Algebraic Structures on Multi Groups I

S.F. El-Hadidi

10 Talal Basha St., San Stefano Alex., Egypt
Helwan University, Cairo, Egypt
Email: sameh2002[at]yahoo.com

Abstract: ***A simple definition of multi-group was derived it helps us to find some properties of multi-groups. Every homomorphic image of a multi-group is a multi-group. Every pre homomorphic image of a multi-group is also a multi-group. Necessary and sufficient conditions for a multiset to be a multi-group were obtained.***

Keywords: multi-set, multigroup

1. Introduction

In classical set theory we know that each element in a set appears one time in a set, that is no copies or duplicates for any object in the set, but in multi sets (or bags) which are set-like structure, but in which an object can appears more than one time, that is the element may have several copies in the multiset (m set) for example $A=\{1,1,1,1,5,5,5,7,7\}$ it is a multiset, and A can be written as:

$A=\{4/1,3/5,2/7\}=\{1,5,7\}_{4,3,2}$ the multiplicity of 1 in A is 4, multiplicity of 5 in A is 3, the multiplicity of 7 in A is 2 for this we write $C_A(1)=4$, $C_A(5)=3$, $C_A(7)=2$.

In general, we can say that the number of occurrences of an element b in the multiset B which is finite and unique is called the multiplicity of b in B, and is denoted by $C_B(b)$ and when $C_B(b)=\text{n}$ we may write $b\in^n B$.

Now let Γ be a set, υ be any positive integer, the family of all multisets over Γ (whose elements taken from Γ) such that the occurrence of each of which not exceeds υ is denoted by $[\Gamma]^{\upsilon}$. We use $[\Gamma]^{\infty}$ if there is no limit of this occurrence.

Definition1.1: "sub multiset":

Assume that A,B are two multisets over the same set Γ then A is said to be sub multiset of B if $C_A(\gamma)\le C_B(\gamma)$ forevery $\gamma\in\Gamma$ and then we write $A\subseteq B$.

The equality holds when the both relations $A\subseteq B, B\subseteq A$ are satisfied and in this case we can write $A=B$.

Definition1.2:"Union":

Let I be an index set and consider the family of multisets $\{A_i : i\in I\}$ over a set Γ then we define their union $\underset{i\in I}{U} A_i$ such that $C_{\underset{i\in I}{UAi}}(\gamma)=\max_{i\in I} C_{Ai}(\gamma)$ for every $\gamma\in\Gamma$

Definition1.3: "Intersection"

We can define the intersection for the family $\{A_i : i\in I\}$ by $\bigcap_{i\in I} A_i$ such that $C_{i\in I}(\gamma)=\min_{i\in I} C_{Ai}(\gamma)$ for every $\gamma\in\Gamma\bigcap_{i\in I} A_i$.

Definition1.4: Complement

For each A_i in $[\Gamma]^{\nu}$ we can define its complement in $[\Gamma]^{\upsilon}$ by A_i^c such that for every $\gamma\in\Gamma$ we have $C_{A_i^c}(\gamma)=\nu-C_{A_i}(\gamma)$.

Definition1.5:
Assume that Γ and Γ' are two non-empty sets and let $f:\Gamma\to\Gamma'$ be a mapping. If A is any multiset in $[\Gamma]^{\upsilon}$ then its image by f is defined as $f(A)$ such that $C_{f(A)}(\gamma')=\max_{f(\gamma)=\gamma'} C_A(\gamma)$ whatever $f^{-1}(\gamma')\ne\phi$ and $C_{f(A)}(\gamma')=0$ whatever $f^{-1}(\gamma')=\phi$.

Definition1.6:
let $f:\Gamma\to\Gamma'$ be a mapping Γ,Γ' are two non-empty sets and let $B\in[\Gamma']^{\upsilon}$ then the inverse image of $B, f^{-1}(B)$ is such that $C_{f^{-1}(B)}(\gamma)=C_B(f(\gamma))$.

2. Multisets over a group

Now let us consider Γ to be a group with identity e, and as before the class of all multisets over Γ, such that the multiplicity does not exceeds an integer υ is denoted by $[\Gamma]^{\upsilon}$.

Definition2.1:

Let H,K be any two members of the class $[\Gamma]^{\upsilon}$ let us define HoK to be such that

$$C_{HoK}(\alpha)=\max_{hk=\alpha}\left\{\min_{h,k\in\Gamma}\left\{C_H(h),C_K(k)\right\}\right\}$$

……………………………………………………… (2.1)

Definition2.2:

For every H in $[\Gamma]^{\upsilon}$ we define H^{-1} as follows $C_H\left(h^{-1}\right)=C_{H^{-1}}(h)$ for every $h\in\Gamma$.

Theorem2.3:

let H,K be any two multisets in $[\Gamma]^{\upsilon}$ then

$$C_{HoK}(u)=\max_{v\in\Gamma}\left\{\min\left\{C_H\left(uv^{-1}\right),C_K(v)\right\}\right\}$$

$$=\max_{v\in\Gamma}\left\{\min\left\{C_H(v),C_K\left(v^{-1}u\right)\right\}\right\}$$

Proof:

Since from the definition of HoK we have:

$$C_{HoK}(u)=\max_{hv=u}\left\{\min_{h,k\in\Gamma}\left\{C_H(h),C_K(v)\right\}\right\}.$$

But now Γ is a group which means that for every u,h in Γ, we can assign one and only element v such that $hv=u$, hence we can write:

$$C_{HoK}(u)=\max_{v\in\Gamma}\left\{\min\left\{C_H\left(uv^{-1}\right),C_K(v)\right\}\right\}$$

$$=\max_{v\in\Gamma}\left\{\min\left\{C_H(v),C_K\left(v^{-1}u\right)\right\}\right\}$$

Theorem2.4:

For every H in Γ^{υ} we have $\left(H^{-1}\right)^{-1}=H$.

Proof:

Let h be any element in Γ, from the definition $C_{H^{-1}}(h)=C_H\left(h^{-1}\right)$, then we deduce that: $C_{\left(H^{-1}\right)^{-1}}(h)=C_{H^{-1}}\left(h^{-1}\right)=C_H(h)$, and hence our required result follows.

Theorem2.5:

For every H,K in $[\Gamma]^{\upsilon}$ we have $(HoK)^{-1}=K^{-1}oH^{-1}$.

Proof:

Since for every $u\in\Gamma$, $C_{HoK}(u)=C_{HoK}\left(u^{-1}\right)$

But,

$$C_{HoK}\left(u^{-1}\right)=\max_{hk=u^{-1}}\left\{\min_{h,k\in\Gamma}\left\{C_H(h),C_K(k)\right\}\right\}.$$

Now since Γ is a group, and when h,k runs over all elements in Γ, then so does h^{-1},k^{-1} and recall the fact $C_H\left(h^{-1}\right)=C_{H^{-1}}(h)$ and $C_{K^{-1}}(k)=C_K\left(k^{-1}\right)$ therefore we can write:

$$C_{HoK}\left(u^{-1}\right)=\max_{(hk)^{-1}=u}\left\{\min_{h^{-1},k^{-1}\in\Gamma}\left\{C_{H^{-1}}\left(h^{-1}\right),C_{K^{-1}}\left(k^{-1}\right)\right\}\right\}$$

$$=\max_{k^{-1}h^{-1}=u}\left\{\min_{k^{-1},h^{-1}\in\Gamma}\left\{C_{K^{-1}}\left(k^{-1}\right),C_{H^{-1}}\left(h^{-1}\right)\right\}\right\}$$

$C_{HoK}\left(u^{-1}\right)=C_{H^{-1}oK^{-1}}(u)$, but $C_{(HoK)^{-1}}(u)=C_{HoK}\left(u^{-1}\right)$, and hence $(HoK)^{-1}=K^{-1}oH^{-1}$.

Theorem 2.6:

The operation "o" on the multisets of $[\Gamma]^{\upsilon}$ is associative that is $(HoK)oL=Ho(KoL)$

Proof:

Let $HoK=D$, therefore $(HoK)oL=DoL$,

$$C_{(HoK)oL}(u)=C_{DoL}(u)=\max_{vl=u}\left\{\min_{v,l\in\Gamma}\left\{C_D(v),C_L(l)\right\}\right\}$$

, but

$$C_D(v)=\max_{hk=v}\left\{\min_{h,k\in\Gamma}\left\{C_H(h),C_K(k)\right\}\right\},$$

$$C_{(HoK)oL}(u)=\max_{vl=u}\left\{\min\left\{\max_{hk=v}\left\{C_H(h),C_K(k)\right\},C_L(l)\right\}\right\}=$$

$$\max_{(hk)l=u}\left\{\min\left\{C_H(h),C_K(k),C_L(l)\right\}\right\},$$

since Γ is a group, then $(hk)l=h(kl)\forall h,k$ and $l\in\Gamma$,

$$C_{(HoK)oL}(u)=\max_{\substack{h(w)=u\\ w=kl}}\left\{\min\left\{C_H(h),C_K(k),C_L(l)\right\}\right\}$$

$$=\max_{hw=u}\left\{\min\left\{C_H(h),\max_{kl=w}\left\{\min C_K(k),C_L(l)\right\}\right\}\right\}$$

$$=\max_{hw=u}\left\{\min\left\{C_H(h),C_{KoL}(v)\right\}\right\}$$

$$=C_{Ho(KoL)}(v).$$

3. Multi Groups and Its Properties

Definition 3.1: A multiset H over a group Γ is a multi-group if for every a,b in Γ, we have:

$$C_H(ab^{-1})\geq\min\left\{C_H(a),C_H(b)\right\}$$
…………………………………………………………………… (3.1)

By $MG(\Gamma)$ we mean the set of all multi-groups over Γ.

Theorem 3.2: let H be any multi-group over Γ, then for every $a\in\Gamma$ we have:

(1) The multiplicity of any element in H not exceeds the multiplicity of the identity element, that is $C_H(a)\leq C_H(e)\ \forall\ a\in\Gamma$

(2) The element and its inverse have the same multiplicity that is $C_H(a^{-1})=C_H(a)$ for every $a\in\Gamma$.

Proof:

(1) Since $H\in MG(\Gamma)$ then for every $a,b\in\Gamma$ we have:

$$C_H(ab^{-1})\geq\min\left\{C_H(a),C_H(b)\right\}$$

Put $a=b$ we get:

$$C_H(aa^{-1})=C_H(e)\geq\min\left\{C_H(a),C_H(a)\right\}=C_H(a)$$

, for every $a\in\Gamma$.

(2) $C_H(a^{-1})=C(ea^{-1})\geq\min\left\{C_H(e),C_H(a)\right\}=C_H(a)$

, thus

$$C_H(a^{-1})\geq C_H(a)$$
……………………………………………………………… (i)

also

$$C_H(a)=C_H(ea)\geq\min\left\{C_H(e),C_H(a^{-1})\right\}=C_H(a^{-1})$$

$$C_H(a)\geq C_H(a^{-1})$$
……………………………………………………………… (ii)

from (ii), (ii) we get $C_H(a^{-1})=C_H(a)$

Lemma 3.3: for every $H\in MG(\Gamma)$ we have

(1) $C_H(a^m)\geq C_H(a)$ for every $a\in\Gamma$.

(2) $H^{-1}=H$.

Proof (1): since a^m can be expressed as $a^{m-1}a$

$$C_H(a^m)=C_H(a^{m-1}a)\geq\min\left\{C_H(a^{m-1}),C_H(a^{-1})\right\}$$

And by using theorem A part (2) we can write

$$C_H(a^m)\geq\min\left\{C_H(a^{m-1}),C_H(a)\right\}$$

Appling the same procedure again, and so on

$$C_H(a^m)\geq\min\left\{C_H(a^{m-2}.a),C_H(a)\right\}$$

$$C_H(a^m)\geq\min\left\{C_H(a^{m-2}),C_H(a^{-1}),C_H(a)\right\}$$

$$C_H(a^m)\geq\min\{\underbrace{C_H(a),C_H(a),\ldots,C_H(a)}_{m\text{-times}}\}=C_H(a)$$

, and hence our required result follows.

(2) From the definition (2.2), we have

$C_{H^{-1}}(a)=C_H(a^{-1})$ ………….………………………… (iii)

and by using theorem (3.2) part (2) we get $C_H(a^{-1})=C_H(a)$ and from (iii) our result follows.

Theorem 3.4: let H be any multiset over a group Γ then H is a multi-group over Γ if and only if

(1) $C_H(ab)\geq\min\left\{C_H(a),C_H(b)\right\}$ for every $a,b\in\Gamma$, and

(2) $C_H(a)=C_H(a^{-1})$ for every $a\in\Gamma$.

Proof: (i) "Necessity": let $H \in MG(\Gamma)$ (i.e. H is a multi-group over the group Γ) then for every $a,b \in \Gamma$ we have:

$$C_H(ab^{-1}) \geq \min\{C_H(a), C_H(b)\}$$

$C_H(ab) \geq \min\{C_H(a), C_H(b^{-1})\}$, and from theorem (3.1) part (2), $C_H(b^{-1}) = C_H(b)$ then

$C_H(ab) \geq \min\{C_H(a), C_H(b)\}$ now

$C_H(a) = C_H(a^{-1})$ follows directly from theorem (3.2) part (2)

(ii) "Sufficiency": let $H \in MS(\Gamma)$ (i.e. H is a multiset over the group Γ) and the two conditions are satisfied that is:

(1) $C_H(ab) \geq \min\{C_H(a), C_H(b)\}$ for every $a,b \in \Gamma$

(2) $C_H(a) = C_H(a^{-1})$ for every $a \in \Gamma$

Now

$C_H(ab^{-1}) \geq \min\{C_H(a), C_H(b^{-1})\}$ and since

$C_H(a) = C_H(a^{-1})$ from (2) then

$C_H(ab^{-1}) \geq \min\{C_H(a), C_H(b)\}$ then the theorem is proved

Definition 3.5: let H be multi-group over the group Γ, $C_H(e) = \zeta$

(1) $H_\upsilon = \{a \in \Gamma : C_H(a) \geq \upsilon, \upsilon, \in Z^+\}$

where Z^+ is the set of all positive integers

(2) $H_{\zeta(e)} = \{a \in \Gamma : C_H(a) = \zeta = C_H(e)\}$

(3) $H_0 = \{a \in \Gamma : C_H(a) \geq 0\}$

Theorem 3.6: let H be multi-group over a group Γ, then each of $H_0, \mathrm{H}_{\zeta(e)}$ and $H_\upsilon, \upsilon \in Z^+$ are sub groups of the group Γ

Proof: (1) to show that H_0 is a sub group of Γ, let $a,b \in H_0$ this mean that $C_H(a) > 0$, $C_H(b) > 0$ thus from the relation $C_H(ab^{-1}) \geq \min\{C_H(a), C_H(b)\} > 0$ we deduce that $C_H(ab^{-1}) > 0 \Rightarrow ab^{-1} \in H_0$ hence H_0 is a sub group of Γ.

(2) To show that $\mathrm{H}_{\zeta(e)}$ is a sub group where $\zeta = C_H(e)$, let $a,b \in \mathrm{H}_{\zeta(e)}$, then

$C_H(a) = C_H(e) = C_H(b) = \zeta$

$C_H(ab^{-1}) = \min\{C_H(a), C_H(b^{-1})\}$

$C_H(ab^{-1}) = \min\{C_H(a), C_H(b)\}$

(because $C_H(b^{-1}) = C_H(b)$ by theorem 3.2 (2))

$C_H(ab^{-1}) = \min\{\zeta, \zeta\} = \zeta = C_H(e)$ thus $ab^{-1} \in \mathrm{H}_{\zeta(e)}$ and so $\mathrm{H}_{\zeta(e)}$ is a sub group of Γ.

(3) let $a,b \in H_\upsilon$ then we can write $C_H(a) = n \geq \upsilon$, $C_H(b) = m \geq \upsilon$

$C_H(ab^{-1}) \geq \min\{C_H(a), C_H(b^{-1})\}$

$C_H(ab^{-1}) \geq \min\{C_H(a), C_H(b)\}$

$C_H(ab^{-1}) \geq \min\{n, m\} = k \geq \upsilon$ and so $C_H(ab^{-1}) \geq \upsilon$ and hence $ab^{-1} \in H_\upsilon$ which implies that H_υ is a sub group of Γ.

Theorem 3.7: A multiset H over group Γ is a multi-group over Γ, if and only if $HoH = H$ and $H = H^{-1}$.

Proof: (1) "Necessity" assume that H is a multi-group over Γ, take $a,b \in \Gamma$ by theorem B(1) we can write:

$C_H(ab) \geq \min\{C_H(a), C_H(b)\}$ for every $a,b \in \Gamma$.

And we get:

$C_H(h) \geq \min_{\substack{a,b \in \Gamma \\ ab=h}}\{C_H(a), C_H(b)\}$ that is

$C_H(h) \geq \max_{\substack{a,b \in \Gamma \\ ab=h}}\{\min\{C_H(a), C_H(b)\}\}$ and since

$C_{HoH}(h)=\max_{\substack{a,b\in\Gamma \\ ab=h}}\left\{\min\left\{C_H(a),C_H(b)\right\}\right\}$ therefore

$C_H(h)\geq C_{HoH}(h)$ which means that

$HoH\subset H\to(*)$

Now by using theorem2.3

$C_H(h)=C_H(ha^{-1}a)$

$C_H(h)\geq\min\left\{C_H(ha^{-1}),C_H(a)\right\}$

$C_H(h)\geq\max_{a\in\Gamma}\left\{\min\left\{C_H(ha^{-1}),C_H(a)\right\}\right\}=C_{HoH}(h)$

$C_H(h)\geq C_{HoH}(h)$

$H\supset HoH\to(**)$

From (*) and (**) we get $HoH=H$

Now since

$C_{H^{-1}}(h)=C_H(h^{-1})\forall h\in\Gamma$ and since H is a multi-group over a group Γ, then by theorem 3.2 part (2)

$C_H(h^{-1})=C_H(h)\forall h\in\Gamma$. Thus

$C_{H^{-1}}(h)=C_H(h^{-1})=C_H(h)$ which implies that

$H=H^{-1}$ as required.

(2) Sufficiency: assume that $HoH=H$ and $H=H^{-1}$ where H is a multiset over Γ (i.e. $H\in MS(\Gamma)$), by theorem 2.3

$C_H(ab^{-1})=C_{HoH}(ab^{-1})=\max_{h\in\Gamma}\left\{\min\left\{C_H(h),C_H(h^{-1}ab^{-1})\right\}\right\}$

put $h=a$ and so

$C_H(ab^{-1})=C_{HoH}(ab^{-1})\geq\min\left\{C_H(a),C_H(b^{-1})\right\}=\min\left\{C_H(a),C_H(b)\right\}$

[Since $H=H^{-1}$, $C_H(b)=C_{H^{-1}}(b)=C_H(b^{-1})$],

thus

$C_H(ab^{-1})\geq\min\left\{C_H(a),C_H(b)\right\}$

$\therefore H\in MG(\Gamma)$.

Theorem 3.8: let H be a multiset over the group Γ, then H is a multi-group over Γ if and only if $HoH^{-1}=H$.

Proof: **"necessity"**: let $H\in MG(\Gamma)$ then by theorem 3.7 we have $HoH=H$ and $H^{-1}=H$ so we can write $HoH^{-1}=H$.

"sufficiency": let $HoH^{-1}=H$ this means that $C_{HoH^{-1}}(a)=C_H(a)$ for every $a\in\Gamma$ therefore we can write using

$C_H(ab^{-1})=C_{HoH^{-1}}(ab^{-1})=\max_{h\in\Gamma}\left\{\min\left\{C_H(ab^{-1}h),C_{H^{-1}}(h^{-1})\right\}\right\}$

and hence

$C_H(ab^{-1})\geq\min\left\{C_H(ab^{-1}h),C_{H^{-1}}(h^{-1})\right\}$

[Since from: $C_{H^{-1}}(a)=C_H(a^{-1})$ i.e. definition (2.2)], then

$C_H(ab^{-1})\geq\min_{h\in\Gamma}\left\{C_H(ab^{-1}h),C_H(h)\right\}$, for $h=b$ we get

$C_H(ab^{-1})\geq\min\left\{C_H(a),C_H(b)\right\}$

And so $H\in MG(\Gamma)$ and hence the theorem is proved.

Theorem 3.9: let H,K be two multi-groups over the same group Γ then $HoK=KoH$ if and only if HoK is a multi-group over Γ.

Proof "Necessity":

Let $HoK=KoH$

Also assume that $KoH=D$

$D^{-1}=(HoK)$ (by theorem 2.5, that is

$(HoK)^{-1}=K^{-1}oH^{-1}$

$D^{-1}=K^{-1}oH^{-1}$

Since $H,K\in MG(\Gamma)$ then by lemma 3.3 (2), we have $H^{-1}=H,K^{-1}=K$

$D^{-1}=K^{-1}oH^{-1}=KoH=D$ and so

$D=D^{-1}$

…………..……………………………………………………

…………………………………………………..………

(1)

$DoD^{-1}=DoD=(HoK)o(HoK)$

$DoD^{-1}=Ho(KoH)oK=Ho(HoK)oK=(HoH)o(KoK)$

$DoD^{-1}=(HoH)o(KoK)$

But since $H,K\in MG(\Gamma)$

$HoH=H,KoK=K$ (by theorem 3.7), thus

$DoD^{-1}=HoK=D$

$DoD^{-1}=D$,

$D^{-1}=D$

thus by theorem 3.7 again we get $D \in MG(\Gamma)$ i.e. $HoK \in MG(\Gamma)$.

(2)"Sufficiency": let $HoK \in MG(\Gamma)$ where $H, K \in MG(\Gamma)$ which implies that $H^{-1} = H$ and also $K^{-1} = K$

Put $HoK = D \in MG(\Gamma)$

$D^{-1} = D$ implies that

$HoK = (HoK)^{-1} = K^{-1}oH^{-1} = KoH$

i.e. $HoK = KoH$ and hence the theorem is proved.

Theorem 3.10: the arbitrary intersection of any family of multi-groups over Γ is again a multi-group over Γ.

Proof: let $\{H_i : i \in I\}$ be a family over Γ where $H_i \in MG(\Gamma)$ for every $i \in I$ and so we have:

$$C_{H_i}(ab^{-1}) = \min\{C_{H_i}(a), C_{H_i}(b)\}, \forall a,b \in \Gamma$$

$$C_{\bigcap_{i\in I} H_i}(ab^{-1}) = \min_{i\in I}\{C_{H_i}(ab^{-1})\}$$

$$C_{\bigcap_{i\in I} H_i}(ab^{-1}) \geq \min_{i\in I}\{\min\{C_{H_i}(a), C_{H_i}(b)\}\}$$

$$C_{\bigcap_{i\in I} H_i}(ab^{-1}) = \min\{\min_{i\in I} C_{H_i}(a), \min_{i\in I} C_{H_i}(b)\}$$

$$C_{\bigcap_{i\in I} H_i}(ab^{-1}) = \min\{C_{\bigcap_{i\in I} H_i}(a), C_{\bigcap_{i\in I} H_i}(b)\}$$

Therefore:

$$C_{\bigcap_{i\in I} H_i}(ab^{-1}) \geq \min\{C_{\bigcap_{i\in I} H_i}(a), C_{\bigcap_{i\in I} H_i}(b)\}$$

Thus

$$\bigcap_{i\in I} H_i \in MG(\Gamma)$$

Remark 3.11: it must be noted that the previous theorem need not be satisfied in the case of union whatever finite or infinite families of multi-groups.

Definition 3.12 "Sub multi-group":

Let $H, K \in MG(\Gamma)$ such that $K \subseteq H$ then we say that K is a sub multi-group of H.

Definition 3.13 "Abelian multiset over Γ":

Let $H \in MS(\Gamma)$ we say that H is abelian multiset over Γ if $C_H(ab) = C_H(ba)$ fore very $a,b \in \Gamma$ and we write $H \in AMS(\Gamma)$.

Definition 3.14 "Abelian multi-group over Γ":

Let $H \in MS(\Gamma)$ then H is said to be abelian over Γ if $H \in AMS(\Gamma)$ that if $C_H(ab) = C_H(ba)$ fore very $a,b \in \Gamma$.

Theorem 3.15: the following statements are equivalent:

(1) $H \in AMS(\Gamma)$

(2) $C_H(a^{-1}ba) = C_H(aba^{-1}) = C_H(b)$

(3) $HoK = KoH$ holds true for every $K \in MS(\Gamma)$

Proof: the proof is carried as follows

$(1) \Rightarrow (2), (2) \Rightarrow (1), (1) \Rightarrow (3), (3) \Rightarrow (1)$

Now $(1) \Rightarrow (2)$ let $H \in AMS(\Gamma)$ then $\forall a,b \in \Gamma$ we have

$$C_H(ab) = C_H(ba)$$

Let $ba = t$

$$C_H(a^{-1}ba) = C_H(a^{-1}t) = C_H(ta^{-1})$$

$$C_H(baa^{-1}) = C_H(b)$$

That is

$$C_H(a^{-1}ba) = C_H(b)$$

………………………………………………………………... (1)

Also let $ab = r$

$$C_H(aba^{-1}) = C_H(ra^{-1}) = C_H(a^{-1}r)$$

$$C_H(a^{-1}ab) = C_H(b)$$

$$C_H(aba^{-1}) = C_H(b)$$

………………………………………………………………… (2)

From $(1),(2)$ we get

$$C_H(aba^{-1}) = C_H(a^{-1}ba) = C_H(b)$$

Now $(2) \Rightarrow (1)$: let

$$C_H(aba^{-1}) = C_H(a^{-1}ba) = C_H(b)$$

Let $ba = t$ then $b = ta^{-1}$ and then $ab = ata^{-1}$

$$C_H(ab) = C_H(ata^{-1}) = C_H(t) = C_H(ba)$$

And hence $H \in AMS(\Gamma)$

Now $(1) \Rightarrow (3)$: let $H \in AMS(\Gamma)$ and $K \in MS(\Gamma)$, then let us take any arbitrary element a in Γ

$$C_{HoK}(a) = \max_{b\in\Gamma}\{\min\{C_H(ab^{-1}), C_K(b)\}\}$$

Since $C_H(rt) = C_H(tr) \forall r,t \in \Gamma$

$$C_{HoK}(a) = \max_{b\in\Gamma}\left\{\min\left\{C_H(b^{-1}a), C_K(b)\right\}\right\}$$

$$C_{HoK}(a) = \max_{b\in\Gamma}\left\{\min\left\{C_K(b), C_H(b^{-1}a)\right\}\right\}$$

$$C_{HoK}(a) = C_{KoH}(a)$$

Therefore $HoK = KoH$

And so $(1) \Rightarrow (3)$ follows.

Now $(3) \Rightarrow (1)$ let $H \in MS(\Gamma)$ such that $HoK = KoH$ holds true for every K in $MS(\Gamma)$.

For any arbitrary two elements a,b in Γ let us define an integers $\upsilon, \upsilon \geq 0$ where $\upsilon = \min\left\{C_H(ab), C_H(ba)\right\}$ and then we can construct a singleton multiset $Q = \left\{\upsilon / b^{-1}\right\}$ which contains the element b^{-1} represented υ times, it is clear that $Q \in MS(\Gamma)$, then from our assumption, $HoK = KoH \forall K \in MS$, and hence $HoQ = QoH$ that is $C_{HoQ}(a) = C_{QoH}(a)\ \forall a \in \Gamma$

$$C_{HoQ}(a) = \max_{hq=a}\left\{\min\left\{C_H(h), C_Q(q)\right\}\right\},$$ using theorem 2.3

$$C_{HoQ}(a) = \max_{b\in\Gamma}\left\{\min\left\{C_H(ab), C_Q(b^{-1})\right\}\right\}$$

$$C_{HoQ}(a) = \max_{b\in\Gamma}\left\{\min\left\{C_H(ab), \upsilon\right\}\right\} = C_H(ab)$$

$$C_{HoQ}(a) = C_H(ab)$$

…………………………………………………………………… (3)

Now

$$C_{HoQ}(a) = \max_{qh=a}\left\{\min\left\{C_Q(q), C_H(h)\right\}\right\}$$

$$C_{HoQ}(a) = \max_{b\in\Gamma}\left\{\min\left\{C_Q(b^{-1}), C_H(ba)\right\}\right\}$$

$$C_{HoQ}(a) = \max_{b\in\Gamma}\left\{\min\left\{\upsilon, C_H(ba)\right\}\right\} = C_H(ba)$$

$$C_{HoQ}(a) = C_H(ba)$$

……………………………………………………………………

…………………………………………………… (4)

From (3) and (4), since $HoQ = QoH$ it follows that $C_H(ab) = C_H(ba)$, i.e. $H \in AMS(\Gamma)$ which completes the proof the theorem.

Theorem 3.16:

Let $f : \Gamma \to \Gamma'$ be a homomorphism from Γ into Γ' then the following statements are satisfied:

(1) every homomorphic image of a multi-group over Γ is a multi-group over Γ'.

(2) every pre homomorphic image of a multi-group over Γ' is a multi-group over Γ.

(3) every homomorphic image of an abelian multi-group over Γ is an abelian multi-group over Γ'.

(4) every pre homomorphic image of an abelian multi-group over Γ' is an abelian multi-group over Γ.

Proof (1): let $f : \Gamma \to \Gamma'$ be a homomorphism from Γ into Γ' and let $H \in MG(\Gamma)$.

Let us take any two arbitrary elements α, β consider the following three cases:

(a) α, β (both of them) in Rang (f), (i.e. $\alpha, \beta \in f(\Gamma)$).

(b) α, β not in Rang (f), (i.e. $\alpha, \beta \notin f(\Gamma)$).

(c) one of them in the Rang (f). Now consider first case (a), (i.e. $\alpha, \beta \in f(\Gamma)$), then we can find $a,b \in \Gamma$ such that $f(a) = \alpha, f(b) = \beta$ and so we have:

$$C_{f(H)}(\alpha\beta^{-1}) = \max_{\gamma\in\Gamma}\left\{C_H(\gamma), \mathrm{f}(\gamma) = \alpha\beta^{-1}\right\}$$

Now since $f : \Gamma \to \Gamma'$ is a homomorphism

$$f(ab^{-1}) = f(a) f(b^{-1}) = f(a).\mathrm{f(b)}^{-1} = \alpha\beta^{-1}$$

and since $f(\gamma) = \alpha\beta^{-1}$

then we can write

$$C_{f(H)}(\alpha\beta^{-1}) = \max\left\{C_H(ab^{-1}), \mathrm{f}(ab^{-1}) = \alpha\beta^{-1}\right\}$$

, since $H \in MG(\Gamma)$

$$C_{f(H)}(\alpha\beta^{-1}) \geq \max_{\substack{f(a)=\alpha \\ f(b)=\beta}}\left\{\min\left\{C_H(a), C_H(b)\right\}\right\}$$

$$= \min\left\{\max_{f(a)=\alpha}\left\{C_H(a), \max_{f(b)=\beta}\left\{C_H(b)\right\}\right\}\right\}$$

$$= \min\left\{C_{f(H)}(\alpha), C_{f(H)}(\beta)\right\}$$ thus

$$\geq \min\left\{C_{f(H)}(\alpha), C_{f(H)}(\beta)\right\}$$

Which means that $f(H) \in MG(\Gamma')$

For the last two cases $(b),(c)$ are trivial.

Proof (2): let $K \in MG(\Gamma')$ and we will show that $f^{-1}(K) \in MG(\Gamma)$, to do this, let

$a,b \in f^{-1}(K)$ (i.e. $a,b \in Range\ f'(K)$), then there exist $\alpha,\beta \in K$. Such that:

$f^{-1}(\alpha)=\alpha$ and $f^{-1}(\beta)=b$ and hence

$\alpha = f(a)$ and $\beta = f(b)$

$C_{f^{-1}(K)}(a)=C_K(f(a))=C_K(\alpha)$ also

$C_{f^{-1}(K)}(b)=C_K(f(a))=C_K(\beta)$

$C_{f^{-1}(K)}(ab^{-1})=C_K(f(ab^{-1}))$

Since f is a homomorphism, then:

$f(\text{pq})=f(p)f(q), f(p^{-1})=f(p)^{-1}$
$\forall p,q \in \Gamma$

$$C_{f^{-1}(K)}(ab^{-1})=C_K(f(a)f(b^{-1}))=C_K\left(f(a)(f(b))^{-1}\right)$$

$$C_{f^{-1}(K)}(ab^{-1})=C_K(\alpha\beta^{-1})$$

Since $K \in MG(\Gamma')$

$$C_K(\alpha\beta^{-1}) \ge \min\{C_K(\alpha),C_K(\beta)\}$$

$$=\min\{C_K(f(a)),C_K(f(b))\}$$

$$=\min\left\{C_{f^{-1}(K)}(a),C_{f^{-1}(K)}(b)\right\}$$

$$C_{f^{-1}(K)}(ab^{-1})=C_K(\alpha\beta^{-1}) \ge \min\left\{C_{f^{-1}(K)}(a),C_{f^{-1}(K)}(b)\right\}$$

Thus $f^{-1}(K) \in MG(\Gamma)$

Also when $a,b \notin Range\ f^{-1}(K)$, and when $a \in Range\ f^{-1}(K)$ but $b \notin Range\ f^{-1}(K)$ are trivial cases.

Proof (3): let A be abelian multi-group over Γ (i.e. $A \in AMG(\Gamma)$) and let $f:\Gamma \to \Gamma'$ be a group homomorphism from the group Γ onto the group Γ'.

Now by part (1), $f(A) \in MG(\Gamma')$ and so it remains to show that $f(A) \in AMG(\Gamma')$ assume that $\alpha,\beta \in \Gamma'$ and from our assumption that f is onto, we can find at least $a \in \Gamma$ for which $f(a)=\alpha$, since for every $t \in \Gamma'$ we have:

$$C_{f(A)}(t)=\max_{\substack{f(\tau)=t\\ \tau\in\Gamma}}\{C_A(\tau)\}$$

put $t=\alpha\beta\alpha^{-1} \in \Gamma'$

$$C_{f(A)}(\alpha\beta\alpha^{-1})=\max_{f(\tau)=\alpha\beta\alpha^{-1}}\{C_A(\tau)\}$$

Since A is abelian multi-group over Γ and by theorem 3.15,

$$C_A(a^{-1}\tau a)=C_A(\tau)$$

And since f is a homomorphism

$$f(\tau)=f(a^{-1}\tau a)=f(a^{-1})f(\tau)f(a)$$

$$f(\tau)=\alpha^{-1}(\alpha\beta\alpha^{-1})\alpha$$

$$f(\tau)=\beta$$

Thus we can write:

$$C_{f(A)}(\alpha\beta\alpha^{-1})=\max_{\substack{\tau\in\Gamma\\ f(a^{-1}\tau a)=\beta}}\{C_A(a^{-1}\tau a)\}$$

$$C_{f(A)}(\alpha\beta\alpha^{-1})=\max_{\substack{\sigma\in\Gamma\\ f(\sigma)=\beta}}\{C_A(\sigma)\}$$

$$C_{f(A)}(\alpha\beta\alpha^{-1})=C_{f(A)}(\beta)$$

That is:

$$C_{f(A)}(\alpha\beta\alpha^{-1})=C_{f(A)}(\beta)$$

and by using (theorem 3.15), we get $f(A)$ is abelian multi-group over Γ (i.e. $f(A) \in AMG(\Gamma')$)

Proof (4): to prove this statement, assume that $K \in AMG(\Gamma')$ where $f:\Gamma \to \Gamma'$ be a homomorphism from Γ into the Γ'.

To show that $f^{-1}(K)$ is again abelian multi-group over Γ, let us take any two arbitrary elements a,b in Γ thus

$$C_{f^{-1}(K)}(ab)=C_K(f(ab))$$

Since f is a homomorphism

$$f(ab)=f(a)f(b)$$

Then $C_{f^{-1}(K)}(ab)=C_K(f(a)f(b))$

Since K is abelian multi-group over Γ'

$C_K(f(a)f(b))=C_K(f(b)f(a))$, Then

$$C_{f^{-1}(K)}(ab)=C_K(f(b)f(a))$$

$$C_{f^{-1}(K)}(ab)=C_K(f(ba))$$

$$C_{f^{-1}(K)}(ab)=C_{f^{-1}(K)}(ba)$$

Which means that $f^{-1}(\text{K})$ is an abelian multi-group over Γ which completes the proof of the theorem.

4. Conclusion

In this note we drive the definition of a multi-group, and its properties the necessary and sufficient conditions for a multiset to be a multi-group. The concept of abelian multisets over a given group, and abelian multi-groups over a given group, were discussed, and homomorphic image of multisets and multi-group, also between abelian multisets and abelian multi-groups were discussed. The same thing for pre-homomorphic image was also discussed.

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