International Journal of Science and Research (IJSR)
ISSN (Online): 2319-7064
Index Copernicus Value (2015): 78.96 | Impact Factor (2015): 6.391

Gaussian Integer Solutions of an Infinite Elliptic Cone $5X^2 + 5Y^2 + 9Z^2 + 46XY - 34YZ - 22XZ = 0$

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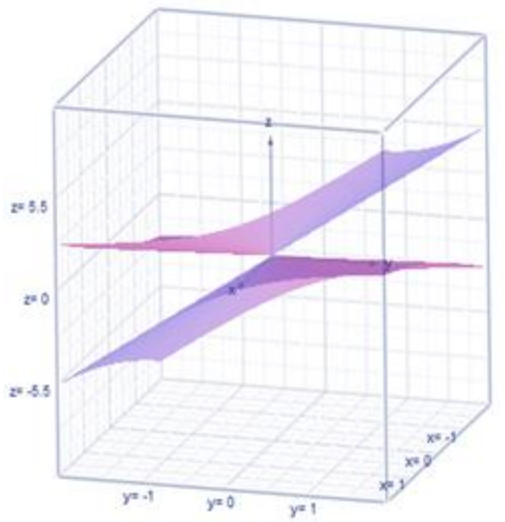
Abstract: ***The ternary quadratic Diophantine equation representing infinite elliptic cone given by $5X^2 + 5Y^2 + 9Z^2 + 46XY - 34YZ - 22XZ = 0$ is analyzed for its non- zero distinct solutions in $Z[i]$. Few different patterns of integer points satisfying the cone under consideration are obtained. Also knowing an integer solution satisfying the given cone a few triples of integers generated from the given solution are exhibited.***

Keywords: Diophantine equation, Ternary quadratic equation, Gaussian integer solution, Infinite elliptic Cone

1. Introduction

The Diophantine equation $x^4 \pm y^4 = z^2$ where x, y and z being Gaussian integers was examined by Hilbert. It was proved that there exist only trivial solutions in $\boldsymbol{Z}[\boldsymbol{i}]$. Elliptic curves have also been used in [3] to prove that the Diophantine equation $x^3 + y^3 = z^3$ has only trivial solutions in Gaussian integers. These results have motivated us to find non- zero distinct Gaussian integer solutions to a homogenous quadratic Diophantine equation in three variables given by $5X^2 + 5Y^2 + 9Z^2 + 46XY - 34YZ - 22XZ = 0$.

Pictorial representation of the equation



2. Method of Analysis

The equation to be solved is

$$5X^2 + 5Y^2 + 9Z^2 + 46XY - 34YZ - 22XZ = 0 \quad (1)$$

The different patterns of solutions to (1) are presented below:

The substitution

$$X = a + i(b + c), Y = 2a + ib, Z = 3a + ic \quad (2)$$

in (2) leads to

$$9a^2 + 7b^2 = c^2 \quad (3)$$

Pattern 1:
Consider (3) as

$$c^2 = 7b^2 + (3a)^2 \quad (4)$$

The solutions are found to be

$$c = 7m^2 + n^2$$
$$b = 2mn$$

$3a = 7m^2 - n^2$, for some $m, n \in \boldsymbol{N}$ and $m > n$.

Since our interest centers on finding Gaussian integer solutions, replace m by $3M$ and n by $2N$ in the above equation. Thus the corresponding solutions to (1) are given by

Both m and n are multiple of 3.
(i.e), Take $m = 3M, n = 3N$, for some $M, N \in \boldsymbol{N}$.

The solutions of (3) are

$$a = 21M^2 - 3N^2$$
$$b = 18MN$$
$$c = 63M^2 + 9N^2$$

In view of a, b, c , the corresponding non- zero distinct Gaussian integer solutions of (1) are

$$X = (21M^2 - 3N^2) + i(63M^2 + 9N^2 + 18MN)$$
$$Y = 2(21M^2 - 3N^2) + i(18MN)$$
$$Z = 3(21M^2 - 3N^2) + i(63M^2 + 9N^2).$$

Pattern 2:
Consider (3) as

$$c^2 - 9a^2 = 7b^2 \quad (5)$$

which can be written in the form of ratio as

$$\frac{c + 3a}{7b} = \frac{b}{c - 3a} = \frac{p}{q}, (say);\ p \neq q \neq 0$$

Expressing this as a system of simultaneous equations

$$3qa - 7pb + cq = 0 \quad (6)$$
$$3ap + bq - cp = 0 \quad (7)$$

and solving (6) and (7) we get

$$a = 7p^2 - q^2,$$
$$b = 6pq,$$
$$c = 3q^2 + 21p^2$$

In view of a, b, c, the corresponding non-zero distinct Gaussian integer solutions of (1) are found to be

$$X = (7p^2 - q^2) + i(3q^2 + 21p^2 + 6pq)$$

$Y = 2(7p^2 - q^2) + i(6pq)$
$Z = 3(7p^2 - q^2) + i(3q^2 + 21p^2)$,
for some $p, q \in \boldsymbol{N}$.

Note 1:
Equation (5) can be also written as

$$\frac{c-3a}{7b} = \frac{b}{c+3a} = \frac{p}{q}, (say);\ p \neq q \neq 0$$

Proceeding as above, we obtain the Gaussian integer solutions of (1) as
$X = (7p^2 - q^2) + i(-21p^2 - 3q^2 - 6pq)$
$Y = 2(7p^2 - q^2) + i(-6pq)$
$Z = 3(7p^2 - q^2) + i(-3q^2 - 21p^2)$, for some $p, q \in \boldsymbol{N}$.

Pattern 3:
Equation (3) can be written as

$$9a^2 + 7b^2 = c^2 * 1 \quad (8)$$

Assume

$$c = 9m^2 + 7n^2, \text{ where } m, n \in \boldsymbol{N}. \quad (9)$$

Write 1 as

$$1 = \frac{(3+i\sqrt{7})(3-i\sqrt{7})}{4^2} \quad (10)$$

Using (9) and (10) in (8) and applying the method of factorization,

$$\left(3a + ib\sqrt{7}\right) = \frac{\left(3m + i\sqrt{7}\,n\right)^2(3 + i\sqrt{7})}{4}$$

Equating real and imaginary factors, we get
$X = (9M^2 - 7N^2 + 14MN) + i(45M^2 + 21N^2 + 18MN)$
$Y = 2(9M^2 - 7N^2 + 14MN) + i(9M^2 - 7N^2 - 18MN)$
$Z = 3(9M^2 - 7N^2 + 14MN) + i(36M^2 + 28N^2)$,
here $m = 2M, n = 2N$ & $M, N \in \boldsymbol{N}$.

Note 2:
Expressing 1 as $1 = \frac{(6+i2\sqrt{7})(6-i2\sqrt{7})}{8^2}$
Proceeding as above, we obtain the solutions of (1) to be
$X = (9M^2 - 7N^2 - 14MN) + i(45M^2 + 21N^2 + 18MN)$
$Y = 2(9M^2 - 7N^2 - 14MN) + i(9M^2 - 7N^2 + 18MN)$
$Z = 3(9M^2 - 7N^2 - 14MN) + i(36M^2 + 28N^2)$,
here $m = 2M, n = 2N$& $M, N \in \boldsymbol{N}$

Pattern 4:
Applying the linear transformation

$$a = u + 7v, b = u - 9v, c = 4w$$

in (3) leads to

$$u^2 + 63v^2 = w^2$$

The solutions are found to be
$u = 63m^2 - n^2, v = 2mn, w = 63m^2 + n^2$, for some $m, n \in \boldsymbol{N}$.

In view of u, v, w , the corresponding non- zero distinct Gaussian integer solutions of (1) are
$X = (63m^2 - n^2 + 14mn) + i(315m^2 + 3n^2 - 18mn)$
$Y = 2(63m^2 - n^2 + 14mn) + i(63m^2 - n^2 - 18mn)$
$Z = 3(63m^2 - n^2 + 14mn) + i(252m^2 + 4n^2)$,
for some $m, n \in \boldsymbol{N}$.

Pattern 5:
Consider (3) as

$$9a^2 = c^2 - 7b^2 \quad (11)$$

Assume a as $a = m^2 - 7n^2$ (12)

Write 9 as $9 = p^2 - 7q^2$ (13)

In view of (12) and (13), (11) is written in the factorizable form as

$$\left(c + \sqrt{7}\,b\right)\left(c - \sqrt{7}\,b\right) = \left(p + \sqrt{7}\,q\right)\left(p - \sqrt{7}\,q\right)\left(m + \sqrt{7}\,n\right)^2\left(m - \sqrt{7}\,n\right)^2$$

Define $\left(c + \sqrt{7}\,b\right) = \left(p + \sqrt{7}\,q\right)\left(m + \sqrt{7}\,n\right)^2$
Equating rational and irrational parts, we get

$$b = q(m^2 + 7n^2) + 2mnp$$
$$c = p(m^2 + 7n^2) + 14mnq$$

To find the value of p, q we proceed as follows:
Let (p_0, q_0) be the initial solution of $p^2 - 7q^2 = 9$.

Employing the linear transformation
$p_1 = p_0 + 3h,\ q_1 = q_0 + h$ respectively, the other values of p and q satisfying the equation (13) are given by

$$p = -8p_0 + 21q_0$$
$$q = -3p_0 + 8q_0$$

In addition, we have an another set of transformation $p_1 = p_0 + 8h,\ q_1 = q_0 + 3h$ respectively, the other values of p, q satisfying the equation (13) are given by

$$p = -127p_0 + 336q_0$$
$$q = -48p_0 + 127q_0$$

A few numerical examples are presented below:

Example 1:
Algebraic expression for 9:

$$9 = \left(4 + \sqrt{7}\right)\left(4 - \sqrt{7}\right)$$

The corresponding solutions of (1) are

$$X = (m^2 - 7n^2) + i(5(m^2 + 7n^2) + 22mn)$$
$$Y = 2(m^2 - 7n^2) + i\left((m^2 + 7n^2) + 8mn\right)$$
$$Z = 3(m^2 - 7n^2) + i(4(m^2 + 7n^2) + 14mn)$$

Example 2:
Algebraic expression for 9:

$$9 = \left((-11) + (-4)\sqrt{7}\right)\left((-11) - (-4)\sqrt{7}\right)$$

The corresponding solutions of (1) are

$$X = (m^2 - 7n^2) + i(-15m^2 - 105n^2 - 78mn)$$
$$Y = 2(m^2 - 7n^2) + i(-4m^2 - 28n^2 - 22mn)$$
$$Z = 3(m^2 - 7n^2) + i(-11m^2 - 77n^2 - 56mn)$$

Example 3:
Algebraic expression for 9:

$$9 = \left((-172) + (-65)\sqrt{7}\right)\left((-172) + (-65)\sqrt{7}\right)$$

The corresponding solutions of (1) are
$X = (m^2 - 7n^2) + i(-237m^2 - 1659n^2 - 1254mn)$
$Y = 2(m^2 - 7n^2) + i(-65m^2 - 455n^2 - 344mn)$
$Z = 3(m^2 - 7n^2) + i(-172m^2 - 1204n^2 - 910mn)$

Generation of solutions:
Suppose that (a_0, b_0, c_0) represents an initial solution to (3). The following triples also satisfy (3).

Choice 1:
Let (a_1, b_1, c_1) be the second solution of (3) where

$$a_1 = a_0 + h, b_1 = b_0 + 2h,\ c_1 = c_0 + 6h \quad (14)$$

In which h is an arbitrary non- zero integer to be determined.
Substituting (14) in (3), we get

$$h = 12c_0 - 28b_0 - 18a_0.$$

Hence the matrix representation of the solution (a_1, b_1, c_1) is,

$$(a_1\ b_1\ c_1)^T = M^1(a_0 \quad b_0 \quad c_0)^T$$

where $M^1 = \begin{pmatrix} -7 & -28 & 12 \\ -36 & -55 & 24 \\ -108 & -168 & 73 \end{pmatrix}$

Proceeding in a similar manner, the general solution $(a_{n+1},\ b_{n+1},\ c_{n+1})$ in the matrix form as given by $(a_{n+1} \quad b_{n+1} \quad c_{n+1})^T = M^{n+1}(a_0 \quad b_0 \quad c_0)^T$, T denotes the transpose of the matrix,

with $M^{2n-1} = \begin{pmatrix} -7 & -28 & 12 \\ -36 & -55 & 24 \\ -108 & -168 & 73 \end{pmatrix}$ and $M^{2n} = I_3$, for every $n \in \boldsymbol{N}$.

Choice 2: The second solution (a_1, b_1, c_1) of (3) is taken as

$a_1 = a_0$
$b_1 = b_0 + 3h$
$c_1 = c_0 + 8h$

Substituting in (3), we get $h = 42b_0 - 16c_0$.

To get a sequence of solutions, consider the matrix representation of solution (b_1, c_1)

$$\text{(i.e.)}. \begin{pmatrix} b_1 \\ c_1 \end{pmatrix} = A \begin{pmatrix} b_0 \\ c_0 \end{pmatrix}$$

where $A = \begin{pmatrix} 127 & -48 \\ 336 & -127 \end{pmatrix}$.

In general, the solution (b_n, c_n) is given by

$$(b_n\ c_n)^T = A^n(b_0\ c_0)^T.$$

It is well known that A^n is represented by the formula

$$A^n = \frac{\alpha^n}{\alpha - \beta}[A - \beta I] + \frac{\beta^n}{\beta - \alpha}[A - \alpha I]$$

where α, β are the Eigen values of A and I is the unit matrix of order 2.

On simplifying, we get

$b_n = (64 + 63(-1)^{n-1})b_0 - 24(1 + (-1)^{n-1})c_0$, &
$c_n = 168(1 + (-1)^{n-1})b_0 - (63 + 64(-1)^{n-1})c_0$ (15)

Therefore equation (15), along with $a_n = a_0$ represent the general solutions of (3).

Choice 3: The second solution (a_1, b_1, c_1) of (3) is taken as

$a_1 = 7a_0 + h$
$b_1 = 7b_0$
$c_1 = 7c_0 + 4h$

Substituting in (3), we get $h = 18a_0 - 8c_0$.

To get a sequence of solutions, consider the matrix representation of solution (a_1, c_1)

$$\text{(i.e.)}. \begin{pmatrix} a_1 \\ c_1 \end{pmatrix} = A \begin{pmatrix} a_0 \\ c_0 \end{pmatrix}$$

where $A = \begin{pmatrix} 25 & -8 \\ 72 & 25 \end{pmatrix}$.

In general, the solution (a_n, c_n) is given by

$$(a_n\ c_n)^T = A^n(a_0\ c_0)^T.$$

On simplifying as above, we get

$a_n = (16(7)^{n-1} + 9(-7)^{n-1})a_0 - 4((7)^{n-1} + (-7)^{n-1})c_0$, &
$c_n = 36((7)^{n-1} + (-7)^{n-1})a_0$
$\quad -(9(7)^{n-1} + 16(-7)^{n-1})c_0$ (16)

Therefore equation (16), along with $b_n = 7^n b_0$ represent the general solutions of (3).

Choice 4: The second solution (a_1, b_1, c_1) of (3) is taken as

$a_1 = a_0$
$b_1 = b_0 + h$
$c_1 = c_0 + 3h$

Substituting in (3), we get $h = 7b_0 - 3c_0$.

To get a sequence of solutions, consider the matrix representation of solution (b_1, c_1)

$$\text{(i.e.)}, \begin{pmatrix} b_1 \\ c_1 \end{pmatrix} = A \begin{pmatrix} b_0 \\ c_0 \end{pmatrix}$$

where $A = \begin{pmatrix} 8 & -3 \\ 21 & -8 \end{pmatrix}$.

In general, the solution (b_n, c_n) is given by

$$(b_n\ c_n)^T = A^n(b_0\ c_0)^T.$$

Proceeding as in the above choices,

$b_n = \frac{1}{2}[(9 - 7(-1)^n)b_0 - 3(1 + (-1)^{n-1})]c_0$, &
$c_n = \frac{1}{2}[21(1 + (-1)^{n+1})b_0 + (-7 + 9(-1)^n)c_0]$ (17)

Therefore equation (17), along with $a_n = a_0$ represent the general solutions of (3).

Substituting the values of a_n, b_n, and c_n in (2), from the choices 2, 3 and 4, a sequence of solutions of (1) can be determined.

3. Conclusion

In this paper, we have presented four different patterns of non- zero distinct Gaussian integer solutions of the infinite elliptic cone given by $5X^2 + 5Y^2 + 9Z^2 + 46XY - 34YZ - 22XZ = 0$. One may search for other patterns of non-zero distinct integer solutions for the same equation. Since the ternary quadratic Diophantine equations are rich in variety, one may search for other choices of Diophantine equations to find their corresponding Gaussian integer solutions.

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