lambda\int_0^\infty e^{-\lambda t}\int_0^t ce^{-\delta s}dsdt=\frac{\lambda c}{\delta}\int_0^\infty\left(1-e^{-\delta t}\right)e^{-\lambda t}dt=\frac{c}{\lambda+\delta}$$

Before the time of the first claim,we can get

$$\lambda\int_0^\infty e^{-\delta t}\int_0^\infty e^{-\delta t}V(b-y)dF(y)dt=\frac{\lambda}{\lambda+\delta}\int_0^\infty V(b-y)dF(y).$$

Thus, we must depict the value of $V(b;b)$, i.e. $V(b)=\frac{c}{\lambda+\delta}+\frac{\lambda}{\lambda+\delta}\int_0^\infty V(b-y)dF(y)$. And put $V(b)$ into equation (9), (10), we can get $V'(b-;b)=V'(b+;b)=1$, so we get $V(x;b)$ is differentiable.

As we all known,although $U_t>c$, before first claim occurs, the surplus process X_t^U maybe be negative.

We can proof that when $X_t^U=0$, the corresponding policy is not optimal. Thus it will occur between two claims (T_{i-1},T_i) , and $dX_t^U=(c-U_t)dt$, $X_{T_i}^U=X_{T_{i-}}^U-Y_{T_i}, T_i\le T$ .If $Z_{T_{i-}}^U-Y_{T_i}\le 0$,then the shareholders should repay $X_T^U=\text{-}\inf\left(X_{T-}^U-Y_T,0\right)$.

So ,the value function satisfies $V\left(X_{T_i}^U\right)=V\left(X_{T_{i-}}^U-Y_{T_i}\right)$, $T_i\le T$. And when

$X^U_{T_{i-}} - Y_{T_i} \le 0,\ \ T_i \le T$, h is the unique solution of the HJB equation, satisfies the following equation:

$$h\left(X^U_{T_i}\right) = h\left(X^U_{T_{i-}} - Y_{T_i}\right), T_i \le T \ . \ (12)$$

Proposition 4.1 Assume that $h(x)$ is the solution of the equation (5), $h(x)$ is increasing and bounded and satisfies the proposition (12), then $\lim_{x\to\infty} h(x) = \frac{\mu_0}{\delta}, h(x) = V(x)$, and the optimal policy is given by the equation(8)

Proof: Because h is bound and increasing, then h is converge,

i.e. $h(\infty) < \infty$.so there exits a series of $x_n \to \infty$, s.t.

$h'(x_n) \to 0$

Let $\mu_n = \mu_{x_n}$ .By the formula (8), and assume $\mu_n \to \mu_0$, let $n \to \infty$ in the formula (5),

Then,

$$0 = (c - \mu_0)h'(x_n) + \lambda\left[\int_0^\infty h(x_n - y)dF(y) - h(x_n)\right] - \delta h(x_n) + \mu_0$$

$$\to -\delta h(\infty) + \mu_0, n \to \infty$$

So $\lim_{x\to\infty} h(x) = \frac{\mu_0}{\delta}$. Let $U = U^*$, using the proposition 3.4 and the HJB equation, we can get

$$\left\{h(X^{U^*}_t)e^{-\delta t} - h(x) + \int_0^t e^{-\delta s}U^*_s ds - e^{-\delta T^{U^*}} X^{U^*}_{T^{U^*}}\right\}$$

is a martingale. Due to the expectation is zero,then

$$h(x) = E\left[h\left(X^{U^*}_t\right)e^{-\delta t} + \int_0^t e^{-\delta s}U^*_s ds - e^{-\delta T^{U^*}} X^{U^*}_{T^{U^*}}\right]$$

Due to $h(x)$ is bound ,through the control convergent theorem, we can get that when $t \to \infty$, $E\left[h\left(X^{U^*}_t\right)e^{-\delta t}\right] \to 0$,and the others are smooth, so change the order between the limit and integral,then

$h(x) = V^{U^*}(x)$. For arbitrary policy U, by the equation (5) ,we can get

$$h(x) \ge E\left[h(X^U_t)e^{-\delta t} + \int_0^t e^{-\delta s}U_s ds - e^{-\delta T^{U^*}} X^{U^*}_{T^{U^*}}\right] \ge E\left[\int_0^t e^{-\delta s}U_s ds - e^{-\delta T} dX_T\right]$$

,

let $t \to \infty, h(x) \ge V^U(x)$,so $h(x) = V(x)$.

It is proved that, when the surplus process is under the barrier b ,the dividend is zero; when the surplus is over the barrier $b\left(X^U_t \ge b\right)$, then we dividend at the maximum the dividend rate μ_0.

5. Exponentially Distributed Claim Sizes

In order to obtain an explicit solution to the HJB equation and an optimal dividend payment policy, we assume that the claim size distribution is given by $F(y) = 1 - e^{-\xi y}, E[y] = \mu = \frac{1}{\xi}$ .To solve this HJB equation, we assume $h(x)$ is a solution on the interval $[0, \infty)$, define a barrier $b := \inf\{x : h'(x) = 1\} > 0$, so the function $h(x;b)$ satisfies the following equation:

$$ch'(x;b) - (\lambda+\delta)h(x;b) + \lambda\int_0^x h(x-y)\xi e^{-\xi y}dy + \lambda\int_x^\infty (x-y)\xi e^{-\xi y}dy = 0, 0 \le x < b \quad (13)$$

$$(c-\mu_0)h'(x;b) - (\lambda+\delta)h(x;b) + \lambda\int_0^x h(x-y)\xi e^{-\xi y}dy + \lambda\int_x^\infty (x-y)\xi e^{-\xi y}dy = 0, x \ge b \ (14)$$

Because

$$(\frac{d}{dx}+s)\int_0^x h(x-y;b)\xi e^{-\xi y}dy = \xi V(x;b)$$

,

$$(\frac{d}{dx}\int_0^\infty (x-y)\xi e^{-\xi y}dy = e^{-\xi x}[1-1]=0$$

So apply $(\frac{d}{dx}+\xi)$ into the formula (13), we can get

$$ch''(x;b)+(\xi c-\lambda-\delta)h'(x;b)-\xi\delta h(x;b)=0, 0\le x<b \quad (15)$$

$$(c-\mu_0)h''(x;b)+(\xi(c-\mu_0)-\lambda-\delta)h'(x;b)-\xi\delta h(x;b)+\xi\mu_0=0, x\ge b \quad (16)$$

We learn from the equation (15),the solution to the differential equation is:

$h(x;b)=C_1e^{v_1x}+C_2e^{v_2x}, 0\le x<b$,where $v_1>0$ and $v_2<0$, v_1,v_2 satisfy the following equation:

$$cv^2+(\xi c-\lambda-\delta)v-\xi\delta=0 \quad (17)$$

i.e. $$v_1=\frac{-\xi c+\lambda+\delta+\sqrt{(-\xi c+\lambda+\delta)^2+4\xi\delta c}}{2c}$$

$$v_2=\frac{-\xi c+\lambda+\delta-\sqrt{(-\xi c+\lambda+\delta)^2+4\xi\delta c}}{2c} \quad (18)$$

Because $\frac{\mu_0}{\delta}$ is a particular solution to the equation (16),then we can get

$$h(x;b)=\frac{\mu_0}{\delta}+De^{vx}, x\ge b \quad (19)$$

Where v is a negative solution to the equation:

$$(c-\mu_0)\beta^2+(\xi(c-\mu_0)-\lambda-\delta)\beta-\xi\delta=0, x\ge b$$

By the bound condition $h'(b)=1$, and the equation (5)、(6), we can get $h(b-;b)=h(b+;b)$ and $h'(b-;b)=h'(b+;b)$ .so we can determine the coefficient $C_i, i=1,2$, so

$$h'(b)=1$$
$$h(b-;b)=h(b+;b) \quad (20)$$
$$h'(b-;b)=h'(b+;b)$$

Solve the equation set, we can get:

$$C_1v_1e^{v_1b}+C_2v_2e^{v_2b}=1$$
$$C_1e^{v_1b}+C_2e^{v_2b}=\frac{\mu_0}{\delta}+De^{vb}$$
$$C_1v_1e^{v_1b}+C_2v_2e^{v_2b}=vDe^{vb}$$
$$C_1v_1e^{v_1b}+C_2v_2e^{v_2b}=vDe^{vb}$$

where,

$$C_1=\frac{\delta v-\mu_0vv_2-\delta v_2}{\delta v(v_1-v_2)}$$
$$C_2=\frac{v_1v\mu_0+v_1\delta-\delta v}{\delta v(v_1-v_2)}$$
$$D=\frac{1}{v}e^{-vb}$$

So , $h(x)=h(x;b)$ can be written as :

$$h(x)=\begin{cases}\frac{\delta v-\mu_0vv_2-\delta v_2}{\delta v(v_1-v_2)}e^{-v_1b}e^{v_1x}+\frac{v_1v\mu_0+v_1\delta-\delta v}{\delta v(v_1-v_2)}e^{-v_2b}e^{v_2x}, 0\le x<b\\ \frac{\mu_0}{\delta}+\frac{1}{v}e^{-vb}e^{vx}, x\ge b\end{cases}$$

We can know that $h(x), x\in(0,\infty)$ is concave. Finally, we need to find the barrier b. Due to $\frac{\partial h(x;b)}{\partial b}=0$, then we can get the positive solution, i.e.

$$(vv_1v_2\mu_0+\delta v_1v_2-\delta vv_1)\ e^{-v_1b}-v_2(v_1v\mu_0+v_1\delta-\delta v)e^{-v_2b}$$

$$b=\frac{1}{v_1-v_2}\ln\frac{vv_1v\mu_0+\delta v_1v_2-\delta vv_1}{v_2(v_1v\mu_0+v_1\delta-\delta v)} \quad (21)$$

Property 5.1: Assume that $F(y)=1-e^{-\xi y}$ and the value function $V(x)$ is concave on the interval $(0,\ \infty)$, then the valve function $V(x)$ and the optimal policy are given as following:

$$V(x)=\begin{cases}\frac{\delta v-\mu_0vv_2-\delta v_2}{\delta v(v_1-v_2)}e^{-v_1b}e^{v_1x}+\frac{v_1v\mu_0+v_1\delta-\delta v}{\delta v(v_1-v_2)}e^{-v_2b}e^{v_2x}, 0\le x<b\\ \frac{\mu_0}{\delta}+\frac{1}{v}e^{-vb}e^{vx}, x\ge b\end{cases}$$

and the optimal dividend barrier b can be given by the equation (21).

Proof: When $x\ge 0$, due to $h(x)$ is the solution of the HJB equation (3.1), by theorem 3.1, we know that ,on the interval $(0,\ \infty)$, $V(x)$ and $h(x)$ are equivalent.

And under the assumptions, the barrier b can be given by the equation (21).

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