International Journal of Science and Research (IJSR)
ISSN (Online): 2319-7064
Index Copernicus Value (2015): 78.96 | Impact Factor (2015): 6.391

Convergence Theorems for Maximal Monotone Operators by Family of Non-Spreading Mappings

Salwa Salman Abed[1], Zena Hussein Maibed[2]

Department of Mathematics, College of Education for Pure Science, Ibn Al-Haithem , University of Baghdad

Abstract: ***In this paper we introduce an iterative schemes of non-spreding and non-expansive mappings in real Hilbert space .Also, we study the strong convergence of these iterative schemes to a point of the set of zeros of maximal monotone multivalued mapping . Finally, there are some consequent of these results in convex analysis.***

1. Introduction and Preliminaries

Let X be a Hilbert space and A be a multivalued mapping with domain

$D(A) = \{\mathbf{x} \in X; Ax \neq \emptyset\}$ and $R(A) = \{y \in D; \exists \mathbf{x} \in$ DA such that $y \ni Ax$.The mapping A is called monotone if:

$$< x_1 - x_2, y_1 - y_1 > \geq 0, \forall x_i \in D(A), \quad \forall y_i \in R(A).$$

Also, any mapping A is called maximal monotone mapping if the graph of A is not properly contained in the graph of any other monotone mapping.

The monotone mappings play a crucial role in modern nonlinear analysis and optimization, see the books[1,2,3,4,5].Consider a resolvent mapping $J_{r_n} = (I + r_n A^{-1})(x)$, where $< r_n >$ is a sequence of positive real numbers.J_{r_n} is single valued non expansive. The metric projection from X on to C is defined as follows: For any $x \in X$ there exists a unique element $P_c(x) \in C$ and salisfy $\|x - P_c(x)\| \leq \|x - y\| \ \forall y \in C$.

Xu[6,7] , studied the convergence of thefollowing iterative scheme

$$x_{n+1} = \alpha_n u + (1 - \alpha_n)T_{x_n}, \quad n = 1,2,3, \ldots \qquad (1)$$

[8] Moudafi, studied the convergence of the iterative schemes and in

$$x_t = tf(x_t) + (1-t)T_{x_t}$$
$$x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n)T_{x_n} \quad \text{as } n \to \infty \qquad (2)$$

where$< \alpha_n >$ be a sequence in (0,1).Xu[9]who extended Moudafiresults.On other hand, Kamimura and Takahashi[10], studied the convergence strongly of the iterative scheme

$$x_{n+1} = \alpha_n u + (1 - \alpha_n)J_{r_{n x_n}}, n \geq 1 \qquad (3)$$

Also, in 2016[11], Abed and Maibed studied the strong convergence of the proximal point scheme

$$x_{n+1} = \alpha_n f_n(x_n) + \beta_n T_n(x_n) + (1 - \gamma_n)J_{r_n}(x_n)$$

Throughout this paper will be areal Hilbert space and C be a nonempty convex closed subset of$X$. The resolventidentity is : $J_\beta(x) = J_\gamma\left(\frac{\gamma}{\beta}x + \left(1 - \frac{\gamma}{\beta}\right)J_\beta(x)\right)$.We recall some definitions and lemmas which will use in the proofs.

Definition (1.1) [12], [13] and [14]

A mapping $T: C \to X$is called

1) firmly non-expansive mapping if for each x, y in C.Then the following inequality holds
$$\| Tx - Ty\|^2 + \|(I - T)x - (I - T)y\|^2 \leq \|x - y\|^2$$
2) Strongly non-expensive mapping if it is non-expensive and for any two sequences in C, $\langle x_n \rangle$and $\langle y_n \rangle$ such that $\langle x_n - y_n \rangle$ is bounded and $\|x_n - y_n\| - \|Tx_n - Ty_n\| \to 0$ it follows that $(x_n - y_n) - (Tx_n - Ty_n) \to 0$.
3) Non-spreading mapping if for each x, y in C.Then the following inequality holds
$$2\|Tx - Ty\|^2 \leq \|x - y\|^2 + \|x - y\|^2$$

Manaka and Takahashi [15], they proved the classes of non-expansive mappings does not contain the classes of non-spreading mappings .Also the classes of non-spreading mappings does not contain the classes of non-expansive mappings**,** consider the following example.Let$T: R \to R$ such that $T(x) = -x$,it is clear that the mapping is non-expansive but not non-spreading when $x = -1/2, y = 1/4$

Remark (1.2)[16]

Every firmly non-expansive mapping is non-sprading**.**

Lemma (1.3) [17]

A mapping T is nonspreding mapping, if for each x, y in C, then the following inequality holds
$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + 2 < x - Tx, y - Ty >$$

Remark (1.4)

1) If T is firmly non-expansive, then it is strongly non-expansive.
2) Any firmly non-expansive (and strongly non-expansive) mapping is non-expansive.

Lemma (1.6) [7]

If $\langle a_n \rangle$ be a sequence of nonnegative real number such that:
$$a_{n+1} \leq (1 - \gamma_n)a_n + \gamma_n S_n \quad , \quad n \geq 0$$
where$\langle \gamma_n \rangle$ is a sequence in (0,1) and $\langle S_n \rangle$ be a sequence in $\mathcal{R}$ such that:

$\sum_{n=0}^{\infty} \gamma_n = \infty$ and $\lim_{n\to\infty} \sup \frac{S_n}{\gamma_n} \leq 0$ or $\sum_{n=1}^{\infty}|S_n| < \infty$. Then $a_n \to 0$ as $n \to \infty$.

Lemma(1.7) [7]

be a sequence of nonnegative real number such that>a_n<If ; $n \geq 0$ $\quad u_n + a_{n+1} \leq (1 - \gamma_n) a_n + \gamma_n S_n$

Where $<S_n>$ be a sequence in$\mathcal{R}$ and $<\gamma_n>$ be a sequence in [0,1] such that

$\lim_{n\to\infty} \sup S_n \leq 0, \sum_{n=0}^{\infty} u_n < \infty$ and $\sum_{n=0}^{\infty} \gamma_n = \infty$.

.Then $a_n \to 0$.

Lemma(1.8) [18]

Let C be a nonempty convex closed subset of X and T be a multivalued mapping on C. If $\langle x_n \rangle$ is convergence weakly to p and $\|x_n - T_{x_n}\| \to 0$ .Then $p \in F(T)$.

Lemma (2.9) : [19]

Let C be a nonempty convex closed subset of X and T is non-expansive multivalued mapping such that $Fix(T) \neq \emptyset$.Then T is demi-closed, i.e.,$x_n \rightharpoonup p$ and $\lim_{n\to\infty} d(x_n, T(x_n)) = 0$.Then $p \in T(p)$

Now, we define the following two classes on mappings:

$$\mathcal{F} = \left\{f_n : C \to C \text{ such that } \|f_n(x) - f_{n+1}(y)\| \le \|f_n(x) - f_n(y)\|^2, n = 1, 2, \dots\right\}$$

$$\mathcal{F}^* = \left\{f_n : C \to C \text{ such that } \sum_{n=0}^{\infty} \|f_n(x)\| < \infty, \; n = 1, 2, \dots\right\}$$

2. Main Results

In this paper we give some types of proximal point schemes of sequences for non-spreading mappings which are different from non-expansive mappings. Also, we study the convergence for this schemes. In the theorems (2.1) and (2.2) consider $< f_n >$ be a sequence of contraction mappingsand $< g_n >$ *be a sequence of non-expansive* mappings on C and $< e_n >$ is real sequence in $(0,\infty)$ such that each sequences belong to $\mathcal{F}^*$.Define the following proximal point scheme as:

$$x_{n+1} = a_n g_n(x) + b_n f_n(x_n) + c_n T_n(x_n) + d_n J_{r_n}(x_n) + e_n \quad (2.1)$$

Where,

$< a_n >, < d_n >$ *are sequence in* $[a, b]$ where $0 < a < b < 1$,

$< b_n >$ *and* $< c_n >$ are sequences in $(0,1]$ such that $a_n + b_n + c_n + d_n = 1$ and

$< r_n >$ *is real sequences in* $(1, \infty)$.

Now, we study the strongly convergent of the proximal point scheme defined in (2.1)

Theorem (2.1) :

Let $< T_n >$ be a sequence of non-spreading mappings on C belong to $\mathcal{F} \cap \mathcal{F}^*$.Suppose that

1) $\left|1 - \frac{r_{n+1}}{r_n}\right| \le k|a_{n+1} - a_n|$, for some $k > 0$ *and* $|d_{n+1} - dn \le cn$

2) $\sum_{n=0}^{\infty} a_n = \infty$ and $\lim_{n\to\infty} \frac{|a_{n+1} - a_n|}{a_n} = \lim_{n\to\infty} e_n = 0$

If $A^{-1}(0) \cap (\cap F(T_n)) \cap (\cap F(f_n)) \neq \emptyset$.Then the proximal point scheme $< x_n >$ converges strongly to a point in $A^{-1}(0)$.

Proof :

Let $p \in A^{-1}(0) \cap (\cap F(T_n)) \cap (\cap F(f_n))$

$$\begin{aligned}\|x_{n+1} - p\| &\le a_n\|g_n(x) - p\| + b_n\|f_n(x_n) - p\| + c_n\|T_n(x_n) - p\| + d_n\|J_{r_n}(x_n) - p\| + \|e_n\| \\ &\le a_n\|x - p\| + \alpha b_n\|x_n - p\| + c_n\|T_n(x_n) - p\| + (1 - (a_n + b_n + c_n))\|x_n - p\| + \|e_n\|\end{aligned}$$

Where $\alpha = \sup\{\alpha_i, i \in N\}$, $0 < \alpha_i < 1$

$$\begin{aligned}\|x_{n+1} - p\| &\le a_n\|x - p\| + (\alpha b_n + (1 - b_n))\|x_n - p\| + \|T_n(x_n) - p\| + \|e_n\| \\ &\le a_n\|x - p\| + (1 - (1 - \alpha)b_n)\|x_n - p\| + \|T_n(x_n) - p\| + \|e_n\| \\ &\le \max\left\{\frac{1}{(1-\alpha)}\|x - p\|, \|x_n - p\|\right\} + \|T_n(x_n) - p\| + \|e_n\| \\ &\le \max\left\{\frac{1}{1-\alpha}\|x - p\|, \|x_{n-1} - p\|\right\} + \|T_n(x_n) - p\| + \|T_{n-1}(x_{n-1}) - p\| + \|e_{n-1}\| + \|e_n\| \\ &\vdots \\ &\le \max\left\{\frac{1}{1-\alpha}\|x - p\|, \|x_0 - p\|\right\} + \sum_{k=0}^{n}\|T_k(x_k) - p\| + \sum_{k=0}^{n}\|e_k\|\end{aligned}$$

But $\sum_{n=0}^{\infty}\|e_n\| < \infty$ and $\sum_{n=0}^{\infty}\|T_n(x_n)\| < \infty \Rightarrow < x_n >$ is bounded sequence. Then there exists a $< x_{nk} >$ subsequence of $< x_n >$ such that $x_{nk} \rightharpoonup \tilde{x}$.

Now, To prove $\tilde{x} \in A^{-1}(0)$

Let

$$V_n = \frac{x_{n+1} - a_n g_n(x) - b_n f_n(x_n) - c_n T_n(x_n) - e_n}{d_n} = J_{r_n}(x_n)$$

And $\omega(x_n) = \omega(V_n)$, So $< V_n >$ is bounded.

But $V_n = J_{r_n}(x_n) \Rightarrow$

$$(I + r_n A)^{-1}(x_n) = V_n \Rightarrow V_n + r_n A(V_n) \ni x_n \Rightarrow \frac{x_n - V_n}{r_n} \in A(V_n)$$

$$\Rightarrow \frac{x_n - x_{n+1} + a_n g_n(x) + b_n f_n(x_n) + c_n T_n(x_n) + e_n}{r_n d_n} \in A(V_n) \quad (2.2)$$

Now, to prove that $\frac{\|x_{n+1} - x_n\|}{r_n} \to 0$ as $n \to \infty$

By using resolventidentity ,and since $< T_n >$ lies in $\mathcal{F}$ we get,

$$\begin{aligned}x_{n+2} - x_{n+1} = d_n &\left(J_{r_{n+1}}(x_{n+1}) - J_{r_n}(x_n)\right) + \left(b_{n+1}f_{n+1}(x_{n+1}) - b_n f_n(x_n)\right) + \left(c_{n+1}T_{n+1}(x_{n+1}) - c_n T_n(x_n)\right) + (a_{n+1} - a_n) \\ &\left(g_n(x) + \frac{e_{n+1}}{a_{n+1}}\right) + a_{n+1}\left(g_{n+1}(x) - g_n(x)\right) + a_n\left(\frac{e_{n+1}}{a_{n+1}} - \frac{e_n}{a_n}\right) + (d_{n+1} - d_n)J_{r_{n+1}}(x_{n+1})\end{aligned}$$

$$\begin{aligned}\|x_{n+2} - x_{n+1}\| = d_n &\left\|J_{r_{n+1}}(x_{n+1}) - J_{r_{n+1}}\left(\left(\frac{r_{n+1}}{r_n}x_n\right) + \left(1 - \frac{r_{n+1}}{r_n}\right)J_{r_n}(x_n)\right)\right\| \\ &+ b_n\|f_{n+1}(x_{n+1}) - f_n(x_n)\| + c_n\|T_{n+1}(x_{n+1}) - T_n(x_n)\| + |a_{n+1} - a_n|\left\|g_n(x) + \frac{e_{n+1}}{a_{n+1}}\right\| \\ &+ a_{n+1}\|g_{n+1}(x) - g_n(x)\| + a_n\left\|\frac{e_{n+1}}{\alpha_{n+1}} - \frac{e_n}{\alpha_n}\right\| + |d_{n+1} - d_n|\|J_{r_{n+1}}(x_{n+1})\| \\ &+ |b_n - b_{n+1}|\|f_{n+1}(x_{n+1})\| + |c_n - c_{n+1}|\|T_{n+1}(x_{n+1})\|\end{aligned}$$

By conditions (i)&(ii) we get

$$
\begin{aligned}
\|x_{n+2}-x_{n+1}\| = d_n \Bigg\| & J_{r_{n+1}}(x_{n+1}) \\
& - J_{r_{n+1}}\left(\left(\frac{r_{n+1}}{r_n}x_n\right)+\left(1-\frac{r_{n+1}}{r_n}\right)J_{r_n}(x_n)\right)\Bigg\| \\
& + b_n\|f_{n+1}(x_{n+1})-f_n(x_n)\| \\
& + c_n\|T_n(x_{n+1})-T_n(x_n)\|^2 \\
& + |a_{n+1}-a_n|\left\|g_n(x)+\frac{e_{n+1}}{a_{n+1}}\right\| \\
& + a_{n+1}\|g_{n+1}(x)-g_n(x)\| \\
& + a_n\left\|\frac{e_{n+1}}{a_{n+1}}-\frac{e_n}{a_n}\right\| \\
& + |d_{n+1}-d_n|\|J_{r_{n+1}}(x_{n+1})\| \\
& + |b_n-b_{n+1}|\|f_{n+1}(x_{n+1})\| \\
& + |c_n-c_{n+1}|\|T_{n+1}(x_{n+1})\| \\
\le d_n \Bigg\| & \frac{r_{n+1}}{r_n}(x_{n+1}-x_n)+\left(1-\frac{r_{n+1}}{r_n}\right)\left(x_{n+1}-J_{r_n}(x_n)\right)\Bigg\| \\
& + b_n\|f_{n+1}(x_{n+1})-f_n(x_n)\| \\
& + c_n\|x_{n+1}-x_n\|^2 \\
& + 2c_n\langle x_{n+1}-T_n(x_{n+1}), x_n-T_n(x_n)\rangle \\
& + |a_{n+1}-a_n|.\left\|g_n(x)+\frac{e_{n+1}}{a_{n+1}}\right\| \\
& + a_{n+1}\|g_{n+1}(x)-g_n(x)\| \\
& + a_n\left\|\frac{e_{n+1}}{a_{n+1}}-\frac{e_n}{a_n}\right\| + c_n\|J_{r_{n+1}}(x_{n+1})\| \\
& + |b_n-b_{n+1}|\|f_{n+1}(x_{n+1})\| \\
& + |c_n-c_{n+1}|\|T_{n+1}(x_{n+1})\| \\
\le (1-a_n)\frac{r_{n+1}}{r_n} & \|x_{n+1}-x_n\| + \left|1-\frac{r_{n+1}}{r_n}\right|\|x_{n+1}-J_{r_n}(x_n)\| \\
& + |a_{n+1}-a_n|.\left\|g_n(x)+\frac{e_{n+1}}{a_{n+1}}\right\| \\
& + a_{n+1}\|g_{n+1}(x)-g_n(x)\| \\
& + a_n\left\|\frac{e_{n+1}}{a_{n+1}}-\frac{e_n}{a_n}\right\| \\
& + b_n\|f_{n+1}(x_{n+1})-f_n(x_n)\| \\
& + c_n\|x_{n+1}-x_n\|^2 \\
& + 2c_n\langle x_{n+1}-T_n(x_{n+1}), x_n-T_n(x_n)\rangle \\
& + c_n\|J_{r_{n+1}}(x_{n+1})\| \\
& + |b_n-b_{n+1}|\|f_{n+1}(x_{n+1})\| \\
& + |c_n-c_{n+1}|\|T_{n+1}(x_{n+1})\| \\
\le (1-a_n)\frac{r_{n+1}}{r_n} & \|x_{n+1}-x_n\| \\
& + a_n\Bigg(\frac{k|a_{n+1}-a_n|}{a_n}\|x_{n+1}-J_{r_n}(x_n)\| \\
& + \frac{|b_n-b_{n+1}|}{a_n}\|f_{n+1}(x_{n+1})\| \\
& + \frac{|c_n-c_{n+1}|}{a_n}\|T_{n+1}(x_{n+1})\| \\
& + \left\|\frac{e_{n+1}}{a_{n+1}}-\frac{e_n}{a_n}\right\| \\
& + \frac{|a_{n+1}-a_n|}{a_n}.\left\|g_n(x)+\frac{e_{n+1}}{a_{n+1}}\right\| \\
& + \frac{a_{n+1}}{a_n}\|g_{n+1}(x)-g_n(x)\| \\
& + \frac{b_n}{a_n}\|f_{n+1}(x_{n+1})-f_n(x_n)\| \\
& + \frac{c_n}{a_n}\left[2\langle x_{n+1}-T_n(x_{n+1}), x_n-T_n(x_n)\rangle\right. \\
& \left. + \|x_{n+1}-x_n\|^2 + \|J_{r_{n+1}}(x_{n+1})\|\right]\Bigg)
\end{aligned}
$$

$$
\begin{aligned}
\frac{\|x_{n+2}-x_{n+1}\|}{r_{n+1}} \le (1-a_n) & \frac{\|x_{n+1}-x_n\|}{r_n} \\
& + a_n\Bigg(\frac{1}{r_{n+1}}\Bigg[\frac{k|a_{n+1}-a_n|}{a_n}\|x_{n+1}-J_{r_n}(x_n)\| \\
& + \frac{|b_n-b_{n+1}|}{a_n}\|f_{n+1}(x_{n+1})\| \\
& + \frac{|c_n-c_{n+1}|}{a_n}\|T_{n+1}(x_{n+1})\| \\
& + \left\|\frac{e_{n+1}}{a_{n+1}}-\frac{e_n}{a_n}\right\| \\
& + \frac{|a_{n+1}-a_n|}{a_n}.\left\|g_n(x)+\frac{e_{n+1}}{a_{n+1}}\right\| \\
& + \frac{a_{n+1}}{a_n}\|g_{n+1}(x)-g_n(x)\| \\
& + \frac{b_n}{a_n}\|f_{n+1}(x_{n+1})-f_n(x_n)\| \\
& + \frac{c_n}{a_n}\{2\langle x_{n+1}-T_n(x_{n+1}), x_n-T_n(x_n)\rangle \\
& + \|x_{n+1}-x_n\|^2 + \|J_{r_{n+1}}(x_{n+1})\|\}\Bigg]\Bigg)
\end{aligned}
$$

By lemma (1.6)we get

$\frac{\|x_{n+1}-x_n\|}{r_n} \to 0$ as $n \to \infty$.

Therefore the equation (2.2) become $0 \in A(V_n)$

$\Rightarrow V_n \in A^{-1}(0)$.

But $A^{-1}(0)$ is closed $\Rightarrow \tilde{x} \in A^{-1}(0)$

$$
\begin{aligned}
\|x_{n+1}-\tilde{x}\| \le\ & a_n\|g_n(x)-\tilde{x}\| + b_n\|f_n(x_n)-\tilde{x}\| \\
& + c_n\|T_n(x_n)-\tilde{x}\| \\
& + d_n\|J_{r_n}(x_n)-\tilde{x}\| + \|e_n\| \\
\le (1-a_n) & \|x_n-\tilde{x}\| + a_n\|g_n(x)-\tilde{x}\| \\
& + \{\|f_n(x_n)-\tilde{x}\| + \|T_n(x_n)-\tilde{x}\| + \|e_n\|\}
\end{aligned}
$$

By using lemma (1.7) ,we get $x_n \to \tilde{x}$ as $n \to \infty$.Then the proximal point scheme$\langle x_n \rangle$converge strongly to $\tilde{x}$ and $\tilde{x} \in A^{-1}(0)$. ■

Now, we study the converge when $< T_n >$be a sequence of any mappings

Theorem (2.2):

Let $< T_n >$ be a sequence of any mappings on Csuch that$< T_n >$,

$< e_n >$ and $< g_n >$belong to $\mathcal{F}^*$. Let the proximal point scheme defined in (2.1) and $< a_n >, < d_n >$as in theorem(2.1) ,$< b_n >$ sequence in $(0,1]$ converges to 0, $< c_n >$ *be a sequence* in $(0,1]$ and $< r_n >$ be a sequences in $(1,\infty)$ .If $\left|1-\frac{r_{n+1}}{r_n}\right| \le k$, for some $k > 0$ such that:

1) $\lim_{n\to\infty}\frac{|a_{n+1}-a_n|}{a_n} = 0$ and $c_n \le b_n$
2) $\sum_{n=0}^{\infty} a_n = \infty$

If $A^{-1}(0) \cap (\cap F(T_n)) \cap (\cap F(f_n)) \neq \emptyset$. Then the proximal point scheme$< x_n >$ converges strongly to a point in $A^{-1}(0)$.

Proof :

As the same proof of theorem (2.1) we get, $< x_n >$ is bounded sequence

Now, to prove that $\frac{\|x_{n+1}-x_n\|}{r_n} \to 0$ as $n \to \infty$

$$\|x_{n+2}-x_{n+1}\| = \|(a_{n+1}-a_n)g_n(x) + a_{n+1}(g_{n+1}(x) - g_n(x)) + b_{n+1}f_{n+1}(x_{n+1}) - b_nf_n(x_n) + c_{n+1}T_{n+1}(x_{n+1}) - c_nT_n(x_n) - d_{n+1}J_{r_{n+1}}(x_{n+1}) + d_nJ_{r_n}(x_n) + e_{n+1} - e_n\|$$

$$= \Big\|(1-a_n)\left(J_{r_{n+1}}(x_{n+1}) - J_{r_n}(x_n)\right) + b_{n+1}\left(f_{n+1}(x_{n+1}) - J_{r_{n+1}}(x_{n+1})\right) + b_n\left(J_{r_n}(x_n) - f_n(x_n)\right) + a_{n+1}(g_{n+1}(x) - g_n(x)) + (a_{n+1}-a_n)\left(g_n(x) - J_{r_{n+1}}(x_{n+1}) + \frac{e_{n+1}}{a_{n+1}}\right) + a_n\left(\frac{e_{n+1}}{a_{n+1}} - \frac{e_n}{a_n}\right) + c_{n+1}\left(T_{n+1}(x_{n+1}) - J_{r_{n+1}}(x_{n+1})\right) + c_n\left(J_{r_n}(x_n) - T_n(x_n)\right)\Big\|$$

$$\|x_{n+2}-x_{n+1}\| \le (1-a_n)\|J_{r_{n+1}}(x_{n+1}) - J_{r_n}(x_n)\| + b_{n+1}\|f_{n+1}(x_{n+1}) - J_{r_{n+1}}(x_{n+1})\| + b_n\|J_{r_n}(x_n) - f_n(x_n)\| + a_{n+1}\|g_{n+1}(x) - g_n(x)\| + |a_{n+1}-a_n|.\left\|g_n(x) - J_{r_{n+1}}(x_{n+1}) + \frac{e_{n+1}}{a_{n+1}}\right\| + a_n\left\|\frac{e_{n+1}}{a_{n+1}} - \frac{e_n}{a_n}\right\| + c_{n+1}\|T_{n+1}(x_{n+1}) - J_{r_{n+1}}(x_{n+1})\| + c_n\|T_n(x_n) - J_{r_n}(x_n)\|$$

Put

$M_n = \left\{\left\|g_n(x) - J_{r_{n+1}}(x_{n+1}) + \frac{e_{n+1}}{a_{n+1}}\right\|, n \in N\right\}$ we get

By using resolvent identity and since the resolvent mapping is non-expansive mapping then we get,

$$\|x_{n+2}-x_{n+1}\| \le (1-a_n)\left\|J_{r_{n+1}}(x_{n+1}) - J_{r_{n+1}}\left(\left(\frac{r_{n+1}}{r_n}x_n\right) + \left(1-\frac{r_{n+1}}{r_n}\right)J_{r_n}(x_n)\right)\right\| + b_{n+1}\|f_{n+1}(x_{n+1}) - J_{r_{n+1}}(x_{n+1})\| + b_n\|J_{r_n}(x_n) - f_n(x_n)\| + a_{n+1}\|g_{n+1}(x) - g_n(x)\| + |a_{n+1}-a_n|M_n + a_n\left\|\frac{e_{n+1}}{a_{n+1}} - \frac{e_n}{a_n}\right\| + c_{n+1}\|T_{n+1}(x_{n+1}) - J_{r_{n+1}}(x_{n+1})\| + c_n\|T_n(x_n) - J_{r_n}(x_n)\|$$

$$\le (1-a_n)\left\|x_{n+1} - \left(\frac{r_{n+1}}{r_n}x_n\right) + \left(1-\frac{r_{n+1}}{r_n}\right)J_{r_n}(x_n)\right\| + b_{n+1}\|f_{n+1}(x_{n+1}) - J_{r_{n+1}}(x_{n+1})\| + b_n\|J_{r_n}(x_n) - f_n(x_n)\| + a_{n+1}\|g_{n+1}(x) - g_n(x)\| + |a_{n+1}-a_n|M_n + a_n\left\|\frac{e_{n+1}}{a_{n+1}} - \frac{e_n}{a_n}\right\| + c_{n+1}\|T_{n+1}(x_{n+1}) - J_{r_{n+1}}(x_{n+1})\| + c_n\|T_n(x_n) - J_{r_n}(x_n)\|$$

$$= (1-a_n)\left\|\frac{r_{n+1}}{r_n}(x_{n+1}-x_n) + \left(1-\frac{r_{n+1}}{r_n}\right)\left(x_{n+1} - J_{r_n}(x_n)\right)\right\| + b_{n+1}\|f_{n+1}(x_{n+1}) - J_{r_{n+1}}(x_{n+1})\| + b_n\|J_{r_n}(x_n) - f_n(x_n)\| + a_{n+1}\|g_{n+1}(x) - g_n(x)\| + |a_{n+1}-a_n|M_n + a_n\left\|\frac{e_{n+1}}{a_{n+1}} - \frac{e_n}{a_n}\right\| + c_{n+1}\|T_{n+1}(x_{n+1}) - J_{r_{n+1}}(x_{n+1})\| + c_n\|T_n(x_n) - J_{r_n}(x_n)\|$$

$$\le (1-a_n)\left[\frac{r_{n+1}}{r_n}\|x_{n+1}-x_n\| + \left|1-\frac{r_{n+1}}{r_n}\right|.\|x_{n+1} - J_{r_n}(x_n)\|\right] + b_{n+1}\|f_{n+1}(x_{n+1}) - J_{r_{n+1}}(x_{n+1})\| + b_n\|J_{r_n}(x_n) - f_n(x_n)\| + a_{n+1}\|g_{n+1}(x) - g_n(x)\| + |a_{n+1}-a_n|M_n + a_n\left\|\frac{e_{n+1}}{a_{n+1}} - \frac{e_n}{a_n}\right\| + c_{n+1}\|T_{n+1}(x_{n+1}) - J_{r_{n+1}}(x_{n+1})\| + c_n\|T_n(x_n) - J_{r_n}(x_n)\|$$

Since $x_{n+1} = a_ng_n(x) + b_nf_n(x_n) + c_nT_n(x_n) + d_nJ_{r_n}(x_n) + e_n$

$$\|x_{n+1} - J_{r_n}(x_n)\| = \|a_ng_n(x) + b_nf_n(x_n) + c_nT_n(x_n) + (d_n - 1)J_{r_n}(x_n) + e_n\| \le a_n\|g_n(x)\| + b_n\|f_n(x_n)\| + c_n\|T_n(x_n)\| + (d_n - 1)\|J_{r_n}(x_n)\| + \|e_n\|$$

Since $\langle g_n\rangle, \langle T_n\rangle, \langle e_n\rangle$ are lies in $\mathcal{F}^*$ and $b_n \to 0$, we get

$\|x_{n+1} - J_{r_n}(x_n)\| \to 0 \quad$ as $\quad n \to \infty$

If $K_{n1} = \{b_{n+1}\|f_{n+1}(x_{n+1}) - J_{r_{n+1}}(x_{n+1})\| + b_n\|J_{r_n}(x_n) -$ fnxn,n∈N

$$K_{n2} = \left\{k\|x_{n+1} - J_{r_n}(x_n)\| + a_{n+1}\|g_{n+1}(x) - g_n(x)\| + a_n\left(\left\|\frac{e_{n+1}}{a_{n+1}} - \frac{e_n}{a_n}\right\| + \frac{|a_{n+1}-a_n|}{a_n}M_n\right), n \in N\right\}$$

$K_{n3} = \{c_{n+1}\|T_{n+1}(x_{n+1}) - J_{r_{n+1}}(x_{n+1})\| + c_n\|T_n(x_n) -$ Jrnxn,n∈N.

Put $\acute{M}_n = K_{n1} + K_{n2} + K_{n3}$.

Since $\|x_{n+1} - J_{r_n}(x_n)\| \to 0$ and $b_n, c_n \to 0$

$\Rightarrow \acute{M}_n \to 0 \qquad$ as $n \to \infty$

$$\|x_{n+2}-x_{n+1}\| \le \left[(1-a_n)\frac{r_{n+1}}{r_n}\right]\|x_{n+1}-x_n\| + \acute{M}_n$$

$$\frac{\|x_{n+2}-x_{n+1}\|}{r_{n+1}} \le (1-a_n)\frac{\|x_{n+1}-x_n\|}{r_n} + a_n\left[\frac{1}{r_{n+1}a_n}\acute{M}_n\right]$$

By lemma (1.6) we get ,

$\frac{\|x_{n+1}-x_n\|}{r_n} \to 0 \quad$ as $\quad n \to \infty$.

By similar to the theorem (2.1) we get, $\tilde{x} \in A^{-1}(0)$

$$\|x_{n+1} - \tilde{x}\| \le a_n\|g_n(x) - \tilde{x}\| + b_n\|f_n(x_n) - \tilde{x}\| + c_n\|T_n(x_n) - \tilde{x}\| + d_n\|J_{r_n}(x_n) - \tilde{x}\| + \|e_n\|$$
$$\le d_n\|x_n - \tilde{x}\| + a_n\|g_n(x) - \tilde{x}\| + b_n\|f_n(x_n) - \tilde{x}\| \; b_n\|T_n(x_n) - \tilde{x}\| + \|e_n\|$$

$\leq (1-a_n)\|x_n-\tilde{x}\| + a_n\left(b_n\left(\frac{\|f_n(x_n)-\tilde{x}\|+\|T_n(x_n)-\tilde{x}\|}{a_n}\right)\right) + \|g_n(x)-\tilde{x}\| + \|e_n\|$. Put $W_n = b_n\left(\frac{\|f_n(x_n)-\tilde{x}\|+\|T_n(x_n)-\tilde{x}\|}{a_n}\right) \Longrightarrow W_n \to 0$ as $n\to\infty$ and

$$\widetilde{W_n} = \|g_n(x)-\tilde{x}\| + \|e_n\| \Rightarrow \sum \widetilde{W_n} < \infty$$

$$\|x_{n+1}-\tilde{x}\| \leq (1-a_n)\|x_n-\tilde{x}\| + a_n W_n + \widetilde{W_n}$$

By using lemma (1.7) ,we get $x_n \to \tilde{x}$ as $n\to\infty$.Then the proximal point scheme converge strongly to $\tilde{x}$ and $\tilde{x} \in A^{-1}(0)$. ■

Theorem (2.3) :

Let $<T_n>$ be a sequence of non – spreading mappings on C belong to $\mathcal{F}^*$ but $<f_n>$ and $<J_{r_n}>$ be a sequence of non – expansive mapping on C belong to $\mathcal{F}^*$. If $<a_n>,<b_n>$ and $<c_n>$ are sequences in (0,1] such that $a_n+b_n-c_n=1$ .Define the proximal point scheme $<x_n>$ as:

$$x_{n+1} = (1-a_n)T_n y_n + (1-b_n)J_{r_n}y_n + c_n f_n(y_n)$$
$$y_n = (1-b_n)x_n + (1-a_n)f_n(x_n) + c_n T_n(x_n)$$

If the conditions are satisfied

1. $\sum_{n=0}^{\infty}(a_n-c_n)=\infty$ and $\lim_{n\to\infty}|b_n-b_{n-1}|=0$
2. $A^{-1}(0)\cap(\cap F(T_n))\cap(\cap F(f_n))\neq\emptyset$.

Then the proximal point scheme $<x_n>$ is converges strongly to common fixed point of T_n and f_n , $\forall n\in N$.

Proof :

Let $p\in A^{-1}(0)\cap(\cap F(T_n))\cap(\cap F(f_n))$.

Since $<f_n>$ and $<J_{r_n}>$ are sequences of nonexpansive mappings and $<T_n>$ be a sequence of non-spreading mappings, we have

$$\|y_n-p\|^2 \leq (1-b_n)\|x_n-p\|^2 + (1-a_n)\|f_n(x_n)-p\|^2 + c_n\|T_n(x_n)-p\|^2$$
$$\leq (1-b_n)\|x_n-p\|^2 + (1-a_n)\|x_n-p\|^2 + c_n[\|x_n-p\|^2 + 2\langle x_n-T_n(x_n), p-T_n(p)\rangle$$
$$\leq ((1-b_n)+(1-a_n)+c_n)\|x_n-p\|^2 = \|x_n-p\|^2$$

$$\|x_n-p\|^2 \leq (1-a_n)\|T_n y-p\|^2 + (1-b_n)\|J_{r_n}(y_n)-p\|^2 + c_n\|f_n(y_n)-p\|^2$$
$$\leq (1-a_n)(\|y_n-p\|^2 + 2\langle y_n-T_n(y_n), p-T_n(p)\rangle) + (1-b_n)\|y_n-p\|^2 + c_n\|y_n-p\|^2$$
$$= \|x_n-p\|^2$$

$<x_n>$ is bounded sequence

So $<y_n>,<T_n(x_n)>$ *and* $<f_n(x_n)>$ also bounded sequence. By add and subtract the amount $(1-b_{n-1})x_n$,we get

$$\|y_n-y_{n-1}\| = \|(1-b_n)x_n + (1-a_n)f_n(x_n) + c_nT_n(x_n) - (1-b_{n-1})x_{n-1} - (1-a_{n-1})f_{n-1}(x_{n-1}) - c_{n-1}T_{n-1}(x_{n-1})\|$$
$$\leq (1-b_{n-1})\|x_n-x_{n-1}\| + |(1-b_n)-(1-b_{n-1})|\|x_n\| + \|(1-a_n)f_n(x_n) - (1-a_{n-1})f_{n-1}(x_{n-1})\| + \|c_nT_n(x_n)-c_{n-1}T_{n-1}(x_{n-1})\|$$
$$\leq (1-b_{n-1})\|x_n-x_{n-1}\| + |b_n-b_{n-1}|\|x_n\| + \|(1-a_n)f_n(x_n) - (1-a_{n-1})f_{n-1}(x_{n-1})\| + c_{n-1}\|T_n(x_n)-T_{n-1}(x_{n-1})\| + |c_{n-1}-c_n|\|T_n(x_n)\|$$

$$\|x_n-x_{n+1}\| \leq b_{n-1}\|T_ny_n-T_{n-1}y_{n-1}\| + \|a_nJ_{r_n}(y_n)-a_{n-1}J_{r_{n-1}}(y_{n-1})\| + \|c_nf_n(y_n)-c_{n-1}f_{n-1}(y_{n-1})\| + |b_{n-1}-b_n|\|T_n(y_n)\|$$
$$\leq b_{n-1}\|T_ny_n-T_{n-1}y_{n-1}\| + a_{n-1}\|J_{r_{n-1}}(y_{n-1})\| + a_n\|J_{r_n}(y_n)\| + c_{n-1}\|f_{n-1}(y_{n-1})\| + c_n\|f_n(y_n)\| + |b_{n-1}-b_n|\|T_n(y_n)\|$$

But $<T_n>,<f_n>$ and $<J_{r_n}>$ belong to $\mathcal{F}^*$

$$\|x_n-x_{n-1}\|\to 0 \quad \text{as} \quad n\to\infty$$

And hence ,since $|b_{n-1}-b_n|\to 0$ as $n\to\infty$

$$\|y_n-y_{n-1}\|\to 0 \quad \text{as} \quad n\to\infty$$

Since, $\|x_{n+1}-p\|\leq\|x_n-p\|\Rightarrow$

$$\|x_{n+1}-p\| \leq \|x_n-p\| + \|x_n-T_nx_n\|$$
$$-\|x_n-T_nx_n\| \leq \|x_n-p\| - \|x_{n+1}-p\|$$
$$\leq |\|x_n-p\|-\|x_{n+1}-p\|| \leq \|x_{n+1}-x_n\| \to 0$$

as $n\to\infty$

$$\|x_n-T_nx_n\|\to 0 \quad \text{as} \quad n\to\infty \tag{2.3}$$

But $<x_n>$ is bounded then there exists x_{nk} subsequence of $<x_n>$ such that $x_{nk}\rightharpoonup\tilde{x}$.

By (2.3) and since $<T_n>$ be a sequence of non-spreading mappings then by using lemma(1.8) we get $\tilde{x}\in\cap F(T_n)$.

As the same way and by using lemma(2.9) we get , $\tilde{x}\in\cap F(f_n)$

$$\|y_n-\tilde{x}\|^2 \leq (1-b_n)\|x_n-\tilde{x}\|^2 + (1-a_n)\|f_n(x_n)-\tilde{x}\|^2 + c_n\|T_n(x_n)-\tilde{x}\|^2$$
$$\leq (1-b_n)\|x_n-\tilde{x}\|^2 + (1-a_n)\|x_n-\tilde{x}\|^2 + c_n\|x_n-\tilde{x}\|^2 + 2<x_n-T_n(x_n), \tilde{x}-T_n(\tilde{x})> = \|x_n-\tilde{x}\|^2$$

$$\|x_{n+1}-\tilde{x}\|^2 \leq (1-a_n)\|T_n(y_n)-\tilde{x}\|^2 + (1-b_n)\|J_{r_n}(y_n)-\tilde{x}\|^2 + c_n\|f_n(y_n)-\tilde{x}\|^2$$
$$\leq (1-a_n)\|y_n-\tilde{x}\|^2 + 2<y_n-T_n(y_n), \tilde{x}-T_n(\tilde{x})> + (1-b_n)\|J_{r_n}(y_n)-\tilde{x}\|^2 + c_n\|y_n-\tilde{x}\|^2$$
$$\leq (1-a_n)\|x_n-\tilde{x}\|^2 + (1-b_n)\|J_{r_n}(y_n)-\tilde{x}\|^2 + c_n\|x_n-\tilde{x}\|^2$$
$$\leq ((1-a_n)+c_n)\|x_n-\tilde{x}\|^2 + (1-b_n)\|J_{r_n}(y_n)-\tilde{x}\|^2$$
$$\leq b_n\|x_n-\tilde{x}\|^2 + (1-b_n)\|J_{r_n}(y_n)-\tilde{x}\|^2$$
$$\leq (1-(a_n-c_n))(\|x_n-\tilde{x}\|^2 + (a_n-c_n)\|J_{r_n}(y_n)-\tilde{x}\|^2$$

By lemma (1.6), $\|x_n-\tilde{x}\|\to 0$ as $n\to\infty$.Then the proximal point scheme converge to $\tilde{x}$ and common fixed point of T_n and f_n , $\forall n\in N$. ■

Theorem (2.4) :

Let $<T_n>$ be a sequence of non – spreading mappings on C and $<f_n>$ be a sequence of non – expansive mapping on C such that $<J_{r_n}>$ and $<T_n>$ belong to $\mathcal{F}^*$. Define the proximal point scheme as :

$$y_n = (1-b_n)x_n + b_nf_n(x_n)$$
$$x_{n+1} = (1-a_n)T_n(y_n) + a_nJ_{r_n}(y_n)$$

Let $<a_n>$ *and* $<b_n>$ are sequences in (0,1]satisfies:

1. $<b_n>$ converges to 0 , $a_n+b_n=1$
2. $\sum b_n=\infty$ and $(\cap F(T_n))\cap(\cap F(f_n))\cap A^{-1}(0)\neq\emptyset$.

Then the proximal point scheme $<x_n>$ converges strongly to common fixed point of T_n, $\forall n\in N$.

Proof:

Let $p \in (\cap F(T_n)) \cap (\cap F(f_n)) \cap A^{-1}(0)$

$$\begin{aligned}\|y_n - p\|^2 &\le (1-b_n)\|x_n - p\|^2 + b_n\|f_n(x_n) - p\|^2 \\ &\le (1-b_n)\|x_n - p\|^2 + b_n\|x_n - p\|^2 \\ &= \|x_n - p\|^2\end{aligned}$$

$$\|x_{n+1} - p\|^2 \le (1-a_n)\|T_n y_n - p\|^2 + a_n\left\|J_{r_n}(y_n) - p\right\|^2$$

$$\begin{aligned}\|y_n - p\|^2 &\le (1-a_n)\|y_n - p\|^2 + a_n\|y_n - p\|^2 \\ &= \|y_n - p\|^2 \le \|x_n - p\|^2\end{aligned}$$

$< x_n >$is bounded sequence , so $< f_n >$, $< J_{r_n} >$ *and* $< y_n >$ also bounded sequence .

$$\begin{aligned}\|y_{n+1} - y_n\| &\le \|(1-b_n)x_n - (1-b_{n-1})x_{n-1} \\ &\quad + (1-b_n)x_{n-1} - (1-b_n)x_{n-1}\| \\ &\quad + \|b_n f_n(x_n) - b_{n-1}f_{n-1}(x_{n-1})\| \\ &\le (1-b_n)\|x_n - x_{n-1}\| \\ &\quad + |b_{n-1} - b_n|\|x_{n-1}\| + b_n\|f_n(x_n)\| \\ &\quad + b_{n-1}\|f_{n-1}(x_{n-1})\|\end{aligned}$$

$$\begin{aligned}\|x_{n+1} - x_n\| &\le \|(1-a_n)T_n(y_n) + a_n J_{r_n}(y_n) \\ &\quad - (1-a_{n-1})T_{n-1}(y_{n-1}) \\ &\quad - a_{n-1}J_{r_{n-1}}(y_{n-1})\| \\ &\le (1-a_n)\|T_n(y_n)\| \\ &\quad + (1-a_{n-1})\|T_{n-1}(y_{n-1})\| \\ &\quad + a_n\left\|J_{r_n}(y_n)\right\| + a_{n-1}\left\|J_{r_{n-1}}(y_{n-1})\right\|\end{aligned}$$

But $< J_{r_n} >$ and $< T_n >$ belong to $\mathcal{F}^*$then we get

$\Rightarrow \|x_n - x_{n-1}\| \to 0$ as $n \to \infty$

Since ,$\|x_n - T_n(x_n)\| \le \|x_n - x_{n+1}\| + \|x_{n+1} - T_n(x_n)\|$

Therefore,

$$\begin{aligned}\|x_n - T_n(x_n)\| &\le \|x_n - x_{n+1}\| \\ &\quad + \|(1-a_n)T_n(y_n) - T_n(x_n)\| \\ &\quad + a_n\left\|J_{r_n}(y_n)\right\| \\ &\le \|x_n - x_{n+1}\| + (1-a_n)\|T_n(y_n)\| + \|T_n(x_n)\| \\ &\quad + a_n\left\|J_{r_n}(y_n)\right\|\end{aligned}$$

As $n \to \infty$, we get

$$\|x_n - T_n(x_n)\| \to 0 \qquad (2.4)$$

Since $< x_n >$ is bounded sequence then there exists a subsequence $< x_{nk} >$ of $< x_n >$ such that $x_{nk} \rightharpoonup \tilde{x}$.

By (2.4) and since$< T_n >$be a sequence of non-spreading mappings then by using (2.4) and lemma(1.8) we get

$\tilde{x} \in \cap F(T_n)$

$$\begin{aligned}\|y_n - \tilde{x}\|^2 &= \|(1-b_n)x_n + b_n f_n(x_n) - \tilde{x}\|^2 \\ &\le (1-b_n)\|x_n - \tilde{x}\|^2 + b_n\|f_n(x_n) - \tilde{x}\|^2\end{aligned}$$

$$\begin{aligned}\|x_{n+1} - \tilde{x}\|^2 &\le (1-a_n)\|T_n(y_n) - \tilde{x}\|^2 + a_n\left\|J_{r_n}(y_n) - \tilde{x}\right\|^2 \\ &\le (1-a_n)\|y_n - \tilde{x}\|^2 \\ &\quad + 2(1-a_n)\langle y_n - T_n(y_n), \tilde{x} - T_n(\tilde{x})\rangle \\ &\quad + a_n\left\|J_{r_n}(y_n) - \tilde{x}\right\|^2.\end{aligned}$$

$$\begin{aligned}\|x_{n+1} - \tilde{x}\|^2 &\le (1-a_n)(1-b_n)\|x_n - \tilde{x}\|^2 \\ &\quad + b_n(b_n\|f_n(x_n) - \tilde{x}\|^2) + \left\|J_{r_n}(y_n) - \tilde{x}\right\|^2\end{aligned}$$

$$\begin{aligned}\|x_{n+1} - \tilde{x}\|^2 &\le (1-b_n)\|x_n - \tilde{x}\|^2 + b_n(b_n\|f_n(x_n) - \tilde{x}\|^2) \\ &\quad + \left\|J_{r_n}(y_n) - \tilde{x}\right\|^2\end{aligned}$$

By lemma (1.7)we get ,

$\|x_{n+1} - \tilde{x}\| \to 0$ as $n \to \infty$.

i.e., the proximal point scheme$\langle x_n \rangle$ converges strongly to common fixed point of T_n, $\forall n \in N$. ■

Theorem (2.5):

Let $< T_n >$ and $< f_n >$ are two sequences of non-spreading mapping on C belong to $\mathcal{F}^*$. and also $< J_{r_n}(x_n) >$. Define the proximal point scheme as the following

$$x_{n+1} = a_n f_n(x_n) + b_n T_n(x_n) + c_n J_{r_n}(x_n)$$

Where $\langle a_n \rangle, \langle b_n \rangle$ and $\langle c_n \rangle$ are sequences in [0,1] such that

1.$a_n + b_n + c_n = 1$

2.$\sum a_n = \infty$and$A^{-1}(0) \cap (\cap F(T_n)) \cap (\cap F(f_n)) \ne \varphi$.

Then the proximal point scheme$\langle x_n \rangle$ converges strongly to a common fixed point of T_n, $\forall n \in N$.

Proof :

Let $p \in A^{-1}(0) \cap (\cap F(f_n)) \cap (\cap F(T_n))$

$$\begin{aligned}\|x_{n+1} - p\|^2 &\le a_n\|f_n(x_n) - p\|^2 + b_n\|T_n(x_n) - p\|^2 \\ &\quad + c_n\left\|J_{r_n}(x_n) - p\right\|^2 \\ &\le a_n[\|x_n - p\|^2 \\ &\quad + 2\langle x_n - f_n(x_n), p - f_n(p)\rangle] \\ &\quad + b_n[\|x_n - p\|^2 \\ &\quad + 2\langle x_n - T_n(x_n), p - T_n p\rangle] + c_n\|x_n - p\|^2 \\ &= \|x_n - p\|^2\end{aligned}$$

$\langle x_n \rangle$is bounded sequence , so $\langle f_n \rangle$ and $\langle J_{rn} \rangle$ also bounded.

Since,

$$\begin{aligned}\|x_n - T_n(x_n)\| &\le a_{n-1}\|f_{n-1}(x_{n-1})\| + c_{n-1}\left\|J_{r_{n-1}}(x_n)\right\| + \\ &\quad b_{n-1}\|T_{n-1}(x_{n-1}) - T_n(x_n)\| \\ &\le a_{n-1}\|f_{n-1}(x_{n-1})\| + c_{n-1}\left\|J_{r_{n-1}}(x_{n-1})\right\| \\ &\quad + \|b_{n-1}T_{n-1}(x_{n-1})\| + \|T_n(x_n)\|\end{aligned}$$

As $n \to \infty$, we get

$$\|x_n - T_n(x_n)\| \to 0 \qquad (2.5)$$

But $\langle x_n \rangle$ is bounded then there exists a $\langle x_{nk} \rangle$ subsequence of $\langle x_n \rangle$ such that $x_{nk} \rightharpoonup \tilde{x}$.

By (2.5) and since$< T_n >$be a sequence of non-spreading mappings then by using lemma(1.8)

$$\Rightarrow \tilde{x} \in \cap F(T_n)$$

$$\begin{aligned}\|x_{n+1} - \tilde{x}\|^2 &\le a_n\|f_n(x_n) - \tilde{x}\|^2 + b_n\|T_n(x_n) - \tilde{x}\|^2 \\ &\quad + c_n\left\|J_{r_n}(x_n) - \tilde{x}\right\|^2 \\ &\le a_n\|f_n(x_n) - \tilde{x}\|^2 \\ &\quad + b_n[\|x_n - \tilde{x}\|^2 \\ &\quad + 2\langle x_n - T_n(x_n), \tilde{x} - T_n(\tilde{x})\rangle] \\ &\quad + c_n\left\|J_{r_n}x_n - \tilde{x}\right\|^2 \\ &\le (1-a_n)\|x_n - \tilde{x}\|^2 + a_n\|f_n(x_n) - \tilde{x}\|^2 \\ &\quad + \left\|J_{r_n}x_n - \tilde{x}\right\|^2\end{aligned}$$

By lemma (1.7), we get $\|x_n - \tilde{x}\| \to 0$.

Then the proximal point scheme converge to $\tilde{x}$ such that $\tilde{x} \in \cap Fix(T_n)$. ■

As future work we may use the idea of this paper in iteration in [20] and [21].

References

[1] H.H. Bauschke and P.L. Combettes, Convex Analysis and Monotone Operator Theory in Hilbert Spaces, Springer-Verlag, 2011.

[2] J.M. Borwein and J.D. Vanderwerff, Convex Functions, Cambridge University Press, 2010.

[3] R.S. Burachik and A.N. Iusem,Set-Valued Mappings and Enlargements ofMonotone Operators, Springer-Verlag, 2008. 24

[4] S. Simons, From Hahn-Banach to Monotonicity, Springer-Verlag, 2008.
[5] C. Z ˇalinescu,Convex Analysis in General Vector Spaces, world Scientific Publishing, 2002.
[6] H.K.Xu," A Nother Control Condition in an Iterative Method for Nonexpansive Mappings, bull ,austral. Math .soc ,65 (2002) ,109-113.
[7] H.K.Xu,"Iterative Algorithm for nonlinear operators",J.london Math.soc .66(2002) 240-256
[8] A.Moudafi, Viscosity Approximation Method for Fixed Point Problems, Journal of Mathematical Analysis and Applications, 241(2000) 46-55
[9] H.K.Xu,"Viscosity Approximation Methods for Nonexpansive Mapping ",J.Math .Anal.Appl..298(2004)279-291.
[10] S. Kamimura, W. Takahashi,Approximating Solutions of Maximal Monotone Operators in Hilbert Spaces, J. Approx. Theory 106 (2000) 226–240.
[11] S.S. Abed ,Z .H. Maibed,Convergence Theorems of Iterative Schemes ForNonexpansive Mappings, journal of advances in mathematics , 12(2016)6845-6851.
[12] H.H. Bauschke and P.L. Combettes, Convex Analysis and Monotone Operator Theory in Hilbert Spaces, Springer-Verlag, 2011.
[13] R. E. Bruck and S. Reich, Nonexpansive Projections and Resolvents of Accretive Operators in Banach spaces, Houston J. Math., vol. 3, pp. 459–470, 1977.
[14] F. Kosaka andW.Takahashi,Fixed Point Theorems for a Classof Nonlinear Mappings Related to Maximal Monotone Operators in Banach Spaces., Arch. Math. (Basel) 91 (2008), 166-177
[15] H.Manaka and W.Takahashi,Weak Convergence Theorems for Maximall Monotone Operators with Nonspreading Mappings in Hilbert Space,CUBO ,A.Math.journal,Vol 13,NO.01(11-24),20011.
[16] K.Aoyamaand F.kohsaka and W.Takahashi.,Proximal Point Methods for Monotone Operator in Banach Space,Taiwanese Journal of Math, Vol.15,NO.1,pp(259-281)2011.
[17] S .Lemoto and W. Takahashi,"Approximating Common Fixed Points of Nonexpansive Mapping andNonspreading Mappings in Hilbert Space"Nonlinear Analysis,71,(2009)2082_2089.
[18] D.A.Ruiz,G.L.Acedo, V.M.Marquez,"Firmly Nonexpansie Mappings",J.Nonlineaanalysis,vol 15(2014)1
[19] M. Eslamian,A. Abkar, "One – Step Iterative Rocess For a Family of Multivalued Mapping", Math. Comput.Modell. 54 (2011) 105 – 111.
[20] S. S. Abed , R. F. Abbas," s-Iteration for General Quasi Multivalued ContractionMappings", IJAMSS, 5(4) 2016, 9-22.
[21] S. S. Abed , R. F. Abbas," Solving a System of Set-valued Operator Equations",Global J. of Math. 8(3)2016, 919-924.