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Application of Homotopy Analysis Method in Model Based Controls

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Abstract: ***Homotopy Analysis Method (HAM) is a semi-analytical technique that has been successfully applied to solve several forms of nonlinear, ordinary as well as partial differential equations. HAM yields a family of solutions at any given order of approximation with selection of appropriate value of an auxiliary linear operator / convergence control parameter, typically denoted as h. In this paper, application of HAM has been specifically proposed for solving nonlinear dynamical systems that are in the input affine form which are quite prevalent in nonlinear controls theory. Dynamic Matrix Control (DMC) is a very popular and proven linear control technique that has been successfully deployed in several process plants. Simplicity in deployment and operation makes DMC a more popular choice than several nonlinear controls techniques. DMC uses the step-response models of a given system to achieve desired control action. The model assumes that the nonlinear plant behaviour can be reasonably approximated with a linear model around a specific region of operation. However, the continuous time varying nature of the plant and changes in the set-point far away from the original operating point pose major challenges for the DMC implementation. This is primarily due to the inadequacy of the original linear step-response model to capture the change in dynamics of a nonlinear plant, because of which the control performance is compromised. The work presented here shows application of the HAM to accomplish both objectives easily so that a better control action is achieved for a given class of non-linear system.***

Keywords: Nonlinear dynamical systems, Input affine systems, Homotopy Analysis Method, Nonlinear controls, Dynamic Matrix Controls

1. Introduction

Importance and limitations of physical experiments, numerical simulations and analytical (approximate) solutions to investigate nonlinear problems have been discussed widely in the literature [1]. Physical experiments consume considerable resources, time and money, where as numerical solutions always pose challenges in terms of singularity and multiple solutions while dealing with nonlinear problems. It is not easy to obtain a closed form solution to nonlinear problems. Approximate analytical solutions have been proposed using perturbation techniques but several short comings of these methods have been discussed in the past. Homotopy Analysis Method is an approximate analytical technique that has been applied successfully across a wide range of nonlinear problems [2-6]. This encompasses application areas such as fluid mechanics, heat and mass transfer, materials and several nonlinear systems exhibiting oscillatory behaviour.

In this paper, we would like to show application of HAM extended to a class of nonlinear dynamical systems whose state-space form is given by,

$$\begin{aligned}\dot{x}(t) &= F(x(t),u(t)) \\ y(t) &= G(x(t))\end{aligned} \quad . \tag{1}$$

Many first principles models shown in Eq. (1) can further be simplified to a control- affine form [7] shown in Eq. (2) where the input appears linearly in the system

$$F(x,u) = f(x) + \Gamma(x)u \tag{2}$$

The control-affine form has a distinct advantage over its variants in the design of model based control law where the input can be manipulated to achieve a desired control action.

One of the very popular and widely accepted model based control scheme in the industry is Dynamic Matrix Control (DMC). This was the first version of Model Predictive Controls (MPC) that uses step response models to compute a sequence of control inputs based on an explicit prediction of outputs within a future horizon [8]. It is a widely accepted fact that the success of model based control schemes are directly linked with the ability of the models to track the dynamics of the process being controlled [9,10]. It is hence not only important for the model to capture dynamics of the process being controlled but also update any changes in the model parameters frequently. The Extended Kalman Filter (EKF) based framework used by Lee & Ricker [10] primarily seeks to track the changing model behaviour across different operating regimes. EKF linearizes the model at every operating point before iteratively computing propagation followed by correction.

In this paper, we propose the use of HAM to solve a nonlinear dynamical problem defined by Eq. (2). Using the exponential base-function, the dynamics of the system is generated for different step inputs. Such a solution will explicitly capture the impact of different amplitude of steps on the output at required operating points as a continuous function of time as given by Eq. (3)

$$y(t) = f(u,t) \tag{3}$$

A dynamic heat transfer system proposed by Abbasbandy [11] has been used to compare the step response coefficients derived from numerical, analytical and HAM based solutions. For a pre-determined sampling rate, the step-response coefficients (SRCs) are estimated from the HAM solution. Improvement in performance of DMC is seen specifically for the HAM based SRC models against the traditional SRC models derived from the linearization approach.

2. Formulation

This section covers the introduction of both the HAM and DMC formulation, where the connection between the dynamic step response coefficients derived from the HAM can be linked with the DMC. The approach has the potential to be extended to other forms of model based control such as Internal Model Control (IMC) and MPCs. For the benefit of the readers the section is divided in to three parts where HAM, DMC and the connection between the both are elaborately discussed.

2.1. Homotopy Analysis Method

There is a plethora of literature on theory of HAM and associated formulation [12,13]. The theory is once again discussed below so as to introduce consistency in the notation used and also to discuss its connections with the control-affine models.

For brevity reasons the non-linear form of Eq. (3) is shown as Eq. (4) which is

$$\mathrm{N}[f(t)] = 0 \tag{4}$$

Where N denotes for a non-linear operator and t an independent variable. $f(t)$ is an unknown function. The homotopy of the above form can be constructed as

$$H[\phi(t;q), f_o(t), h, q] = (1-q)\mathbf{L}[\phi(t;q) - f_o(t)] - qhH(t)\mathrm{N}[\phi(t;q)] \tag{5}$$

$q \in [0,1]$ shown in Eq. (5) is an embedding parameter, h is a non-zero auxiliary parameter, $H(t)$ is an auxiliary function $(H(t) \neq 0)$, L is an auxiliary linear operator, $f_o(t)$ is an initial approximation of $f(t)$ that satisfies the initial conditions. Likewise, $\phi(t;q)$ is also the function that satisfies initial conditions.

Eq. (6) is called the zero-th order deformation equation and is formulated by setting the homotopy to zero

$$(1-q)\mathbf{L}[\phi(t;q) - f_o(t)] = qhH(t)\mathrm{N}[\phi(t;q)] \tag{6}$$

When q is set to zero, Eq. (6) is simplified to the form below

$$\mathbf{L}[\phi(t;0) - f_o(t)] = 0 \tag{7}$$

From the definition of L it follows that

$$\phi(t;0) = f_o(t) \tag{8}$$

Similarly, when $q = 1$,

$$N[\phi(t;1)] = 0 \tag{9}$$

It is important to note that Eq. (8) satisfies the initial condition of the differential equation. As q varies between 0 to 1, the $\phi(t;q)$ varies continuously from the initial approximation $f_o(t)$ to the final solution. This is carried out by developing a series of linear approximations to the non-linear equation. The mth order linear approximation $f_m(t)$ is given as

$$f_m(t) = \frac{1}{m!}\frac{\partial^m \phi(q;t)}{\partial q^m}\bigg|_{q=0} \tag{10}$$

Eq. (10) is known as the mth order deformation derivative. The $\phi(t;q)$ when expanded using Taylor series with respect to q

$$\phi(t;q) = \phi(t;0) + \sum_{m=1}^{\infty} \frac{1}{m!}\frac{\partial^m \phi(q;t)}{\partial q^m}\bigg|_{q=0} q^m \tag{11}$$

Substituting Eqs. (8) and (10) in to Eq. (11)

$$\phi(t;q) = f_o(t) + \sum_{m=1}^{\infty} f_m(t) q^m \tag{12}$$

If the auxiliary linear operator, initial guess, h and the auxiliary function are chosen so that the series converges when q=1,

$$f(t) = f_o(t) + \sum_{m=1}^{\infty} f_m(t) \tag{13}$$

To deduce $f_m(t)$, Eq. (6) is differentiated with respect to q

$$(1-q)\mathbf{L}\left(\frac{\partial \phi(t;q)}{\partial q}\right) - \mathbf{L}(\phi(t;q) - f_o(t)) = hH(t)N\left[\phi(t;q) + qhH(t)\frac{\partial N(\phi(t;q))}{\partial q}\right] \tag{14}$$

Setting q=0 and

$$\mathbf{L}[f_1(t)] = hH(t)N[f_o(t)] \tag{15}$$

If the process is extended further, the form can be expressed as

$$\mathbf{L}[f_m(t) - \chi_m f_{m-1}(t)] = hH(t)\frac{1}{(m-1)!}\frac{\partial^{m-1} N[\phi(t;q)]}{\partial q^{m-1}}\bigg|_{q=0} \tag{16}$$

By introducing a parameter χ_m such that

$$\chi_m = \begin{cases} 1 & \text{if } m \leq 1 \\ 0 & \text{if } m > 1 \end{cases}$$

and rearranging Eq. (16) we can obtain

$$f_m(t) = \chi_m f_{m-1}(t) + \mathbf{L}^{-1}\left\{ hH(t)\frac{1}{(m-1)!}\frac{\partial^{m-1}N(\phi(t;q))}{\partial q^{m-1}}\bigg|_{q=0}\right\} \quad (17)$$

The final solution is obtained through

$$f(t) = \sum_{m=0}^{\infty} f_m(t) \quad (18)$$

Typically, the solution includes finite number of approximation unlike as shown in Eq. (18). The convergence of the solution is judged based on the graph of the sum of finite number of terms evaluated at a specific value of *t* against auxiliary parameter, *h*. At the point of convergence, the graph tends to go horizontal to the axis of parameter *t*. The above formulation of HAM can be used in deducing the necessary transfer-function for the DMC. The next section discusses in detail the nature of the model needed for formulating the DMC control-law.

2.2. Dynamic Matrix Control

DMC uses the step response model for controlling the plant. The step response coefficients g_i are collected by exciting the plant with a step test. It is assumed that the plan is stable and the model is adequately able to capture dynamics around the region of operation. The response collected with the changes in inputs is obtained by

$$y(t) = \sum_{i=1}^{k} g_i \Delta u(t-i) \quad (19)$$

The step response model helps in deducing pth step ahead prediction with *n* control actions and the response is given by

$$\hat{y} = G\Delta u + f \quad (20)$$

Where $\hat{y}$ is the predicted output, G the step response matrix holding system dynamics and f being the free response of the system.

Representation of the above terms in the form of matrix is given by

$$\hat{y} = \begin{bmatrix} \hat{y}(t+1|t) \\ \hat{y}(t+1|t) \\ \vdots \\ \hat{y}(t+p|t) \end{bmatrix}_p \quad G = \begin{bmatrix} g_1 & 0 & \cdots & 0 \\ g_2 & g_1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ g_n & g_{n-1} & \cdots & g_1 \\ \vdots & \vdots & \ddots & \vdots \\ g_p & g_{p-1} & \cdots & g_{p-n+1} \end{bmatrix}_{p\times n} \quad (21)$$

$$\Delta u = \begin{bmatrix} \Delta u(t) \\ \Delta u(t+1) \\ \vdots \\ \Delta u(t+n-1) \end{bmatrix}_n \quad f = \begin{bmatrix} f(t,1) \\ \hat{y}(t,2) \\ \vdots \\ \hat{y}(t,p) \end{bmatrix}_p$$

The free response of the system is thereby computed as

$$f(t,k) = y_n(t) + \sum_{i=1}^{n}(g_{k+i} - g_i)\Delta u(t-i) \quad (22)$$

The control law is deduced using the objective function given by Eq. (23) which is obtained by minimizing deviation of the future output ($\hat{y}(t+j|t)$ from the set-point trajectory ($w(t+j)$) and is given by

$$J = \sum_{j=1}^{p}\left[\hat{y}(t+j|t) - w(t+j)\right]^2 + \sum_{j=1}^{n}\lambda\left[\Delta u(t+j-1)\right]^2 \quad (23)$$

The objective function ensures minimizing the difference between the estimated output along a prediction horizon (p) with n control moves. The control law hence reduces to the form shown in Eq. (24)

$$\Delta u = \left[(G^T G + \lambda 1)^{-1} G^T (w - f)\right]_n \quad (24)$$

It is important to note that the effectiveness of the control action depends on how accurately the step-response model is able to capture the dynamics and the gain of the system to be controlled. Most of the times, the step-response model is obtained by carrying a small step test in the plant or from the off-line simulation of the plant model. Although exhaustive literature is available on online adaptation, tuning and control, the subject has had limited success in industry due to several operating constraints and complexity. Leveraging the HAM framework for deployment of DMC is covered in the subsequent sections.

2.3. HAM based DMC Implementation

In this paper, we propose to leverage the control-affine structure shown in Eq. (2). This helps in deducing the step-response behaviour of the system using HAM. In process plants, change in the set-point directs the DMC to bring necessary changes in the manipulated variable so that the system output is driven successively towards the target. It is hence important to capture a more accurate step-response model of the system which the controller uses to deduce the future input sequence. In the HAM framework, we propose to achieve this by selecting an exponential base-function. This is also based on the fact that the system being controlled exhibits an asymptotically stable behaviour. To illustrate HAM based step-response models, a simplified form of control-affine model has been chosen such where $\Gamma(x) = c$ in Eq. (3). Combining Eqs. (2) and (3), a simplified form of a dynamical system is shown in Eq. (25)

$$\dot{x} = f(x) + cu \quad (25)$$

$f(x)$ could a linear or non-linear representation of the state-variable x and $\Gamma(x) = c$ is a special case of control-affine structure. Let u_s and x_s represent initial steady-state state of the system. Deviation from the steady-state essentially can be described as

$$\dot{x} = \left[f(x) - f(x_s)\right] + c\left[u - u_s\right] \quad (26)$$

$\Delta = u - u_s$ represent the amplitude of the step-change in the input. By rearranging the terms in Eq. (26)

$$\frac{1}{c\Delta}\dot{x} = 1 + \frac{[f(x) - f(x_s)]}{c\Delta} \tag{27}$$

The RHS of the Eq. (27) can be assigned to a pseudo-variable $(\aleph)$ and solved using HAM. This modified form shown in Eq. (27) offers some distinct advantages while dealing with different nonlinear forms of $f(x)$ described by Eq. (25). However, it is to be noted that depending on the nature of the model, appropriate modifications of Eq. (25) need to be used. The next section shows an example of a nonlinear system which was solved using HAM to derive the step-response model.

3. HAM based Step-response Model

In this section, a liquid level system exhibiting nonlinear dynamics has been chosen as an example. The system has been a popular example in the area of process modelling and simulation in the literature [14-16]. For an incompressible fluid, the basic model for the liquid level system is given by

$$A\frac{d\hbar}{dt} = F_i - k\sqrt{\hbar} \tag{28}$$

The afore mentioned system is a continuous flow system with volumetric flow rate of incompressible fluid equal to F_i and flow coefficient- k. The $\hbar$ defines height of the liquid in the tank with constant cross-sectional area A. It is to be noted that the model of the system is a nonlinear first-order differential equation.

The corresponding steady-state representation of the above system is given by

$$F_{i,s} = k\sqrt{\hbar_s} \tag{29}$$

Hence, the deviation of the system from the original steady-state on applying a step-change with amplitude Δ is given by

$$A\frac{d\hbar}{dt} = \Delta - k\left(\sqrt{\hbar} - \sqrt{\hbar_s}\right) \tag{30}$$

or

$$\frac{A}{\Delta}\frac{d\hbar}{dt} = 1 - \frac{k}{\Delta}\left(\sqrt{\hbar} - \sqrt{\hbar_s}\right) \tag{31}$$

Assigning the RHS of Eq. (31) to $\aleph$ and by re-writing the differential w.r.t to the newly assigned variable, the Eq. (31) can be written as

$$\left(\frac{2A\Delta}{k^2}\right)\aleph\frac{d\aleph}{dt} - \frac{2A}{k}\left(\frac{\Delta}{k} + \sqrt{\hbar_s}\right)\frac{d\aleph}{dt} - \aleph = 0 \tag{32}$$

For a given step-change, the terms in the bracket of Eq. (32) are constants and hence the equation is ready to be solved using HAM. The modified form will hence be

$$\alpha\aleph\frac{d\aleph}{dt} - (\alpha + \beta)\frac{d\aleph}{dt} - \aleph = 0 \text{ where } \aleph(0) = 1 \tag{33}$$

indicating that at $t = 0$ the system is at initial steady-state with $\hbar = \hbar_s$

such that $\alpha = \dfrac{2A\Delta}{k^2}$ & $\beta = \dfrac{2A\sqrt{\hbar_s}}{k}$.

The base function for Eq. (33) is taken as $\left\{ e^{-nt} \,\middle|\, n = 1,2,3,\ldots \right\}$ in the form where

$$\aleph(\tau) = \sum_{n=1}^{\infty} d_n e^{-nt} \tag{34}$$

where d_n is a coefficient to be determined. As shown in Eq. (15), the linear operator is chosen as

$$\mathbf{L}\,[\phi(t\,;q)] = \frac{\partial\phi(t\,;q)}{dt} + \phi(t\,;q) \tag{35}$$

such that

$$\mathbf{L}\,[c_1 e^{-t}] = 0 \tag{36}$$

Where c_1 is constant. From Eq. (33)

$$N[\phi(t\,;q)] = \alpha\phi(t\,;q)\frac{\partial\phi(t\,;q)}{\partial t} - (\alpha + \beta)\frac{\partial\phi(t\,;q)}{\partial t} - \phi(t\,;q) \tag{37}$$

As per Eq. (33) the initial approximation should be in the form

$y_0(t) = e^{-t}$ and the initial condition of the zeroth -order deformation Eq. (6) is

$$\phi(0\,;\,q) = 0$$

$$R_m\left(\vec{\aleph}_{m-1}\right) = \alpha\sum_{n=0}^{m-1}\aleph_n(t)\aleph'_{m-1-n}(t) - (\alpha + \beta)\aleph'_{m-1}(t) - \aleph_{m-1}(t) \tag{38}$$

The m^{th} order deformation for $m \geq 1$ is

$$\aleph_m(t) = \chi_m\aleph_{m-1}(t) + he^{-t}\int_0^t e^t H(t) R_m\left(\vec{\aleph}_{m-1}\right)dt + c_1 e^{-t} \tag{39}$$

The respective solutions for different orders of approximation are obtained as below

$$\aleph_1(\mathrm{t}) = \mathrm{e}^{-\mathrm{t}}\mathrm{h}(-\mathrm{t} + \mathrm{t}\alpha + \mathrm{t}\beta + \alpha(-1 + \mathrm{Cosh}(\mathrm{t}) - \mathrm{Sinh}(\mathrm{t})))$$

and

$$\aleph_2(\mathrm{t}) = \frac{1}{2}\mathrm{e}^{-3\mathrm{t}}\,\mathrm{h}2\,(\alpha + \mathrm{e}^{\mathrm{t}}\,(-\alpha + \mathrm{t}\,(-1 + \alpha + \beta)))\,(3\alpha + e^{t}\,(-3\alpha - 2\beta + \mathrm{t}\,(-1 + \alpha + \beta))) \\ + e^{-t}\,\mathrm{h}\,(-\mathrm{t} + \mathrm{t}\alpha + \mathrm{t}\,\beta + \alpha\,(-1 + \mathrm{Cosh}(\mathrm{t}) - \mathrm{Sinh}(\mathrm{t})))$$

The m[th] order approximation of $\aleph(t)$ is given by

$$\aleph(t)=\aleph_0+\sum_{n=1}^{m+1}\aleph_n q^n\Big|_{q=1} \qquad (40)$$

For third-order approximation Eq. (40) could be written as

$$\begin{aligned}\aleph(t)=&\frac{1}{2}e^{-3t}h^2(\alpha+e^t(-\alpha+t(-1+\alpha+\beta)))(3\alpha+e^t(-3\alpha-2\beta+t(-1+\alpha+\beta)))\\&+\frac{1}{6}e^{-t}h^2(-16h\alpha^3+16e^{-3t}h\alpha^3+ht^3(-1+\alpha+\beta)^3\\&-3t^2(-1+\alpha+\beta)^2(-1+3h\alpha+2h\beta)+9\alpha^2(-1+5h\alpha+2h\beta)\\&-6\alpha(7h\alpha^2+\beta(-1+h\beta)+\alpha(-3+6h\beta))+3t(-1+\alpha+\beta)(7h\alpha^2+2\beta(-1+h\beta)\\&+\alpha(-4+8h\beta))+9e^{-2t}\alpha^2(1+h(-5\alpha-2\beta+3t(-1+\alpha+\beta)))\\&+6e^{-t}\alpha(7h\alpha^2+2ht^2(-1+\alpha+\beta)^2+\beta(-1+h\beta)+\alpha(-3+6h\beta)-2t(-1+\alpha+\beta)\\&(-1+2h(2\alpha+\beta))))+e^{-t}h(-t+t\alpha+t\beta+\alpha(-1+\cosh(t)-\sinh(t)))\end{aligned}$$

Remark 1:
It is to be noted that this formulation specifically serves to estimate response for a non-linear system perturbed with step-input of definite amplitude. Therefore, deviation of the system from the previous steady-state can be estimated for any amplitude of the input from the previously known steady-state of the system. Response of the system from the earlier steady-state is estimated by using Eq. (31) where

$$\hbar(t)=\left(\frac{\Delta}{k}\left[1-\aleph(t)\right]+\sqrt{\hbar_s}\right)^2$$

Remark 2:
It is also important to deduce appropriate value of h before using the proposed formulation.

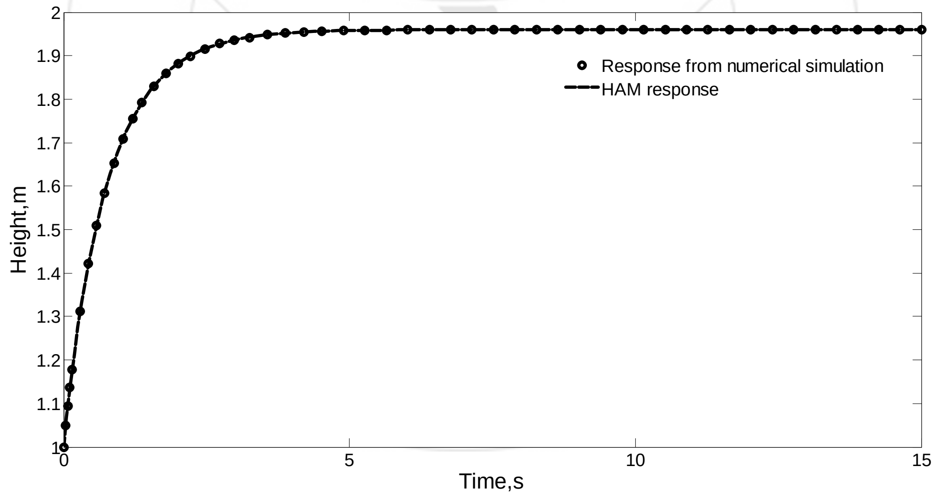

Figure 1: Comparison of step-response behavior for a positive step-change

The response obtained from numerical simulation is compared with the one predicted by HAM in Fig.1. The parameters of the system are chosen to be $A=0.75$ m^2, k =2.5 m$^{2.5}$/s, $\hbar_s$ =1 m. The initial input governing this steady-state is $F_{i,s}$ =2.5 m^3/s. The response shown in Fig.1 is for the Δ =+1 m^3/s, which corresponds to F_i = 3.5m^3/s. For the same set of parameters, Fig.2 shows the behaviour predicted when a negative step change of $\Delta=-1.5$ m^3/s was introduced.

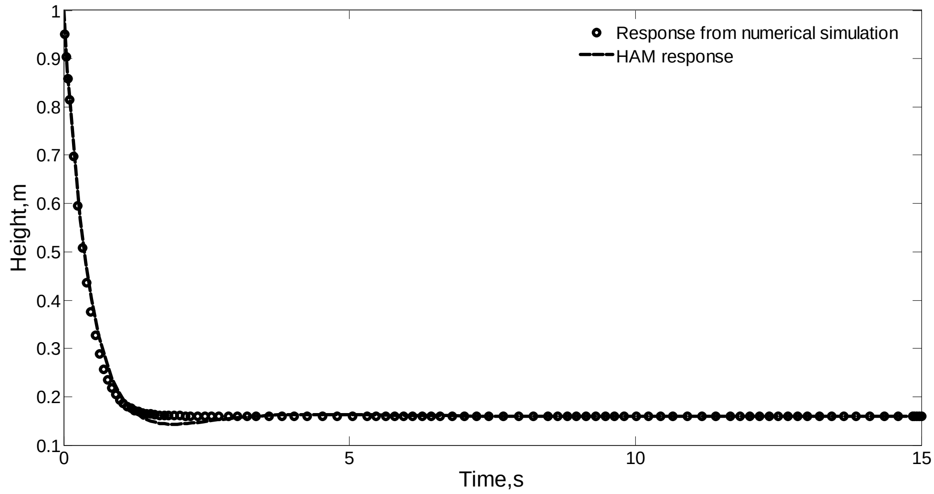

Figure 2: Comparison of step-response behavior for negative step-change

For both the simulations, h was chosen to be 0.9 and a 3rd order approximation was found to be adequate to capture response of the system for step changes with different amplitude and direction.

4. Case-Studies

In this section, we show the application of HAM for DMC. The formulation presented in the previous section illustrates that by manipulating the original form of the differential equation, it is possible to deduce a homotopy model with the help of a lumped parameter that is independent of the initial state of the process and the corresponding input. The formulation also benefits in estimating the trajectory of the step-response by unfolding the lumped parameter and by using the amplitude of step-changes. This approach is extended to the formulation of DMC in the sections to follow.

4.1. Heater System with Variable Specific Heat

The heating/cooling system is one of the most frequently used unit operation in process industries. The process stream that is to be cooled or heated is fed in to a jacketed vessel. The jacket either houses an electrical element or a fluid to achieve heating or cooling operation. The case considered for this study is a bi-linear system with process stream's heat capacity being a strong function of the temperature [11]. Although the initial structure of this model was proposed earlier by Abbasbandy [11], we have made a minor modification to the formulation which is discussed in Appendix-1.

Remark 3:
It is important for the readers to note that there is a subtle difference between the model discussed by Abbasbandy and the form discussed in this paper. It is important to consider this change as the dependency of specific heat with temperature cannot be independently treated outside the derivative term. However, we can establish equivalence of both the model structures with certain valid assumptions and manipulations

The bilinear model for the heater/cooler system from Appendix 1 follows to be

$$\rho V c_p \frac{dT}{dt} + UA(T - T_a) = 0 \text{ such that } T(0) = T_i \quad (41)$$

Where $c_p = c_a[1 + \beta(T - T_a)]$

Transformation applied to Eq. (41) translates to

$$(1 + \in \aleph)\frac{d\aleph}{d\tau} + \aleph = 0 \text{ with } \aleph(0) = 1 \quad (42)$$

Where

$$\aleph = \frac{T - T_a}{\Delta}, \tau = \frac{tUA}{\rho V c_p}, \in = \beta\Delta \text{ and } \Delta = T_i - T_a$$

As discussed in the previous section, 5th order approximation using HAM could be seen as sufficient to capture the step-response of the system. Hence,

$$\aleph_0 = e^{-\tau}$$

$$\aleph_1 = h \in e^{-\tau}(\cosh(\tau) - 1 - \sinh(\tau))$$

$$\aleph_2 = \frac{3}{2}h^2 \in^2 e^{-3\tau} + \left(h^2 \in -2h^2 \in^2\right)e^{-2\tau} + \left(-h \in -h^2 \in + \frac{h^2 \in^2}{2} + h \in \cosh(\tau) - h \in \sinh(\tau)\right)e^{-\tau}$$

$$\aleph_3 = \frac{8}{3}h^3 \in^3 e^{-4\tau} + \left(3h^2 \in^2 + 3h^3 \in^2 - \frac{9}{2}h^3 \in^3\right)e^{-3\tau} + \left(2h^2 \in + h^3 \in -4h^2 \in^2 -4h^3 \in^2 + 2h^3 \in^3\right)e^{-2\tau}$$
$$\left(-h \in -2h^2 \in -h^3 \in + h^2 \in^2 + h^3 \in^2 - \frac{1}{6}h^3 \in^3 + h \in \cosh(\tau) - h \in \sinh(\tau)\right)e^{-\tau}$$

and

$$\aleph_4 = \frac{125}{24}h^4 \in^4 e^{-5\tau} + \left(8h^3 \in^3 + 8h^4 \in^3 - \frac{32}{3}h^4 \in^4\right)e^{-4\tau} +$$
$$\left(3h^2 \in + 3h^3 \in + h^4 \in - 6h^2 e^2 - 12h^3 \in^2 - 6h^4 \in^2 + 6h^3 \in^3 + 6h^4 \in^3 - \frac{4}{3}h^4 \in^4\right)e^{-2\tau} +$$
$$\left(\frac{9}{2}h^2 \in^2 + 9h^3 \in^2 + \frac{9}{2}h^4 e^2 - \frac{27}{2}h^3 \in^3 + \frac{27}{2}h^4 \in^3 + \frac{27}{4}h^4 \in^4\right)e^{=3\tau} +$$
$$\left(\frac{3}{2}h^2 \in^2 + 3h^3 \in^2 + \frac{3}{2}h^4 \in^2 - \frac{1}{2}h^3 \in^3 - \frac{1}{2}h^4 \in^3 + \frac{1}{24}h^4 \in^4 + h \in \cosh(\tau) - h \in \sinh(\tau)\right)e^{-\tau}$$

The net response is hence obtained by Eq. (43) such that

$$\aleph(t) = \sum_{n=0}^{4} \aleph_n \qquad (43)$$

It is to be noted that the actual temperature profile of the process is obtained by unfolding Eq. (43)

$$T(t) = \aleph(t)\Delta + T_a \qquad (44)$$

Figs. (3) and (4) compare the actual step-response behaviour of the system against the HAM based estimation.

The list of parameters used for the simulation is as follows.

Table 1: Parameters for heater/cooler simulation

Name	A m^2	U w/m^2C	V m^3	β 1/°C	ρ kg/m^3	c_a kJ/kg°C
Values	0.7854	150	0.5	0.035	500	2

The specific heat used in this simulation is a strong function of temperature and hence leads to changes in the dynamic response for any step-change in the input. Fig. (3) shows the accelerated change in the response of the system although the final steady-state achieved by the linear and bilinear system are the same. The initial steady-state (T_i) was assumed to be 15 °C and the Δ was 20 °C corresponding to the $T_a = 35$ °C.

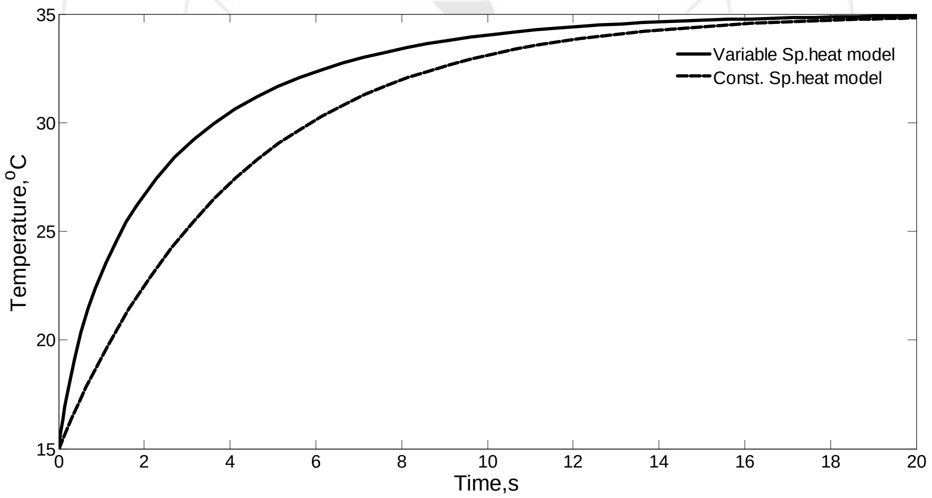

Figure 3: Difference between dynamic response of the bilinear and the linear model due to the variable and constant specific heat terms respectively

Comparison in the dynamic response predicted by HAM against the numerical solution to the step-changes has been shown in Fig.4. The response has been obtained for the same set of parameters reported in Table 1.

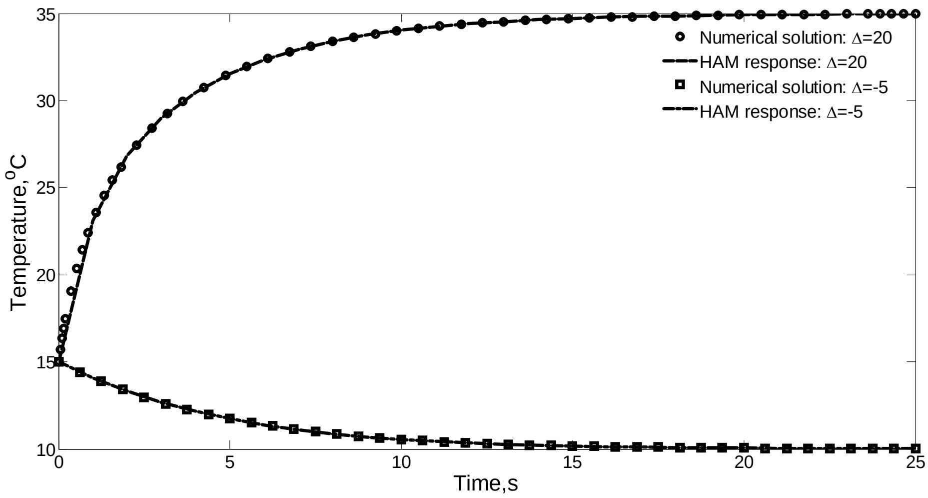

Figure 4: Comparison between the HAM predicted dynamics vs. the actual response for the step changes of different amplitude and direction

From Fig. 4 it is evident that that 5^{th} order approximation model could comfortably track the system behaviour for various step changes. The HAM model is hence concluded to be adequate to be used for DMC formulation to track systems response in the servo mode.

The DMC simulation for the heater/cooler assumes the process to be initially at the steady-state which corresponds to 15°C. All parameters of the model are assumed to be the same as reported in Table 1. The traditional DMC approach uses a linear model that is identified in the vicinity to the initial operating point. The model thus identified assumes the transfer function in the Laplace Domain as

$$\frac{T(s)}{T_a(s)} = \frac{1}{4.2441s+1}$$

The control and prediction horizon for the DMC is set to be 10 and 30 respectively with the sampling time of 1 sec. The tuning factor (λ) is set at 0.5 The set-point of the process with an initial operating temperature of 15°C is changed to 23°C at 50^{th} instant and a second change in set-point was imposed around 200^{th} instant. The DMC is expected to manipulate the jacket temperature (T_a) such that the set-point is tracked smoothly and instantly. However, it could be see from the Fig. (5) that the traditional DMC struggles to settle around the set-point showing a huge oscillation around 50^{th} instant and a sluggish tracking similarly is seen around 200^{th} instant. This is primarily due to large model-plant mismatch and hence inability of the model to achieve the desired performance. It also relates to the obvious understanding that for a model based controls system, the control action is only as good as the model would be.

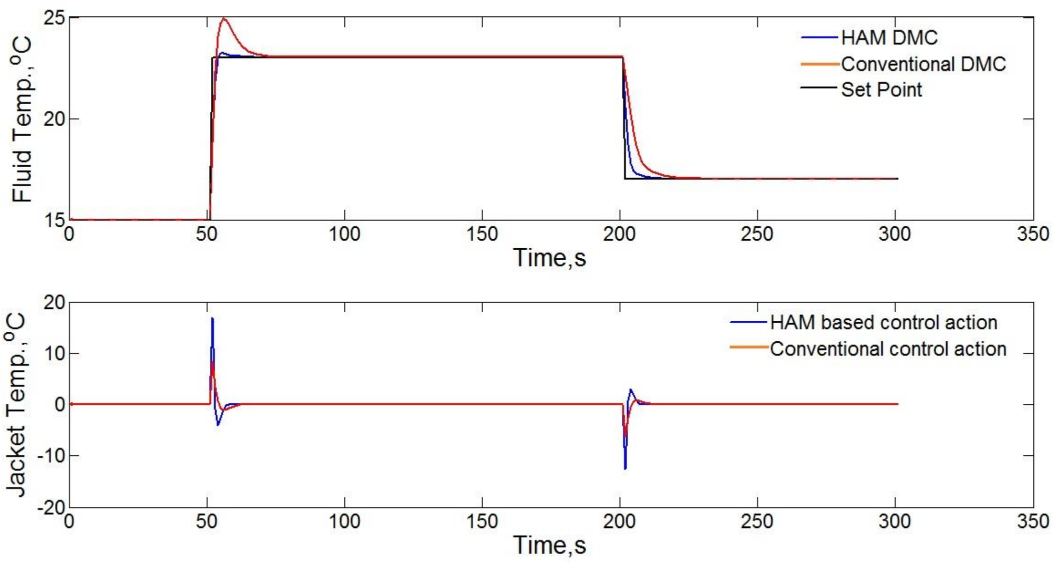

Figure 5: Comparison between the conventional DMC vs HAM-DMC

The limitation of the traditional DMC framework was successfully overcome by using HAM based prediction. Whenever the change in set-point needs to be tracked, the corresponding step-response model was determined by using the approach discussed before. The updated model serves to re-determine the parameters of the DMC so that the new set-point change for a non-linear system is tracked effortlessly, smoothly and instantly. Although, a similar step-response could be obtained by directly using numerical methods, the approach becomes computationally intensive for the programmable logic controllers (PLCs) in the plant. The advantage of the proposed HAM based models hence can be leveraged effortlessly as it just translates to substitution of set-points and the model parameters in the algebraic equations of the restricted order.

4.2. Application of HAM-DMC to pH control

This example illustrates application of HAM based DMCs for control of pH system which is highly non-linear [17]. The scheme assumes pH control of a highly alkaline concentrated sodium chloride solution in mixed tank system with addition of concentrated hydrochloric acid (HCl). The mixing tank hence receives two feeds, the first one being the stream to be treated and the other one being 0.1 M HCl. Conditions of the system is provided in Table 2.

Table 2: Composition of the incoming streams to the mixing tank

Ions	Na mEq/l	Cl, mEq/l	HCO_3 mEq/l	CO_3 mEq/l	Flow m^3/min	HCl mEq/l	pH
Feed stream,F_1	22.62	8.243	7.522	13.581	0.166	-	11
Acid stream,F_2	-	-	-	-	-	100	0

Remark 4:

a) Unlike the example discussed in (Henson & Seborg,1994), this problem assumes the reactor to be a mixing tank with constant volume (*V*) of 5 m^3. Hence, level control is not taken in to consideration
b) The incoming feed-stream and the mixed stream flowing out are both assumed to be electrically neutral
c) This is assumed to be a closed system with no transfer of carbon-dioxide with the surrounding. Hence, the total inorganic carbon (C_T) is assumed to constant.

Assuming the density of all streams to be constant, the component balance for each element hence is given by

$$V\frac{dc_i}{dt} = F_1 c_{1,i} + F_2 c_{2,i} - (F_1 + F_2)c_i$$

Or

$$\frac{dc_i}{dt} = \frac{F_1}{V}(c_{1,i} - c_i) + \frac{F_2}{V}(c_{2,i} - c_i) \qquad (45)$$

Where the subscript (i) reference to the elements namely Sodium (Na^+), Chloride (Cl^-), Bicarbonate (HCO_3^-), Carbonate (CO_3^{2-}) and Hydrogen ion (H^+) in mmol/liter. The differential equation above uses the algebraic constraint discussed in Remark b such that

$$[H^+] + [Na^+] = [Cl^-] + [HCO_3^-] + 2[CO_3^{2-}] + [OH^-] \qquad (46)$$

Where $[\ ]$ indicates the molar concentration in mmol/liter. Eq. (46) further can be simplified as

$$[HCO_3^-] = \frac{C_T}{1 + \frac{k_1}{[H^+]} + \frac{k_1 k_2}{[H^+]^2}}$$

$$,[CO_3^{2-}] = \frac{C_T}{1 + \frac{[H^+]}{k_1} + \frac{k_2}{[H^+]}} \text{ and } [OH^-] = \frac{10^{-11}}{[H^+]}$$

The parameters of the model are provided in Table 3.

Table 3: Constants used for the simulation

Const.	Eq.Const, k_1 mmol/l	Eq.Const, k_2 mmol/l	TIC C_T mmol/l
Values	5.1215×10^{-4}	8.2599×10^{-8}	7.6101

The pH profile follows completely a nonlinear gain against acid addition. This is evident from the Fig.6 where the pH is shown to change from 11 to 2 with different volume of acid added. As shown, the gain continuously changes below 9, 7.3, 5 and 3.

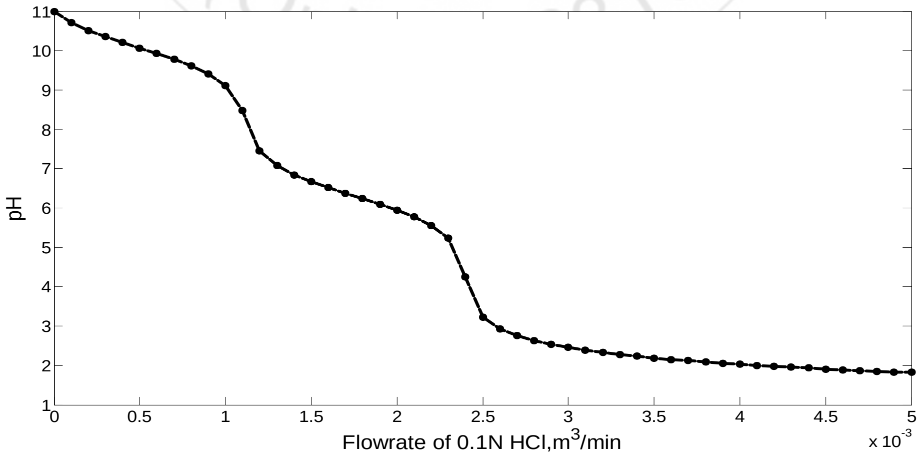

Figure 6: Steady-state pH profile against acid addition

Depending on the set-point, it is hence important for the model to guide the controller continuously by updating the gain as well as the dynamics.

The HAM solution to the mixing and pH model

The deviation form of the pH model using Eq. (27) is show in Appendix 2. The model assumes the form as shown in Eq. (47).

$$\frac{d\aleph_i}{dt} = \frac{F_1}{V}\aleph_i^2 - \frac{(F_1 + F_1\aleph_{i,o} - \Delta)}{V}\aleph_i + \frac{(F_1\aleph_{i,o} - \Delta)}{V} \quad (47)$$

The χ_i is the transformed variable and the concentration of a given component can be derived from Eq. (48)

$$c_i = \frac{c_{1,i} - \aleph_i c_{2,i}}{1 - \aleph_i} \quad (48)$$

A sixth order approximation was found to be sufficient to capture pH response to the different step-changes in acid flowrate.

$\aleph(t) = \sum_{n=0}^{5} \aleph_n$ such that

$$\aleph(0) = \aleph_o + te^{-t}$$

$$\aleph(1) = 0.0249h - 0.0332he^{-t} + 0.9668hte^{-t} + 0.0083h + 0.0$$
$$+ 0.2F_2\left(-h + ht + he^{-t} + hte^{-t}\right)$$

and so on. The optimal value of h was found to be -0.625. The estimated step response behaviour for step changes in 0.1 N HCl from 0.001 to 0.022 m^3/min is shown in Fig 7. using both HAM and nonlinear solvers.

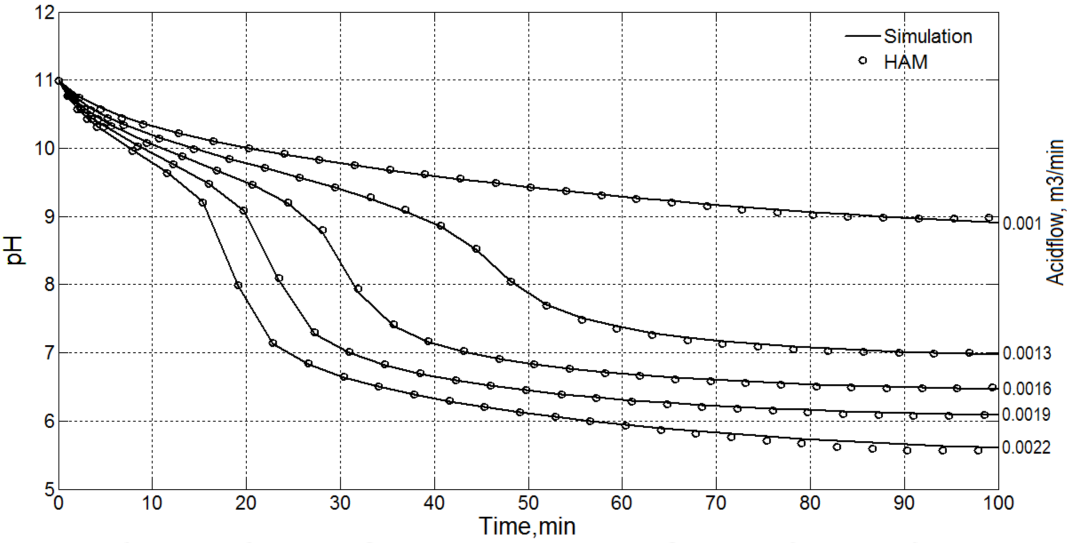

Figure 7: Comparison of dynamic pH response predicted using HAM and nonlinear solver for different step-changes in acid addition

The picture shows that the HAM was able to predict the responses satisfactorily, which is essential for the DMC algorithm. Fig. 7. also shows the fact that the step response behavior is not linear such that the gain and dynamics change significantly for constant increment in acid flow. Performance of DMC that has been designed for unit change in pH is shown in Fig.8. The initial pH of the process is at pH of 11. At 110th min, the set-point is changed to pH of 10. A first order transfer function with gain of 535 and time constant of 29.5 min was found to be the local model suitable for a unit change in pH. The step-response coefficients derived from the model were used to achieve change in pH from 11 to 10 which was found to be smooth and instantaneous.

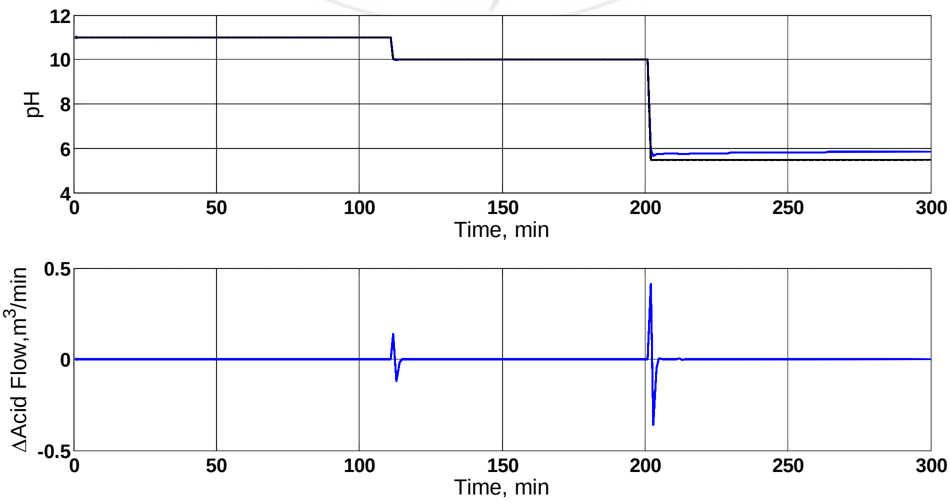

Figure 8: pH response using conventional DMC approach

However, when the pH set-point was changed from 10 to 5.5, the model was found to be inadequate to control the pH resulting in a bias that continued to digress from the set-point. The change in set-point was imposed on 200th min and the response continued to digress even after 300th min. The pH model being highly nonlinear in nature requires new step response coefficients corresponding to the new set-point imposed. The proposed HAM model could conveniently track the set-point. Results of the HAM-DMC is shown in Fig 9.

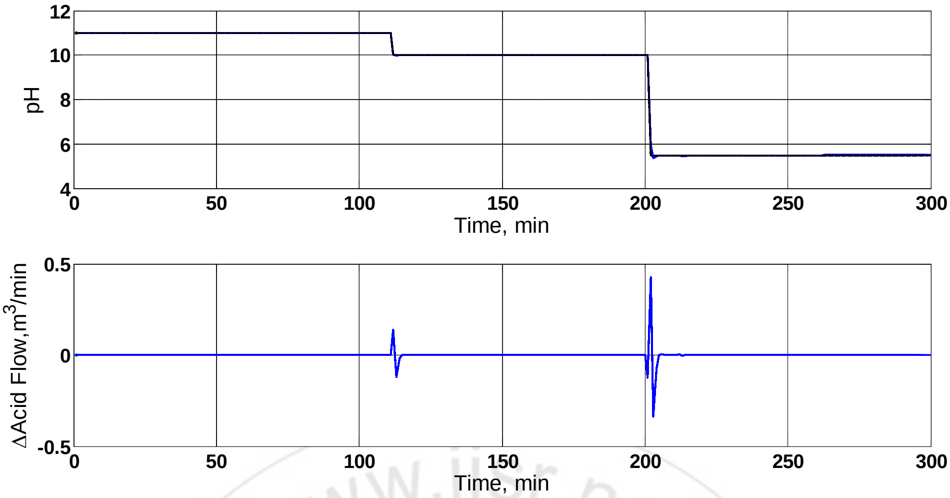

Figure 9: Set point tracking using HAM-DMC framework

From the response, it is evident that the proposed approach could enable smooth tracking despite a big change in set-point for a nonlinear system.

5. Conclusion

HAM is a proven and a popular technique for solving non-linear differential and partial differential equations. This paper provides a new framework where the HAM has been used to solve a class of non-linear dynamical systems by which significant improvement in the DMC can be realized. The approach presented is applicable to a class of control-affine structures. The proposed transformation primarily helps in deducing the step-response behaviour of a non-linear system that is independent of its initial state and amplitude/nature of the input. This is achieved by choosing the exponential base function so that the resulting model can be directly incorporated in to the DMC framework. The most significant feature of this approach is the use of simple algebraic step-response model resulting from the HAM solution. Using this structure, the step-response model can be readily updated depending on the change in set-point imposed on the controller. When the traditional linear DMC model shows limitations to cope-up with the model-plant mismatch at different operating points, the HAM can quickly estimate the new model corresponding to the shift in the set-point and provide better control at every new operating point. Although this work proposes solution to non-linear models to re-estimate DMC models, the performance of DMC is not compared with the non-linear MPC. It is primarily based on the premise that a better DMC action can be achieved using HAM based algebraic solutions wherever the PLCs have limited processing capacity to deal with non-linear solvers. Two case-studies have been provided to illustrate merits of the proposed approach. This could be further extended to multi-input multi-output (MIMO) systems along with constraints.

Appendix-1

For a jacketed vessel, the heat exchanged with a cooler medium is given by

$$\dot{Q} = UA(T_a - T)$$

T_a is the coolant temperature in the jacket, T is the temperature of the process fluid in the tank, U is the overall heat-transfer coefficient and A is the surface area.

$\dot{Q}$ is the rate of accumulation of heat and is given by

$$\dot{Q} = \frac{d\left[mc(T - T_r)\right]}{dt}$$

The fluid of m being heated in the vessel is considered to be having specific heat c as a linear function of temperature such that $c = c_a\left[1 + b(T - T_a)\right]$ where b is a constant. For a constant volume system, the above expression could be simplified as

$$\rho V \frac{d\left(c_a\left[1 + b(T - T_a)\right](T - T_r)\right)}{dt} = UA(T_a - T)$$

If $T_r = T_a$, then the above equation could be simplified as

$$\rho V c_a\left[1 + 2b(T - T_a)\right]\frac{dT}{dt} + UA(T - T_a) = 0$$

Replacing $2b$ with β

$$\rho V c_a\left[1 + \beta(T - T_a)\right]\frac{dT}{dt} + UA(T - T_a) = 0$$

Appendix-2

For a given component, the mixing model simplifies to $V\frac{dC}{dt} = F_1(c_1 - c) + F_2(c_2 - c)$. To establish the dependency on acid added to the concentration profile, the form recommended in Eq. (27) is again deployed.

$\frac{V}{c_2 - c}\frac{dC}{dt} = F_1 \frac{c_1 - c}{c_2 - c} + F_2$. If c_o is the concentration corresponding to the steady-state $F_{2,s}$, then

$$\frac{V}{c_2 - c}\frac{dC}{dt} = F_1\left(\frac{c_1 - c}{c_2 - c} - \frac{c_1 - c_o}{c_2 - c_o}\right) + \Delta \text{ where}$$

$$\Delta = F_2 - F_{2,s}$$

By assigning $\frac{c_1 - c}{c_2 - c} = \aleph$ and $\frac{c_1 - c_o}{c_2 - c_o} = \aleph_o$, and by differentiating with respect to c, the above equation simplifies to

$$\frac{d\aleph}{dt} = \frac{F_1}{V}\aleph^2 - \frac{(F_1 + F_1\aleph_o - \Delta)}{V}\aleph + \frac{(F_1\aleph_o - \Delta)}{V}$$

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