

# Optimal Dividend Barrier in the Classical Risk Model Perturbed by Diffusion

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**Abstract:** In this paper we consider a diffusion perturbed classical compound Poisson risk model in the presence of a constant dividend barrier. An integro-differential equation with certain boundary conditions of the discounted dividend payments prior to ruin is derived and solved. We also consider few particular examples to offer optimal dividend barrier.

**Keywords:** Compound Poisson process; Diffusion Process; Discounted dividend payments; Integro-differential equation

## 1. Introduction

The classical risk model perturbed by a diffusion was first introduced by Gerber (1970) and has been further studied by many authors during the last few years; e.g., Dufresne and Gerber (1991), Gerber and Landry (1998), Wang and Wu (2000), Wang (2001), Tsai (2001, 2003), Tsai and Willmot (2002a, b), Zhang and Wang (2003), Chiu and Yin (2003), and the references therein.

The barrier strategy was initially proposed by De Finetti (1957) for a binomial model. More general barrier strategies for a compound Poisson risk process have been studied in a number of papers and books. These references include Buhlmann (1970), Segerdahl (1970), Gerber (1973, 1979, 1981), Gerber (1979), Paulsen and Gjessing (1997), Albrecher and Kainhofer (2002), Højgaard (2002), Lin et al. (2003), Dickson and Waters (2004), Li and Garrido (2004), and Albrecher et al. (2005). The main focus is on optimal dividend payouts and problems associated with time of ruin, under various barrier strategies and other economic conditions. For the risk model modeled by a Brownian motion, Gerber and Shiu (2004) give some very explicit calculations on the moments and distribution of the discounted dividends paid until ruin.

## 2. The model

Consider the following classical surplus process perturbed by a diffusion

$$X(t) = x + pt - \sum_{i=1}^{N_t} U_i + \sigma B(t), \quad t \geq 0, \quad (2.1)$$

(1)  $x = X(0) \geq 0$  is the initial surplus.

(2)  $\{N_t; t \geq 0\}$  is a Poisson process with parameter  $\lambda$ ,

denoting the total number of claims from an insurance portfolio.

(3)  $\{U_i\}$  independent of  $\{N_t; t \geq 0\}$ , are positive i.i.d.

random variables with common distribution function,

$P(x) = 1 - \bar{P}(x) = P(U_i \leq x)$ , density function  $p(x)$ ,

moments  $\mu_j = \int_0^\infty x^j p(x) dx$ , for  $j = 0, 1, 2, \dots$ , and the

Laplace transform  $\hat{P}(s) = \int_0^\infty e^{-sx} p(x) dx$ .

(4)  $\{B_t; t \geq 0\}$  is a standard Wiener process that is

independent of the aggregate claims process  $S(t) := \sum_{i=1}^{N_t} U_i$

and  $\sigma > 0$  is the dispersion parameter.

(5)  $p = \lambda \mu_1 (1 + \theta)$  is the premium rate per unit time,

$\theta > 0$  is the relative security loading factor.

In this paper, a barrier strategy is considered by assuming that there is a horizontal barrier of level  $b \geq x$  such that when the surplus reaches level  $b$ , dividends are paid continuously such that the surplus stays at level  $b$  until it becomes less than  $b$ . Let  $X_b(t)$  be the modified surplus

process with initial surplus  $X_b(0) = x$  under the above barrier strategy. We define the ruin time of the company as

$$\tau = \inf \{t : X_b(t) \leq 0\}.$$

Let  $q > 0$  be the force of interest for valuation and define

$$D_b(x) = \int_0^\tau e^{-qt} dL_t, \quad 0 \leq x \leq b$$

to be the present value of all dividends until time of ruin  $\tau$ , where  $L_t$  is the aggregate dividends paid by time  $t$ . We use the symbol  $V_b(x)$ ,  $0 \leq x \leq b$ , for the expectation of  $D_b(x)$ :

$$V_b(x) = E[D_b(x)], \quad 0 \leq x \leq b.$$

### 3. An Integro-differential Equations of $V_b(x)$

In this section, we will show that  $V_b(x)$  satisfies an integro-differential equation with certain boundary conditions as follows.

$$\frac{\sigma^2}{2} V_b''(x) + pV_b'(x) - (\lambda + q)V_b(x) + \lambda \int_0^x V_b(x-y)p(y)dy = 0 \tag{3.1}$$

with the boundary conditions

$$V_b(0) = 0 \tag{3.2}$$

$$V_b'(b) = 1 \tag{3.3}$$

**Proof:** Consider the infinitesimal interval from 0 to  $dt$ . Conditioning, one obtains that

$$V_b(x) = e^{-qdt} \{P(W_1 > dt)E[V_b(x + pdt + \sigma B(dt))] + P(W_1 \leq dt)E[V_b(x + pdt + \sigma B(dt) - U_1)]\} \tag{3.4}$$

$$E[V_b(x + pdt + \sigma B(dt))] = V_b(x) + [pV_b'(x) + \frac{\sigma^2}{2}V_b''(x)]dt + o(dt)$$

while

$$E[V_b(x + pdt + \sigma B(dt) - U_1)] = E\left[\int_0^{x+pd+\sigma B(dt)} V_b(x + pdt + \sigma B(dt) - y)p(y)dy\right]$$

then substituting these formulas into (3.4), subtracting  $V_b(x)$  from both sides, interpreting  $dt$  and  $o(dt)$  terms, canceling out common factors, and letting  $dt \rightarrow 0$ , we prove that the integro-differential equation (3.1) holds.

The boundary condition (3.2) is obvious: If  $X(0) = 0$ , ruin is immediate and no dividends are paid.

To prove the boundary condition (3.3), let  $\varepsilon > 0$  and  $V_{b,\varepsilon}(x)$  be the expected discounted dividends paid until ruin in the

**Lemma 1** If the density function  $p(x)$  is continuously differentiable in  $(0, \infty)$ , then  $V_b(x)$  is twice continuously differentiable in  $x$  in the interval  $(0, b)$ .

**Theorem 2** Suppose  $p(x)$  is continuously differentiable on  $(0, \infty)$ , then  $V_b(x)$  satisfies the following homogenous integro-differential equation for  $0 < x < b$ :

Since

$$e^{-qdt} = 1 - qdt + o(dt)$$

$$P(W_1 > dt) = 1 - \lambda dt + o(dt)$$

$$P(W_1 \leq dt) = \lambda dt + o(dt)$$

Taylor's expansion (Lemma 1 shows that  $V_b(x)$  is twice continuously differentiable in  $x$ ) gives

following risk model in the presence of the a dividend barrier  $b$ ,

$$X_\varepsilon(t) = x + (p + p_\varepsilon)t - \sum_{i=1}^{N_{t,\varepsilon}} U_i - \varepsilon N_{t,\varepsilon},$$

where  $N_{t,\varepsilon}$  is a Poisson process with parameter  $\lambda_\varepsilon > 0$ , and  $p_\varepsilon$  is such that  $p + p_\varepsilon > \lambda\mu_1 + \varepsilon\lambda_\varepsilon$ . It is well known

that  $\sum_{i=1}^{N_{t,\varepsilon}} U_i + \varepsilon N_{t,\varepsilon}$  is also a compound

Poisson process. Gerber and Shiu (1998, Eq. (7.4)) shows that  $V'_{b,\varepsilon}(b) = 1$ . Now

we choose  $\varepsilon$ ,  $\lambda_\varepsilon$ , and  $p_\varepsilon$  such that  $Var[\varepsilon N_{t,1}] = \sigma^2 t$

and  $E[p_\varepsilon t - \varepsilon N_{t,1}] = 0$ . These two conditions yield

$$\lambda_\varepsilon = \frac{\sigma^2}{\varepsilon^2} \text{ and } p_\varepsilon = \frac{\sigma^2}{\varepsilon}. \text{ It is easy to prove that, when}$$

$$\varepsilon \rightarrow 0^+,$$

$E[e^{z(p_\varepsilon t - \varepsilon N_{t,1})}] \rightarrow e^{z^2 \sigma^2 t / 2}$ . This shows that the process

$\{p_\varepsilon t - \varepsilon N_{t,1}; t \geq 0\}$  converges weakly to  $\{\sigma B(t); t \geq 0\}$ ,

therefore, the surplus process  $\{X_\varepsilon(t); t \geq 0\}$  converges

$$\frac{\sigma^2}{2} V_b''(x) + p V_b'(x) - (\lambda + q) V_b(x) + \lambda \int_0^x V(x-y) \beta e^{-\beta y} dy = 0 \quad (4.1.1)$$

Since

$$\int_0^x V(x-y) \beta e^{-\beta y} dy = \beta e^{-\beta x} \int_0^x V(y) e^{\beta y} dy$$

So (4.1.1) change into

$$\frac{\sigma^2}{2} V_b''(x) + p V_b'(x) - (\lambda + q) V_b(x) + \lambda \beta e^{-\beta x} \int_0^x V(y) e^{\beta y} dy = 0 \quad (4.1.2)$$

Furthermore we differentiate (4.1.2) with respect to  $x$ , we get

$$\frac{\sigma^2}{2} V_b'''(x) + p V_b''(x) - (\lambda + q) V_b'(x) + \lambda \beta [-\beta e^{-\beta x} \int_0^x V(y) e^{\beta y} dy + e^{-\beta x} \cdot V(x) \cdot e^{\beta x}] = 0$$

$$\frac{\sigma^2}{2} V_b'''(x) + p V_b''(x) - (\lambda + q) V_b'(x) - \lambda \beta^2 e^{-\beta x} \int_0^x V(y) e^{\beta y} dy + \lambda \beta V(x) = 0$$

So

$$\int_0^x V(y) e^{\beta y} dy = \frac{\frac{\sigma^2}{2} V_b'''(x) + p V_b''(x) - (\lambda + q) V_b'(x) + \lambda \beta V_b(x)}{\lambda \beta^2 e^{-\beta x}}$$

So (4.1.2) change into:

$$\frac{\sigma^2}{2} V_b''(x) + p V_b'(x) - (\lambda + q) V_b(x) + \lambda \beta e^{-\beta x} \cdot \frac{\frac{\sigma^2}{2} V_b'''(x) + p V_b''(x) - (\lambda + q) V_b'(x) + \lambda \beta V_b(x)}{\lambda \beta^2 e^{-\beta x}} = 0$$

weakly to the surplus process  $\{X(t); t \geq 0\}$ . Then we

conclude that  $\lim_{\varepsilon \rightarrow 0^+} V_{b,\varepsilon}(x) = V_b(x)$ , and

$$\lim_{\varepsilon \rightarrow 0^+} V'_{b,\varepsilon}(b) = V'_b(b) = 1. \quad \square$$

## 4. Example

### 4.1 Exponentially Distributed Claim Sizes

Now consider the case when claim sizes are exponentially

distributed with parameter  $\beta$ , that is  $p(y) = \beta e^{-\beta y}$  for

$y > 0$  and  $\sigma \neq 0$ . The equation (3.1) change into:

Then we can get

$$\frac{\sigma^2}{2}V_b'''(x) + \left(\frac{\sigma^2\beta}{2} + p\right)V_b''(x) + [p\beta - (\lambda + q)]V_b'(x) - \beta qV_b(x) = 0$$

which is a third-order differential equation with constant coefficients.

It follows that  $V_b(x)$  takes the form

$$V_b(x) = C_1e^{r_1x} + C_2e^{r_2x} + C_3e^{r_3x}, \quad x > 0$$

where  $r_1, r_2$  and  $r_3$  are the three roots of the equation

$$\frac{\sigma^2}{2}r^3 + \left(\frac{\sigma^2\beta}{2} + p\right)r^2 + [p\beta - (\lambda + q)]r - \beta q = 0$$

(4.1.3)

$$\begin{cases} C_1 + C_2 + C_3 = 0 \\ C_1r_1e^{r_1b} + C_2r_2e^{r_2b} + C_3r_3e^{r_3b} = 1 \\ C_1\left(\frac{1}{2}\sigma^2r_1^2 + pr_1\right) + C_2\left(\frac{1}{2}\sigma^2r_2^2 + pr_2\right) + C_3\left(\frac{1}{2}\sigma^2r_3^2 + pr_3\right) = 0 \end{cases}$$

With some algebraic calculations, we can show that the value function of a firm that chooses a target cash reserves level  $b$  is given by

$$V_b(x) = \frac{f(x)}{f'(b)} \quad \text{for } x \leq b$$

with

$$f(x) = (\alpha_3 - \alpha_2)e^{r_1x} + (\alpha_1 - \alpha_3)e^{r_2x} + (\alpha_2 - \alpha_1)e^{r_3x}$$

$C_1, C_2$  and  $C_3$  are fully determined by the following boundary conditions:

$$\begin{cases} V_b(0) = 0 \\ V_b'(b) = 1 \\ \frac{1}{2}\sigma^2V_b''(0) + pV_b'(0) = 0 \end{cases}$$

or equivalently

$$\alpha_1 = \frac{1}{2}\sigma^2r_1^2 + pr_1, \quad \alpha_2 = \frac{1}{2}\sigma^2r_2^2 + pr_2, \quad \text{and}$$

$$\alpha_3 = \frac{1}{2}\sigma^2r_3^2 + pr_3$$

We check straight forward that  $V_b(x)$  is a solution of (4.1.1). We have thus found the firm value when it adopts a barrier strategy. Maximizing  $V_b(x)$  with respect to  $b$ , it turns out that the optimal barrier  $b^*$  satisfies  $f''(b^*) = 0$ ,

$$f''(b^*) = r_1^2e^{r_1b^*}(\alpha_3 - \alpha_2) + r_2^2e^{r_2b^*}(\alpha_1 - \alpha_3) + r_3^2e^{r_3b^*}(\alpha_2 - \alpha_1) = 0$$

Now, let  $\lambda = 10, \beta = 2, p = 15, q = 0.1, \sigma = 1$ ,

The roots of equation (4.1.3) are

$$r_1 = 0.01, \quad r_2 = -1.3067, \quad r_3 = -30.7033, \quad \text{Then}$$

$$b^* = 6.6262.$$

#### 4.2 Mixed exponentially distributed claim sizes

Let us consider the case when claim sizes are mixed exponentially distributed, that is

$$p(y) = \frac{1}{2}e^{-y} + e^{-2y}. \quad \text{The equation (3.1) change into:}$$

$$\frac{\sigma^2}{2} V_b''(x) + pV_b'(x) - (\lambda + q)V_b(x) + \lambda \int_0^x V(x-y) \left( \frac{1}{2} e^{-y} + e^{-2y} \right) dy = 0 \quad (4.2.1)$$

Equivalent to

$$\frac{\sigma^2}{2} V_b''(x) + pV_b'(x) - (\lambda + q)V_b(x) + \frac{1}{2} \lambda \int_0^x V(x-y) e^{-y} dy + \lambda \int_0^x V(x-y) e^{-2y} dy = 0 \quad (4.2.2)$$

Since

$$\int_0^x V(x-y) e^{-y} dy = e^{-x} \int_0^x V(y) e^y dy, \quad \int_0^x V(x-y) e^{-2y} dy = e^{-2x} \int_0^x V(y) e^{2y} dy$$

So (4.2.2) change into

$$\frac{\sigma^2}{2} V_b''(x) + pV_b'(x) - (\lambda + q)V_b(x) + \frac{1}{2} \lambda e^{-x} \int_0^x V(y) e^y dy + \lambda e^{-2x} \int_0^x V(y) e^{2y} dy = 0 \quad (4.2.3)$$

Furthermore we differentiate (4.2.3) with respect to  $x$ , we get

$$\frac{\sigma^2}{2} V_b'''(x) + pV_b''(x) - (\lambda + q)V_b'(x) - \frac{1}{2} \lambda e^{-x} \int_0^x V(y) e^y dy + \frac{1}{2} \lambda V(x) - 2\lambda e^{-2x} \int_0^x V(y) e^{2y} dy + \lambda V(x) = 0$$
 So

$$\frac{1}{2} \lambda e^{-x} \int_0^x V(y) e^y dy + \lambda e^{-2x} \int_0^x V(y) e^{2y} dy = \frac{\sigma^2}{2} V_b'''(x) + pV_b''(x) - (\lambda + q)V_b'(x) + \frac{3}{2} V_b(x) - \lambda e^{-2x} \int_0^x V(y) e^{2y} dy$$

So (4.2.3) change into:

$$\frac{\sigma^2}{2} V_b'''(x) + \left( \frac{\sigma^2}{2} + p \right) V_b''(x) + (p - \lambda - q) V_b'(x) + \left( \frac{3}{2} - \lambda - q \right) V_b(x) - \lambda e^{-2x} \int_0^x V(y) e^{2y} dy = 0 \quad (4.2.4)$$

Furthermore we differentiate (4.2.4) with respect to  $x$ , we get

$$\lambda e^{-2x} \int_0^x V(y) e^{2y} dy = -\frac{1}{2} \left[ \frac{\sigma^2}{2} V_b^{(4)}(x) + \left( \frac{\sigma^2}{2} + p \right) V_b'''(x) + (c - \lambda - q) V_b''(x) + \left( \frac{3}{2} - \lambda - q \right) V_b'(x) - \lambda V_b(x) \right]$$
 So

(4.2.4) change into:

$$\frac{\sigma^2}{2} V_b^{(4)}(x) + \left( \frac{3}{4} \sigma^2 + \frac{p}{2} \right) V_b'''(x) + \left( \frac{\sigma^2}{2} + \frac{3}{2} p - \frac{\lambda + q}{2} \right) V_b''(x) + \left[ p - \frac{3}{2} (\lambda + q) + \frac{3}{4} \right] V_b'(x) + \left( \frac{3}{2} - \frac{3}{2} \lambda - q \right) V_b(x) = 0 \quad (4.2.5)$$

While  $\sigma = 0$ , (4.2.5) change into:

$$\frac{p}{2} V_b'''(x) + \left( \frac{3}{2} p - \frac{\lambda + q}{2} \right) V_b''(x) + \left[ p - \frac{3}{2} (\lambda + q) + \frac{3}{4} \right] V_b'(x) + \left( \frac{3}{2} - \frac{3}{2} \lambda - q \right) V_b(x) = 0$$

which is a third-order differential equation with constant coefficients.

It follows that  $V_b(x)$  takes the form

$$V_b(x) = A_1 e^{n_1 x} + A_2 e^{n_2 x} + A_3 e^{n_3 x}, \quad x > 0$$

where  $n_1, n_2$  and  $n_3$  are the three roots of the equation

$$\frac{p}{2}n^3 + \left(\frac{3}{2}p - \frac{\lambda+q}{2}\right)n^2 + \left[p - \frac{3}{2}(\lambda+q) + \frac{3}{4}\right]n + \frac{3}{2} - \frac{3}{2}\lambda - q = 0 \quad (4.2.6)$$

$A_1$ ,  $A_2$  and  $A_3$  are fully determined by the following boundary conditions:

$$\begin{cases} V_b(0) = 0 \\ V_b'(b) = 1 \\ V_b'(0) = 0 \end{cases} \Rightarrow \begin{cases} A_1 + A_2 + A_3 = 0 \\ A_1 n_1 e^{n_1 b} + A_2 n_2 e^{n_2 b} + A_3 n_3 e^{n_3 b} = 1 \\ A_1 n_1 + A_2 n_2 + A_3 n_3 = 0 \end{cases}$$

With some algebraic calculations, we can show that the value function of a firm that chooses a target cash reserves level  $b$  is given by

$$V_b(x) = \frac{g(x)}{g'(b)} \text{ for } x \leq b$$

With

$$g(x) = (n_3 - n_2)e^{n_1 x} + (n_1 - n_3)e^{n_2 x} + (n_2 - n_1)e^{n_3 x}$$

We check straight forward that  $V_b(x)$  is a solution of

(4.2.1). We have thus found the firm value when it adopts a

barrier strategy. Maximizing  $V_b(x)$  with respect to  $b$ , it

turns out that the optimal barrier  $b^*$  satisfies  $g''(b^*) = 0$ ,

$$g''(b^*) = (n_3 - n_2)n_1^2 e^{n_1 b^*} + (n_1 - n_3)n_2^2 e^{n_2 b^*} + (n_2 - n_1)n_3^2 e^{n_3 b^*} = 0$$

Now, let  $\lambda = 1$ ,  $p = 2$ ,  $q = 0.1$ ,  $\sigma = 0$ , The roots of equation (4.2.6) are

$n_1 = 0.0772$ ,  $n_2 = -0.7145$ ,  $n_3 = -1.8128$ . Then

$b^* = 0.9160$ .

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