Algebraic Structures on Multi Groups I

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Abstract: A simple definition of multi-group was derived it helps us to find some properties of multi-groups. Every homomorphic image of a multi-group is a multi-group. Every pre homomorphic image of a multi-group is also a multi-group. Necessary and sufficient conditions for a multiset to be a multi-group were obtained.

Keywords: multi-set, multigroup

1. Introduction

In classical set theory we know that each element in a set appears one time in a set, that is no copies or duplicates for any object in the set, but in multi sets (or bags) which are set-like structure, but in which an object can appears more than one time, that is the element may have several copies in the multiset (m set) for example $A = \{1,1,1,1,5,5,5,7,7\}$ it is a multiset, and A can be written as:

 $A = \{4/1, 3/5, 2/7\} = \{1, 5, 7\}_{4,3,2} \text{ the multiplicity}$ of 1 in A is 4, multiplicity of 5 in A is 3, the multiplicity of 7 in A is 2 for this we write $C_A(1) = 4$, $C_A(5) = 3$, $C_A(7) = 2$.

In general, we can say that the number of occurrences of an element b in the multiset B which is finite and unique is called the multiplicity of b in B, and is denoted by $C_B(b)$ and when $C_B(b) = n$ we may write $b \in B^n$.

Now let Γ be a set, υ be any positive integer, the family of all multisets over Γ (whose elements taken from Γ) such that the occurrence of each of which not exceeds υ is denoted by $[\Gamma]^{\upsilon}$. We use $[\Gamma]^{\infty}$ if there is no limit of this occurrence.

Definition1.1: "sub multiset":

Assume that A, B are two multisets over the same set Γ then A is said to be sub multiset of B if $C_A(\gamma) \leq C_B(\gamma)$ forevery $\gamma \in \Gamma$ and then we write $A \subset B$.

The equality holds when the both relations $A \subseteq B, B \subseteq A$ are satisfied and in this case we can write A = B.

Definition1.2:"Union":

Let I be an index set and consider the family of multisets $\{A_i : i \in I\}$ over a set Γ then we define their union $\bigcup_{i \in I} A_i$ such that $\underset{i \in I}{C} (\gamma) = \max_{i \in I} C_{Ai} (\gamma)$ for every $\gamma \in \Gamma$

Definition1.3: "Intersection"

We can define the intersection for the family
$$\{A_i : i \in I\}$$
 by $\bigcap_{i \in I} A_i$ such that $\prod_{i \in I} (\gamma) = \min_{i \in I} C_{A_i} (\gamma)$ for every $\gamma \in \Gamma \bigcap_{i \in I} A_i$.

Definition1.4: Complement

For each A_i in $[\Gamma]^v$ we can define its complement in $[\Gamma]^v$ by A_i^c such that for every $\gamma \in \Gamma$ we have $C_{A_i^c}(\gamma) = v - C_{A_i}(\gamma)$.

Definition1.5:

Assume that Γ and Γ' are two non-empty sets and let $f: \Gamma \to \Gamma'$ be a mapping. If A is any multiset in $[\Gamma]^{\nu}$ then its image by f is defined as f(A) such that $\underset{f(A)}{C}(\gamma') = \underset{f(\gamma)=\gamma'}{\max}C_A(\gamma)$ whatever $f^{-1}(\gamma') \neq \phi$ and $\underset{f(A)}{C}(\gamma') = 0$ whatever $f^{-1}(\gamma') = \phi$.

Definition1.6:

let $f: \Gamma \to \Gamma'$ be a mapping Γ, Γ' are two non-empty sets and let $B \in [\Gamma']^{\nu}$ then the inverse image of $B, f^{-1}(B)$ is such that $\underset{f^{-1}(B)}{C}(\gamma) = C_B(f(\gamma)).$

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2. Multisets over a group

Now let us consider Γ to be a group with identity e, and as before the class of all multisets over Γ , such that the multiplicity does not exceeds an integer U is denoted by $[\Gamma]^{\nu}$.

Definition2.1:

Let H, K be any two members of the class $[\Gamma]^{\nu}$ let us define HoK to be such that $\sum_{HoK} (\alpha) = \max_{hk = \alpha} \left\{ \min_{h,k \in \Gamma} \left\{ C_H(h), C_K(k) \right\} \right\}$... (2.1)

Definition2.2:

For every H in $[\Gamma]^{\nu}$ we define H^{-1} as follows $C_{H}(h^{-1}) = C_{H^{-1}}(h) \text{ for every } h \in \Gamma.$

Theorem2.3:

let H, K be any two multisets in $[\Gamma]^{\nu}$ then

$$\underbrace{C}_{HoK}(u) = \max_{v \in \Gamma} \left\{ \min \left\{ C_{H}(uv^{-1}), C_{K}(v) \right\} \right\}$$

$$= \max_{v \in \Gamma} \left\{ \min \left\{ C_{H}(v), C_{K}(v^{-1}u) \right\} \right\}$$

Proof:

Since from the definition of HoK we have:

$$\underbrace{C}_{HoK}(u) = \max_{hv = u} \left\{ \min_{h,k \in \Gamma} \left\{ \underbrace{C}_{H}(h), \underbrace{C}_{K}(v) \right\} \right\}.$$

But now Γ is a group which means that for every u, h in Γ , we can assign one and only element v such that hv = u, hence we can write:

Theorem2.4:

For every H in Γ^{υ} we have $(H^{-1})^{-1} = H$. **Proof:**

Let h be any element in Γ , from the definition $C_{H^{-1}}(h) = C_{H}(h^{-1}),$ then we deduce that: $C_{(H^{-1})^{-1}}(h) = C_{H^{-1}}(h^{-1}) = C_{H}(h)$, and hence our

required result follows.

Theorem2.5:

H, K in $[\Gamma]^{\nu}$ everv For we have $(HoK)^{-1} = K^{-1}oH^{-1}.$

Proof:

Since for every $u \in \Gamma$, $\underset{HoK}{C} (u) = \underset{HoK}{C} (u^{-1})$

But,

$$\sum_{HoK} (u^{-1}) = \max_{hk = u^{-1}} \left\{ \min_{h,k \in \Gamma} \left\{ C_H(h), C_K(k) \right\} \right\}.$$

Now since Γ is a group, and when h, k runs over all elements in Γ , then so does h^{-1}, k^{-1} and recall the fact

$$\underbrace{C}_{H}\left(h^{-1}\right) = \underbrace{C}_{H^{-1}}\left(h\right) \qquad \text{and} \underbrace{C}_{K^{-1}}\left(k\right) = \underbrace{C}_{K}\left(k^{-1}\right)$$
therefore we can write:
$$\begin{bmatrix} & & \\ & &$$

$$\sum_{HoK} (u^{-1}) = \max_{(hk)^{-1} = u} \left\{ \min_{h^{-1}, k^{-1} \in \Gamma} \left\{ C_{H^{-1}}(h^{-1}), C_{K^{-1}}(k^{-1}) \right\} \right\}$$

$$= \max_{k^{-1}h^{-1} = u} \left\{ \min_{k^{-1}, h^{-1} \in \Gamma} \left\{ C_{K^{-1}}(k^{-1}), C_{H^{-1}}(h^{-1}) \right\} \right\}$$

$$C_{HoK}(u^{-1}) = C_{H^{-1}oK^{-1}}(u), \text{ but } C_{(HoK)^{-1}}(u) = C_{HoK}(u^{-1})$$
, and hence $(HoK)^{-1} = K^{-1}oH^{-1}.$

Theorem 2.6:

The operation "o" on the multisets of $[\Gamma]^{\nu}$ is associative that is (HoK)oL = Ho(KoL)**Proof:**

Let
$$HoK = D$$
, therefore $(HoK)oL = DoL$,

$$\sum_{(HoK)oL} (u) = \sum_{DoL} (u) = \max_{vl=u} \left\{ \min_{v,l\in\Gamma} \left\{ C_D(v), C_L(l) \right\} \right\}$$
, but

$$C_{D}(v) = \max_{hk=v} \left\{ \min_{h,k\in\Gamma} \left\{ C_{H}(h), C_{K}(k) \right\} \right\},\$$

Volume 6 Issue 5, May 2017

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DOI: 10.21275/ART20173240

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International Journal of Science and Research (IJSR) ISSN (Online): 2319-7064 Index Copernicus Value (2015): 78.96 | Impact Factor (2015): 6.391

$$\frac{C}{(HoK)_{oL}}(u) = \max_{\forall l=u} \left\{ \min \left\{ \max_{hk=v} \left\{ C_{H}(h), C_{K}(k) \right\} \right\}, C_{L}(l) \right\} = \max_{(hk)l=u} \left\{ \min \left\{ C_{H}(h), C_{K}(k), C_{L}(l) \right\} \right\},$$
since Γ is a group, then $(hk)l = h(kl) \forall h, k$ and $l \in \Gamma$,
$$\frac{C}{(HoK)_{oL}}(u) = \max_{\substack{h(w)=u\\ w=kl}} \left\{ \min \left\{ C_{H}(h), C_{K}(k), C_{L}(l) \right\} \right\}$$

$$= \max_{hw=u} \left\{ \min \left\{ C_{H}(h), \max_{kl=w} \left\{ \min C_{K}(k), C_{L}(l) \right\} \right\} \right\}$$

$$= \max_{hw=u} \left\{ \min \left\{ C_{H}(h), C_{K}(v) \right\} \right\}$$

3. Multi Groups and Its Properties

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Definition 3.1: A multiset H over a group Γ is a multigroup if for every a, b in Γ , we have:

By $MG(\Gamma)$ we mean the set of all multi-groups over Γ .

Theorem 3.2: let H be any multi-group over Γ , then for every $a \in \Gamma$ we have:

(1) The multiplicity of any element in H not exceeds the multiplicity of the identity element, that is $C_{H}(a) \leq C_{H}(e) \forall a \in \Gamma$

(2) The element and its inverse have the same multiplicity

that is
$$C_H(a^{-1}) = C_H(a)$$
 for every $a \in \Gamma$.
Proof:

(1) Since $H \in MG(\Gamma)$ then for every $a, b \in \Gamma$ we have: $C_{H}(ab^{-1}) \ge \min \left\{ C_{H}(a), C_{H}(b) \right\}$ Put a = b we get: $C_{H}(aa^{-1}) = C_{H}(e) \ge \min \left\{ C_{H}(a), C_{H}(a) \right\} = C_{H}(a)$, for every $a \in \Gamma$.

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Proof: (i) "Necessity": let $H \in MG(\Gamma)$ (i.e. H is a multi-group over the group Γ) then for every $a, b \in \Gamma$ we have:

$$C_{H}(ab^{-1}) \ge \min \left\{ C_{H}(a), C_{H}(b) \right\}$$

$$C_{H}(ab) \ge \min \left\{ C_{H}(a), C_{H}(b^{-1}) \right\}, \quad \text{and} \quad \text{from}$$
theorem (3.1) part (2), $C_{H}(b^{-1}) = C_{H}(b)$ then

$$C_{H}(ab) \ge \min \left\{ C_{H}(a), C_{H}(b) \right\} \text{ now}$$

$$C_{H}(a) = C_{H}(a^{-1}) \text{ follows directly from theorem (3.2)}$$
part (2)

(ii) "Sufficiency": let $H \in MS(\Gamma)$ (i.e. H is a multiset over the group Γ) and the two conditions are satisfied that is:

(1)
$$C_{H}(ab) \ge \min \left\{ C_{H}(a), C_{H}(b) \right\}$$
 for every
 $a, b \in \Gamma$
(2) $C_{H}(a) = C_{H}(a^{-1})$ for every $a \in \Gamma$

Now

$$C_{H}(ab^{-1}) \ge \min\left\{C_{H}(a), C_{H}(b^{-1})\right\} \text{ and since}$$

$$C_{H}(a) = C_{H}(a^{-1}) \text{ from (2) then}$$

$$C_{H}(ab^{-1}) \ge \min\left\{C_{H}(a), C_{H}(b)\right\} \text{ then the theorem}$$
is proved

is proved

Definition 3.5: let *H* be multi-group over the group Γ , $C_H(e) = \zeta$

(1)
$$H_{\upsilon} = \left\{ a \in \Gamma : C_{H}(a) \ge \upsilon, \upsilon, \in Z^{+} \right\}$$

where Z^+ is the set of all positive integers

(2)
$$H_{\zeta(e)} = \left\{ a \in \Gamma : C_{H}(a) = \zeta = C_{H}(e) \right\}$$

(3)
$$H_{0} = \left\{ a \in \Gamma : C_{H}(a) \ge 0 \right\}$$

Theorem 3.6: let H be multi-group over a group Γ , then each of H_0 , $H_{\zeta(e)}$ and H_v , $v \in Z^+$ are sub groups of the group Γ

Proof: (1) to show that H_0 is a sub group of Γ , let $a, b \in H_0$ this mean that

C(a) > 0, C(b) > 0 thus from the relation $C_{\mu}(ab^{-1}) \ge \min \left\{ C_{\mu}(a), C_{\mu}(b) \right\} > 0$ we deduce that $C(ab^{-1}) > 0 \Longrightarrow ab^{-1} \in H_0$ hence H_0 is a sub group of Γ . (2) To show that $H_{\zeta(e)}$ is a sub group where $\zeta = C(e), \quad \text{let} \quad a,b \in \mathbf{H}_{\zeta(e)},$ then $C_{\mu}(a) = C_{\mu}(e) = C_{\mu}(b) = \zeta$ $C_{u}(ab^{-1}) = \min \{C_{u}(a), C_{u}(b^{-1})\}$ $C_{u}(ab^{-1}) = \min\{C_{u}(a), C_{u}(b)\}$ (because $C_{\mu}(b^{-1}) = C_{\mu}(b)$ by theorem 3.2 (2)) $C_{u}\left(ab^{-1}\right) = \min\left\{\zeta,\zeta\right\} = \zeta = C_{u}\left(e\right)$ thus $ab^{-1} \in \mathbf{H}_{\zeta(e)}$ and so $\mathbf{H}_{\zeta(e)}$ is a sub group of Γ . (3) let $a, b \in H$ then we can write $C_H(a) = n \ge v$, $C(b) = m \ge v$ $C_{u}\left(ab^{-1}\right) \geq \min\left\{C_{u}\left(a\right), C_{u}\left(b^{-1}\right)\right\}$ $C_{u}(ab^{-1}) \geq \min\left\{C_{u}(a), C_{u}(b)\right\}$ $C_{u}\left(ab^{-1}\right) \geq \min\left\{n,m\right\} = k \geq v$ and so $C(ab^{-1}) \ge v$ and hence $ab^{-1} \in H$ which implies that H is a sub group of Γ . **Theorem 3.7**: A multiset H over group Γ is a multigroup over Γ , if and only if HoH = Hand $H = H^{-1}$.

Proof: (1) "Necessity" assume that H is a multi-group over Γ , take $a, b \in \Gamma$ by theorem B(1) we can write:

$$C_{H}(ab) \ge \min \left\{ C_{H}(a), C_{H}(b) \right\} \quad \text{for every}$$

 $a,b \in I$. And we get:

$$C_{H}(h) \geq \min_{\substack{a,b\in\Gamma\\ab=h}} \{C_{H}(a), C_{H}(b)\} \text{ that is}$$

$$C_{H}(h) \geq \max_{\substack{a,b\in\Gamma\\ab=h}} \{\min\{C_{H}(a), C_{H}(b)\}\} \text{ and since}$$

Volume 6 Issue 5, May 2017

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International Journal of Science and Research (IJSR) ISSN (Online): 2319-7064 Index Copernicus Value (2015): 78.96 | Impact Factor (2015): 6.391

$$\begin{array}{l}
C_{HoH}(h) = \max_{\substack{a,b \in \Gamma \\ ab=h}} \left\{ \min \left\{ \begin{array}{l} C_{H}(a), C_{H}(b) \right\} \right\} & \text{therefore} \\
\end{array}$$

$$\begin{array}{l}
C_{H}(h) \geq C_{HoH}(h) & \text{which means that} \\
\end{array}$$

$$\begin{array}{l}
HoH \subset H \rightarrow (*) \\
\text{Now by using theorem2.3} \\
C_{H}(h) = C_{H}(ha^{-1}a) \\
\end{array}$$

$$\begin{array}{l}
C_{H}(h) \geq \min \left\{ C_{H}(ha^{-1}), C_{H}(a) \right\} \\
C_{H}(h) \geq \max_{a \in \Gamma} \left\{ \min \left\{ C_{H}(ha^{-1}), C_{H}(a) \right\} \right\} = C_{HoH}(h) \\
\end{array}$$

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 $C_{H}(h) \ge C_{HoH}(h)$ $H \supset HoH \rightarrow (**)$

From (*) and (**) we get HoH = HNow since

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$$C_{H^{-1}}(h) = C_{H}(h^{-1}) \forall h \in \Gamma$$
 and since H is a multi-

group over a group Γ , then by theorem 3.2 part (2) $C_{H}(h^{-1}) = C_{H}(h) \forall h \in \Gamma$. Thus $C_{H^{-1}}(h) = C_{H}(h^{-1}) = C_{H}(h)$ which implies that

 $H = H^{-1}$ as required.

(2) Sufficiency: assume that HoH = H and $H = H^{-1}$ where H is a multiset over Γ (i.e. $H \in MS(\Gamma)$), by theorem 2.3

$$C_{H}(ab^{-1}) = C_{HoH}(ab^{-1}) = \max_{h\in\Gamma} \left\{ \min\left\{ C_{H}(h), C_{H}(h^{-1}ab^{-1}) \right\} \right\}$$

put h = a and so

$$C_{H}\left(ab^{-1}\right) = C_{HoH}\left(ab^{-1}\right) \ge \min\left\{C_{H}\left(a\right), C_{H}\left(b^{-1}\right)\right\} = \min\left\{C_{H}\left(a\right), C_{H}\left(b\right)\right\}$$

[Since $H = H^{-1}$, $C_H(b) = C_{H^{-1}}(b) = C_H(b^{-1})$], thus

$$C_{H}(ab^{-1}) \geq \min\left\{C_{H}(a), C_{H}(b)\right\}$$

$$\therefore H \in MG(\Gamma).$$

Theorem 3.8: let H be a multiset over the group Γ , then H is a multi-group over Γ if and only if $HoH^{-1} = H$.

Proof: "necessity": let $H \in MG(\Gamma)$ then by theorem 3.7 we have HoH = H and $H^{-1} = H$ so we can write $HoH^{-1} = H$. "sufficiency": let $HoH^{-1} = H$ this means that $C_{HoH^{-1}}(a) = C_{H}(a)$ for every $a \in \Gamma$ therefore we can write using

$$C_{H}(ab^{-1}) = C_{HoH^{-1}}(ab^{-1}) = \max_{h\in\Gamma} \left\{ \min\left\{ C_{H}(ab^{-1}h), C_{H^{-1}}(h^{-1}) \right\} \right\}$$

and hence

$$C_{H}(ab^{-1}) \ge \min\left\{C_{H}(ab^{-1}h), C_{H^{-1}}(h^{-1})\right\}$$

[Since from: $C_{H^{-1}}(a) = C_{H}(a^{-1})$ i.e. definition (2.2)]

then

$$C_{H}(ab^{-1}) \ge \min_{h \in \Gamma} \left\{ C_{H}(ab^{-1}h), C_{H}(h) \right\}, \quad \text{for}$$

$$h = b \text{ we get}$$

$$C_{H}(ab^{-1}) \geq \min\left\{C_{H}(a), C_{H}(b)\right\}$$

And so $H \in MG(\Gamma)$ and hence the theorem is proved.

Theorem 3.9: let H, K be two multi-groups over the same group Γ then HoK = KoH if and only if HoK is a multi-group over Γ . **Proof "Necessity"**:

Let
$$HoK = KoH$$

Also assume that $KoH = D$
 $D^{-1} = (HoK)$ (by theorem 2.5, that is
 $(HoK)^{-1} = K^{-1}oH^{-1}$
 $D^{-1} = K^{-1}oH^{-1}$
Since $H, K \in MG(\Gamma)$ then by lemma 3.3 (2), we
have $H^{-1} = H, K^{-1} = K$
 $D^{-1} = K^{-1}oH^{-1} = KoH = D$ and so
 $D = D^{-1}$

 $DoD^{-1} = DoD = (HoK)o(HoK)$ $DoD^{-1} = Ho(KoH)oK = Ho(HoK)oK = (HoH)o(KoK)$

 $DoD^{-1} = (HoH)o(KoK)$ But since $H, K \in MG(\Gamma)$ HoH = H, KoK = K (by theorem 3.7), thus $DoD^{-1} = HoK = D$ $DoD^{-1} = D$, $D^{-1} = D$

Volume 6 Issue 5, May 2017

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thus by theorem 3.7 again we get $D \in MG(\Gamma)$ i.e.

 $HoK \in MG(\Gamma).$ (2)"Sufficiency": let $HoK \in MG(\Gamma)$ where $H, K \in MG(\Gamma)$ which implies that $H^{-1} = H$ and also $K^{-1} = K$ Put $HoK = D \in MG(\Gamma)$ $D^{-1} = D$ implies that $HoK = (HoK)^{-1} = K^{-1}oH^{-1} = KoH$ i. HoK = KaH and a so that the set of th

i.e. HoK = KoH and hence the theorem is proved. **Theorem 3.10**: the arbitrary intersection of any family of multi-groups over Γ is again a multi-group over Γ . **Proof**: let $\{H_i : i \in I\}$ be a family over Γ where $H_i \in MG(\Gamma)$ for every $i \in I$ and so we have:

$$C_{H_{i}}(ab^{-1}) = \min\left\{C_{H_{i}}(a), C_{H_{i}}(b)\right\}, \forall a, b \in \Gamma$$

$$C_{\cap H_{i}}(ab^{-1}) = \min_{i \in I}\left\{C_{H_{i}}(ab^{-1})\right\}$$

$$C_{\cap H_{i}}(ab^{-1}) \ge \min_{i \in I}\left\{\min\left\{C_{H_{i}}(a), C_{H_{i}}(b)\right\}\right\}$$

$$C_{\cap H_{i}}(ab^{-1}) = \min\left\{\min_{i \in I}C_{H_{i}}(a), \min_{i \in I}C_{H_{i}}(b)\right\}$$

$$C_{\cap H_{i}}(ab^{-1}) = \min\left\{C_{\cap H_{i}}(a), C_{\cap H_{i}}(b)\right\}$$
Therefore:

$$\frac{C}{\substack{\bigcap H_{i} \\ i \in I}} \left(ab^{-1} \right) \geq \min \left\{ \frac{C}{\substack{\bigcap H_{i} \\ i \in I}} \left(a \right), \frac{C}{\substack{\bigcap H_{i} \\ i \in I}} \left(b \right) \right\}$$
Thus
$$\prod_{i \in I} H_{i} \in MG(\Gamma)$$

Remark 3.11: it must be noted that the previous theorem need not be satisfied in the case of union whatever finite or infinite families of multi-groups.

Definition 3.12 "Sub multi-group": Let $H, K \in MG(\Gamma)$ such that $K \subseteq H$ then we say that K is a sub multi-group of H. **Definition 3.13 "Abelian multiset over** Γ ": Let $H \in MS(\Gamma)$ we say that H is abelian multiset

over Γ if $C_{H}(ab) = C_{H}(ba)$ fore very $a, b \in \Gamma$ and we write $H \in AMS(\Gamma)$. Definition 3.14 "Abelian multi-group over Γ ":

Let $H \in MS(\Gamma)$ then H is said to be abelian over

$$\Gamma \text{ if } H \in AMS(\Gamma) \text{ that if } C_H(ab) = C_H(ba)$$

fore very $a, b \in \Gamma$.

Theorem 3.15: the following statements are equivalent:

(1) $H \in AMS(\Gamma)$

(2)
$$C_{H}\left(a^{-1}ba\right) = C_{H}\left(aba^{-1}\right) = C_{H}\left(b\right)$$

(3) HoK = KoH holds true for every $K \in MS(\Gamma)$

Proof: the proof is carried as follows (1) \Rightarrow (2),(2) \Rightarrow (1),(1) \Rightarrow (3),(3) \Rightarrow (1) Now (1) \Rightarrow (2) let $H \in AMS$ (Γ) then $\forall a, b \in \Gamma$ we have $C_{H}(ab) = C_{H}(ba)$ Let ba = t $C_{H}(a^{-1}ba) = C_{H}(a^{-1}t) = C_{H}(ta^{-1})$ $C_{H}(baa^{-1}) = C_{H}(b)$ That is $C_{H}(a^{-1}ba) = C_{H}(b)$(1) Also let ab = r

From (1),(2) we get

$$C_{H}(aba^{-1}) = C_{H}(a^{-1}ba) = C_{H}(b)$$
Now (2) \Rightarrow (1): let

$$C_{H}(aba^{-1}) = C_{H}(a^{-1}ba) = C_{H}(b)$$
Let $ba = t$ then $b = ta^{-1}$ and then $ab = ata^{-1}$

$$C_{H}(ab) = C_{H}(ata^{-1}) = C_{H}(t) = C_{H}(ba)$$
And hence $H \in AMS(\Gamma)$
Now (1) \Rightarrow (3): let $H \in AMS(\Gamma)$ and

 $K \in MS(\Gamma)$, then let us take any arbitrary element a in Γ

$$C_{HoK}(a) = \max_{b \in \Gamma} \left\{ \min \left\{ C_H(ab^{-1}), C_K(b) \right\} \right\}$$

Volume 6 Issue 5, May 2017

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Since
$$C_{H}(rt) = C_{H}(tr) \forall r, t \in \Gamma$$

 $C_{HoK}(a) = \max_{b \in \Gamma} \left\{ \min \left\{ C_{H}(b^{-1}a), C_{K}(b) \right\} \right\}$
 $C_{HoK}(a) = \max_{b \in \Gamma} \left\{ \min \left\{ C_{K}(b), C_{H}(b^{-1}a) \right\} \right\}$
 $C_{HoK}(a) = C_{KoH}(a)$

Therefore HoK = KoH

And so $(1) \Longrightarrow (3)$ follows.

Now $(3) \Rightarrow (1)$ let $H \in MS(\Gamma)$ such that HoK = KoH holds true for every K in $MS(\Gamma)$.

For any arbitrary two elements a, b in Γ let us define an integers $\upsilon, \upsilon \ge 0$ where

 $\upsilon = \min\left\{ \begin{array}{l} C (ab), C (ba) \\ H (ba) \end{array} \right\} \text{ and then we can}$ construct a singleton multiset $Q = \left\{ \upsilon / b^{-1} \right\}$ which contains the element b^{-1} represented υ times, it is clear that $Q \in MS(\Gamma)$, then from our assumption, $HoK = KoH \forall K \in MS$, and hence HoQ = QoH that is $\begin{array}{c} C \\ HoQ \end{array} = \begin{array}{c} C \\ QoH \end{array} \left\{ a \right\} \forall a \in \Gamma$ $\begin{array}{c} C \\ HoQ \end{array} = \begin{array}{c} max \\ hq=a \end{array} \left\{ \min\left\{ C \\ H \end{array} \left\{ h \right\}, C \\ Q \end{array} \left\{ q \right\} \right\} \right\},$ using theorem 2.3

$$C_{HoQ}(a) = \max_{b \in \Gamma} \left\{ \min \left\{ C_{H}(ab), C_{Q}(b^{-1}) \right\} \right\}$$
$$C_{HoQ}(a) = \max_{b \in \Gamma} \left\{ \min \left\{ C_{H}(ab), \nu \right\} \right\} = C_{H}(ab)$$
$$C_{HoQ}(a) = C_{H}(ab)$$

$$C_{HoQ}(a) = \max_{qh=a} \left\{ \min \left\{ C_Q(q), C_H(h) \right\} \right\}$$

$$C_{HoQ}(a) = \max_{b \in \Gamma} \left\{ \min \left\{ C_Q(b^{-1}), C_H(ba) \right\} \right\}$$

$$C_{HoQ}(a) = \max_{b \in \Gamma} \left\{ \min \left\{ v, C_H(ba) \right\} \right\} = C_H(ba)$$

$$C_{HoQ}(a) = C_H(ba)$$

From (3) and (4), since HoQ = QoH it follows that $C_H(ab) = C_H(ba)$, i.e. $H \in AMS(\Gamma)$ which completes the proof the theorem.

Theorem 3.16:

Let $f: \Gamma \to \Gamma'$ be a homomorphism from Γ into Γ' then the following statements are satisfied:

(1) every homomorphic image of a multi-group over Γ is a multi-group over Γ' .

(2) every pre homomorphic image of a multi-group over Γ' is a multi-group over Γ .

(3) every homomorphic image of an abelian multi-group over Γ is an abelian multi-group over Γ' .

(4) every pre homomorphic image of an abelian multigroup over Γ' is an abelian multi-group over Γ .

Proof (1): let $f : \Gamma \to \Gamma'$ be a homomorphism from Γ into Γ' and let $H \in MG(\Gamma)$.

Let us take any two arbitrary elements α, β consider the following three cases:

(a) α, β (both of them) in Rang (f), (i.e. $\alpha, \beta \in f(\Gamma)$).

(b) α, β not in Rang (f), (i.e. $\alpha, \beta \notin f(\Gamma)$).

(c) one of them in the Rang (f). Now consider first case (a), (i.e. $\alpha, \beta \in f(\Gamma)$), then we can find $a, b \in \Gamma$ such that $f(a) = \alpha, f(b) = \beta$ and so we have:

$$C_{f(H)}(\alpha\beta^{-1}) = \max_{\gamma \in \Gamma} \left\{ C_{H}(\gamma), f(\gamma) = \alpha\beta^{-1} \right\}$$

Now since $f : \Gamma \to \Gamma'$ is a homomorphism $f(ab^{-1}) = f(a)f(b^{-1}) = f(a).f(b)^{-1} = \alpha\beta^{-1}$

and since
$$f(\gamma) = \alpha \beta^{-1}$$

then we can write

$$\frac{C}{f(H)} (\alpha \beta^{-1}) = \max \left\{ C_{H} (ab^{-1}), f(ab^{-1}) = \alpha \beta^{-1} \right\}$$
, since $H \in MG(\Gamma)$

$$\frac{C}{f(H)} (\alpha \beta^{-1}) \ge \max_{\substack{f(a)=\alpha \\ f(b)=\beta}} \left\{ \min \left\{ C_{H} (a), C_{H} (b) \right\} \right\}$$

$$= \min \left\{ \max_{\substack{f(a)=\alpha \\ f(H)}} \left\{ C_{H} (a), \max_{\substack{f(b)=\beta \\ H}} \left\{ C_{H} (b) \right\} \right\} \right\}$$

$$= \min \left\{ C_{f(H)} (\alpha), C_{f(H)} (\beta) \right\}$$
Which means that $f(H) \in MG(\Gamma')$
For the last two cases $(b), (c)$ are trivial.
Proof (2): let $K \in MG(\Gamma)$ to do this let

Volume 6 Issue 5, May 2017

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 $a,b \in f^{-1}(K)$ (i.e. $a,b \in Range f'(K)$), then there exist $\alpha, \beta \in K$. Such that: $f^{-1}(\alpha) = \alpha$ and $f^{-1}(\beta) = b$ and hence $\alpha = f(a)$ and $\beta = f(b)$ $C_{f^{-1}(K)}(a) = C_{K}(f(a)) = C_{K}(\alpha) \text{ also}$ $C_{F^{-1}(K)}(b) = C_{K}(f(a)) = C_{K}(\beta)$ $C_{f^{-1}(K)}(ab^{-1}) = C_{K}(f(ab^{-1}))$ Since f is a homomorphism, then: $f(pq) = f(p)f(q), f(p^{-1}) = f(p)^{-1}$ $\forall p,q \in \Gamma$ $C_{K^{-1}(K)}(ab^{-1}) = C_{K}(f(a)f(b^{-1})) = C_{K}(f(a)(f(b))^{-1})$ $C_{f^{-1}(K)}\left(ab^{-1}\right) = C_{K}\left(\alpha\beta^{-1}\right)$ Since $K \in MG(\Gamma')$ $C(\alpha\beta^{-1}) \geq \min \left\{ C(\alpha), C(\beta) \right\}$ $=\min\left\{C\left(f\left(a\right)\right),C\left(f\left(b\right)\right)\right\}$ $=\min\left\{ \sum_{f^{-1}(K)} (a), \sum_{f^{-1}(K)} (b) \right\}$ $\sum_{f^{-1}(K)} (ab^{-1}) = C_{K} (\alpha\beta^{-1}) \ge \min \left\{ C_{f^{-1}(K)}(a), C_{f^{-1}(K)}(b) \right\}$ Thus $f^{-1}(K) \in MG(\Gamma)$

Also when $a,b \notin Range f^{-1}(K)$, and when $a \in Range f^{-1}(K)$ but $b \notin Range f^{-1}(K)$ are trivial cases.

Proof (3): let A be abelian multi-group over Γ (i.e. $A \in AMG(\Gamma)$) and let $f: \Gamma \to \Gamma'$ be a group homomorphism from the group Γ onto the group Γ' . Now by part (1), $f(A) \in MG(\Gamma')$ and so it remains to show that $f(A) \in AMG(\Gamma')$ assume that $\alpha, \beta \in \Gamma'$ and from our assumption that f is onto, we can find at least $\alpha \in \Gamma$ for which $f(\alpha) = \alpha$, since for every $t \in \Gamma'$ we have:

$$C_{f(A)}(t) = \max_{\substack{f(\tau)=t\\\tau\in\Gamma}} \left\{ C_A(\tau) \right\}$$

put $t = \alpha \beta \alpha^{-1} \in \Gamma'$

 $C_{f(A)}(\alpha\beta\alpha^{-1}) = \max_{f(\tau)=\alpha\beta\alpha^{-1}}\left\{C_{A}(\tau)\right\}$

Since A is abelian multi-group over Γ and by theorem 3.15,

And since
$$f$$
 is a homomorphism
 $f(\tau) = f(a^{-1}\tau a) = f(a^{-1})f(\tau)f(a)$
 $f(\tau) = f(a^{-1}\tau a) = f(a^{-1})f(\tau)f(a)$
 $f(\tau) = \alpha^{-1}(\alpha\beta\alpha^{-1})\alpha$
 $f(\tau) = \beta$
Thus we can write:
 $C_{f(A)}(\alpha\beta\alpha^{-1}) = \max_{\substack{\tau \in \Gamma \\ f(a^{-1}\tau a) = \beta}} \left\{ C_A(a^{-1}\tau a) \right\}$
 $C_{f(A)}(\alpha\beta\alpha^{-1}) = \max_{\substack{\sigma \in \Gamma \\ f(\sigma) = \beta}} \left\{ C_A(\sigma) \right\}$
 $C_{f(A)}(\alpha\beta\alpha^{-1}) = C_{f(A)}(\beta)$
That is:
 $C_{f(A)}(\alpha\beta\alpha^{-1}) = C_{f(A)}(\beta)$
and by using (theorem 3.15), we get $f(A)$ is above

elian multi-group over Γ (i.e. $f(A) \in AMG(\Gamma')$) **Proof** (4): to prove this statement, assume that $K \in AMG(\Gamma')$ where $f: \Gamma \to \Gamma'$ be а homomorphism from Γ into the Γ' . To show that $f^{-1}(K)$ is again abelian multi-group over Γ , let us take any two arbitrary elements a,b in Γ thus $C_{f^{-1}(K)}(ab) = C_{K}(f(ab))$ Since f is a homomorphism f(ab) = f(a)f(b)Then $\underset{f = 1/K}{C} (ab) = \underset{K}{C} (f(a)f(b))$ Since K is abelian multi-group over Γ' $C_{K}(f(a)f(b)) = C_{K}(f(b)f(a)), \text{ Then}$ $C_{K}(ab) = C(f(b)f(a))$

$$C_{f^{-1}(K)}(ab) = C_{K}(f(b)f(a))$$

$$C_{f^{-1}(K)}(ab) = C_{K}(f(ba))$$

$$C_{f^{-1}(K)}(ab) = C_{f^{-1}(K)}(ba)$$

Which means that $f^{-1}(\mathbf{K})$ is an abelian multi-group over Γ which completes the proof of the theorem.

Volume 6 Issue 5, May 2017

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DOI: 10.21275/ART20173240

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4. Conclusion

In this note we drive the definition of a multi-group, and its properties the necessary and sufficient conditions for a multiset to be a multi-group. The concept of abelian multisets over a given group, and abelian multi-groups over a given group, were discussed, and homomorphic image of multisets and multi-group, also between abelian multisets and abelian multi-groups were discussed. The same thing for pre-homomorphic image was also discussed.

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