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# A Parametric Programming Problem for Selecting the Optimal Scheme of Waiting Time of the Stuffs for Work

**Jang-Ryol[1], Ri Chol-Ryong[2], Kim Yong[3]**

[1]Sunan economic University, Pyongyang, D.P.R. of Korea

[2]Library of Science, State Academy of Science, Pyongyang, D.P.R. of Korea

[3]Department of Mathematics, University of Science, Pyongyang, D.P.R. of Korea

**Abstract*: In this paper, we propose a discrete parametric programming problem in which the feasible region is a set of the waiting time schemes, and beginning and closing times of the stuffs for work are alternated strictly with an invariable ratio (parameter). The objective function is represented by min-max operator. We study change of feasible region and minimum of the objective function according to change of the parameter $\lambda$. Moreover we study relations of the parametric programming problem and a multi-objective problem.***

**Keywords:** Discrete parametric min-max problem, Optimal control technique of filming process, Alternative production organization, Variable supply constraints of resources, cutting in multiple processing technique of software.

## 1. Introduction

Generally, parametric programming problems are those of mathematical programming problems which investigate how to change the optimal solutions according to the change of parameters contained in the problems.

The discrete parametric min-max problems appear in many fields of sciences, engineering, and productions.

A solving method for the linear min-max problems using the principle of simplex methods of linear programming has been studied in [1]. The discrete min-max problem with linear constraints and continuously differentiable objective functions $f_{i}(x),\ \ i=\overline{1,m}$ can be solved by using derived gradient methods [1, 2]. The papers [3, 4] proposed an algorithm for solving the location problem of the immediate service facilities described as a piecewise linear min-max problem.

In [5] authors considered the so-called machine-time scheduling problem for minimizing the losses due to violations of requirements at the beginning time of works in the manufacture production. They reduced the problem to a discrete min-max problem. Also, the resource allocation problems and position problems of service center can be formulate as the non-linear min-max problems, see [6, 7]. The solving methods for the discrete min-max problems in the case where $f_{i}(x)$ is quasi-convex, concave or quasi-monotonous are studied in [1, 8],and these methods are applied to the optimal design of the ultra-short waves' circuit, and the several technical control problems [8,9, 2]. Authors of [12,13] suggested the generalized discrete min-max fractional programming, and obtained the so-called parameter-free sufficient optimality conditions and the duality results.

Model and solving methods of the discrete min-max problems can be applied to solve system of convex inequalities $f_{i}(x)\le 0,\ \ i=\overline{1,m}$, since it is equivalent to the minimization problem of the function

$\varphi(x)=\max\left\{\max\limits_{1\le i\le m} f_{i}(x),\ 0\right\}$ See [1].

The heuristic search methods including the hereditary algorithms,ACO (Ant ColonyOptimization) are also widely used for solving the discrete min-max problems [10,14,15,16].

On the other hand, the discrete min-max principle is used to treat multiple objective optimization problems, and is studied in the relations between optimal solutions of the discrete min-max problems and Pareto optimal solutions of multi-objective programming problems [11].

In this paper we propose and analyze a discrete parametric programming problem in which the feasible region is set of the waiting time's scheme and objective function is represented by min-max operator or multi objective form, and beginning and closing times of the stuffs for work is alternated strictly in an invariable ratio (parameter).

This form of problem has occurred in a number of practical problems including alternative production organization which is enacted strictly according to the variable supply limit of resources, the design of CNC multi-complex processing team, the cutting in multiple processing technique of software, etc.

Now let's try to show an actual problem in the realization of reasonable control systems of the 1-comma filming process with a control computer.

The reasonable control systems of the line-drawing camera equipment, with a control computer require that the stuffs for work disposed by each implements have different process time and their orders of priority are arbitrary, finishing up quickly and accurately in the most short time for the guarantee crank speed.

An important way of ensuring the film's quality and crank speed, lowering the filming costs is that to make exactly the moving steps of camera implements, shorten the 1-comma crank time and prevent a lot of consumption of electricity at first starting time under the situation that many electric motors are used at the same time.

Generally, in the cinemas, the number of running scenes (moving picture) for one second is 24pieces, namely 24-comma. So, after all, it is important for us to shorten the 1-comma crank time in order to attain a high crank speed.
Usually, in taking a motion picture, it is important to keep up well the speeds of the different moving implements so that the effects designed previously may occur quickly and accurately, making to carry out mutually and independently the several actions at the same time. This problem can be solved only by controlling the stepping electro motors and the several moving implements with a control computer on the base of the reasonable control algorithms. However, the processing speeds of stuffs for work of the several moving implements are defined by the corresponding step's numbers of the stepping electro motors and these numbers are determined variously according to the structures of contents of scenarios. Besides, in the action process of the step electric motors there arephenomena of losing their impulses at the beginning and closing parts of work, and to remove these we must make the speed rates lower than the responding frequencies of the step electric motors, or equal to the latters. However, the running distances (step's number) and expending times of step electric motors become very long, since the frequencies are very low.

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1) We prevent the loss of steps by letting the step electric motor start smoothly with lower speed rate than the maximum responding frequency.
2) Next, let the step electric motor accelerate smoothly and gradually to the working frequency (the chain working frequency is usually 250~300 steps $/s$ ).
3) And then, let it pass for the most part of the whole distance with the chain working frequency.
4) After a while, let the speed rate decrease to that lower than the responding frequency before approaching to the end point.
5) Lastly, let it remove its sliding to stop exactly at the end point by letting the stepping electric motor run with speed rate less than the responding frequency for a short time.

Therefore, in the 1-comma filming process based on automatic accelerating and decelerating speed control principle as above, there occurs inevitably accelerating and decelerating timeat the work's starting and closing time of each moving implement, which is usually given in proportion to the number of its moving steps

And this process is required to be handled so as for its accelerating and decelerating time not to happen at the same time.

Let's consider the following optimization's problem on the basis of an actual task discussed above.

In an automatic filming process, let's suppose that a 1-comma filming **K** is completed by operating all the m machines $R_i, (i=\overline{1,m})$ performing the different tasks.

This time each machine $R_i, (i=\overline{1,m})$ deals individually with independent stuffs for work by means of step electric motors connected to control computer.

Let $\theta$ (second)denote the time required for one step's moving of each machine, then the time required for the 1-comma filming equal to $t'_i=\theta\cdot\tau_i,\ i=\overline{1,m}$. Denote as $t_i, (i=\overline{1,m})$ the rearrangement of the above data $t'_i,\ (i=\overline{1,m})$ in order of size(namely, $t_1 \ge t_2 \ge \cdots \ge t_m$ ).

The vector $t=(t_1,\ t_2,\ \cdots,\ t_m)$ is said to be time data of the stuffs for work.

Obviously, the time data of stuffs for work $t=(t_1,\ t_2,\ \cdots,\ t_m)$ is given differently according to changing the 1-comma filming scenarios **K**.

Now, let $\lambda$ be an allowed ratio of the accelerating and decelerating times of each machine $R_i, (i=\overline{1,m})$ required at the beginning and closing time of the works. We prevent the loss of steps by letting the step electric motor start smoothly with lower speed rate than the maximum responding frequency.
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In the whole process from the time of each machine beginning to works to the time of 1-comma filming ending each machine must be located at the suitable position, and it is denoted by waiting time $R_i, (i=\overline{1,m})$ decision variables $x=(x_1, x_2, \cdots, x_m)$, in dimension of which machine under the strict constraint condition that the accelerating and decelerating control cannot be performed at the same moment.

Each machine $R_i$ requires thetime $f_{i(x)} = x_i + t_i$, $(i=\overline{1,m})$ to handle its stuff for work.

We use the notation $x+t=(f_{1(x)}, f_{2(x)}, \cdots, f_{m(x)})$.

Now, the problem determining the waiting time scheme $x=(x_1, x_2, \cdots, x_m)$ of each machine $R_i$ so thattotal time to end 1-comma filming process (the time taken to the machine whose operation ends last) may be minimized, is formulated as a parametric discrete min-max problem ormulti-objective problem. More generally, we can consider problems of the following form.

We use following sets:

$$\hat{X} = \{x=(x_1, x_2, \cdots, x_m) \in R_+^m \ \big|\ x_i \ne x_j,\ (i \ne j),\ i,j=\overline{1,m}\},$$

$$\tilde{T} = \{t=(t_1, t_2, \cdots, t_m) \big|\ (t_i \in R_+ \setminus \{0\}, i=\overline{1,m}) \wedge (t_i > t_{i+1},\ i=\overline{1,m-1})\}.$$

For any $\lambda \in (0, 0.5)$ define following sets.

$$A_i = [x_i,\ x_i+\lambda t_i) \cup [x_i+(1-\lambda)t_i,\ x_i+t_i),\quad i=\overline{1,m}$$

$$C_i = [x_i+\lambda t_i,\ x_i+(1-\lambda)t_i),\quad i=\overline{1,m}$$

$$B_i = A_i \cup C_i,\quad i=\overline{1,m}.$$

A vector $t=(t_1, t_2, \cdots, t_m) \in \tilde{T}$ is called a time data of stuff for work. The feasible set is as follows:

$$G_{(\lambda;t)} = \{x \in \hat{X} \ \big|\ A_i \cap A_j = \phi,\ (i \ne j),\ i,j=\overline{1,m}\}.$$

For $x \in G_{(\lambda;t)}, t \in \tilde{T}$, the $f_{i(x)} = x_i + t_i$ is called the performance time of i-th stuff for work.

For the feasible region and objective function defined above, we consider the following problems.

$$\begin{cases} \min\limits_{x} \max\limits_{1\le i\le m} f_{i(x)} \\ x \in G_{(\lambda;t)} \end{cases} \quad (1.1)$$

$$\begin{cases} \min\limits_{x} (\ f_{1(x)},\ f_{2(x)}, \cdots, f_{m(x)}\ ) \\ x \in G_{(\lambda;t)} \end{cases} \quad (1.2)$$

**[Definition 1.1]** Let $(\lambda, t) \in (0, 0.5) \times \tilde{T}$.

If $\min\limits_{x\in G_{(\lambda;t)}} \max\limits_{1\le i\le m} f_{i(x)} = t_1, t_1 < \min\limits_{x\in G_{(\lambda;t)}} \max\limits_{1\le i\le m} f_{i(x)} < \sum\limits_{i=1}^{m} t_i$,

$\min\limits_{x\in G_{(\lambda;t)}} \max\limits_{1\le i\le m} f_{i(x)} = \sum\limits_{i=1}^{m} t_i$

Then we call the $x = \arg \min\limits_{x\in G_{(\lambda;t)}} \max\limits_{1\le i\le m} f_{i(x)}$ $\lambda$-foremost optimal solution, $\lambda$-middle optimal solution and $\lambda$-free optimal solution respectively. We denote the set of all $\lambda$-foremost optimal solutions, the set of all $\lambda$-middle optimal solutions and the set of all $\lambda$-free optimal solutions by $\hat{G}_{(\lambda;t)}$, $\bar{G}_{(\lambda;t)}$ and $\tilde{G}_{(\lambda;t)}$ respectively.

In this paper we study following problems.

First, we study the relations of variations of the feasible sets and optimal values of objective functions according to parameter $\lambda$.

Second, we propose a solving method of parametric discrete min-max problem (1.1) in the relation of multi objective problem (1.2).

Third, we study the theoretical and methodological problems how to obtain the critical intervals of parameter corresponding to the foremost, the middle and the worst (free) situations in the problem (1.1).

The paper is composed of as follows. In section 2, we give some definitions and consider the relations of variations of the feasible sets and optimal values of objective functions according to parameter $\lambda$, and prove some results in connection with sufficient optimality conditions. In section 3, based on the results of section 2, we obtain a solution of parametric discrete min-max problem (1.1) in the relation of multi objective problem (1.2). In section 4, we describe the research results of the third problem above.

## 2. Parameter-sufficient optimality conditions

First, we describe the basic conceptions and notations.

**[Definition 2.1]** Let $\lambda \in (0, 0.5)$ and $t \in \tilde{T}$ be given.

We denote $(1-2\lambda)t_i - t_j$ as $\Delta_{ij}$ for all $i, j$ $(i < j, i = 1, 2, \cdots, m-1, \ j = 2, 3, \cdots m)$.

If $\Delta_{ij} \geq 0$, then we say that $B_i$ is assignable $B_j$ by $\lambda$ in the first order and then $\Delta_{ij}$ is called first order freedom degree of assignation to $B_i$ of $B_j$. If $\Delta_{ij} < 0$, then we say that $B_i$ exclude $B_j$ by $\lambda$ in the first order and then $\Delta_{ij}$ is called first order freedom degree of exclusion by $B_i$ of $B_j$.

When $\Delta_{ij} > 0$, we denote the value $\Delta_{ij} - \lambda t_\ell$ simple as $\varepsilon_{ij\ell}$ for all $(i, j, \ell) \in \{1, \cdots, m-2\} \times \{2, \cdots, m-1\} \times \{3, \cdots m\}$ suchthat $i < j < \ell$.

If $\varepsilon_{ij\ell} \geq 0$, then we say that $B_i$ is assignable $B_j$ and $B_\ell$ by $\lambda$ in the second order and then $\varepsilon_{ij\ell}$ is called second order freedom degree of assignation to $B_i$ of $B_j$ and $B_\ell$.

If $\varepsilon_{ij\ell} < 0$, then we say that $B_i$ exclude $B_j$ and $B_\ell$ by $\lambda$ in the second order and then $\varepsilon_{ij\ell}$ is called second order freedom degree of exclusion of $B_j$ and $B_\ell$ to $B_i$.

From above definition, if $\Delta_{ij}$ is the first order freedom degree of assignation ($\Delta_{ij} \geq 0$), then $\lambda \leq \dfrac{t_i - t_j}{2t_i}$, if $\Delta_{ij}$ is the first order freedom degree of exclusion ($\Delta_{ij} < 0$), then $\lambda > \dfrac{t_i - t_j}{2t_i}$, and then the value $\dfrac{t_i - t_j}{2t_i}$ is called the first order ratio of assignation (exclusion) of $B_j$ to $B_i$, respectively.

Similarly, if $\varepsilon_{ij\ell} \geq 0$ then $\lambda \leq \dfrac{t_i - t_j}{2t_i + t_\ell}$, if $\varepsilon_{ij\ell} < 0$ then $\lambda > \dfrac{t_i - t_j}{2t_i + t_\ell}$.

$\dfrac{t_i - t_j}{2t_i}$ is called the first order ratio of assignation of $B_j$ to $B_\ell$.

And $\dfrac{t_i - t_j}{2t_i + t_\ell}$ is called second order ratio of assignation (exclusion) of $B_j$ and $B_\ell$ to $B_i$, respectively.

We denote $\dfrac{t_i - t_j}{2t_i}$ by $\lambda_{ij}$, $\dfrac{t_i - t_j}{2t_i + t_\ell}$ by $\mu_{ij\ell}$.

In particular, if $j = i+1$ and $\ell = i+2$, then we denote simply as $\Delta_{ij}$ (is called $i$-th first order adjoining freedom degree of assignation (exclusion)) and as $\varepsilon_{ij\ell}$ (is called $i$-th second order adjoining freedom degree of assignation (exclusion)) by $\Delta_i$ and $\varepsilon_i$, respectively.

Moreover, $\dfrac{t_i - t_{i+1}}{2t_i}$ and $\dfrac{t_i - t_{i+1}}{2t_i + t_{i+2}}$ are called $i$-th first order adjoining assignation (exclusion) ratio, and $i$-th second order adjoining assignation (exclusion) ratio, respectively. We denote $\dfrac{t_i - t_{i+1}}{2t_i}$ and $\dfrac{t_i - t_{i+1}}{2t_i + t_{i+2}}$ as $\lambda_i$, $\mu_i$, respectively. We use the following notations:

$$\lambda_* = \min_{1 \leq i \leq m-1} \lambda_i, \quad \lambda^* = \max_{1 \leq i \leq m-1} \lambda_i \quad (2.1)$$

$$\mu^* = \max_{1 \leq i \leq m-2} \mu_{i\,i+1\,m} \quad (2.2)$$

Notice that there exists $i_0 \in \{1, 2, \cdots, m-1\}$ such that for any $j \in \{i_0+1, i_0+2, \cdots, m\}$ $\Delta_{i_0 j} < \Delta_{i_0 j+1}$.

We denote simply this relation as ($\Delta_{i_0 j} \uparrow$).

**[Definition 2.2]** Let $\omega = (\lambda_1, \lambda_2, \cdots, \lambda_{m-1})$ ,where $\lambda_i$ is $i$-th first order adjoining assignationratio $(i = \overline{1, m-1})$.The components in $[\lambda_1, \lambda^*]$ of the vector are called assignation points of $\lambda_1$ corresponding to $\omega$. We call the number of assignation points of $\lambda_1$ the order of $\lambda_1$ and denote as $\underset{\lambda_1}{index}^+ \omega$.

In like manner, we call the components in $[\lambda_*, \lambda_1)$ exclusion points of $\lambda_1$ corresponding to $\omega$ and call their number the order (of exclusion) of $\lambda_1$ and denote as $\underset{\lambda_1}{index}^- \omega$.

Moreover, we denote the process reducing the number of exclusion points of $\lambda_1$ as $(\underset{\lambda_1}{index}^- \omega \downarrow)$.

**[Lemma 2.1]** For any $\lambda', \lambda'' \in (0, 0.5)$ such that

$\lambda' > \lambda''$ ,the followings hold true:

$G_{(\lambda'';t)} \supset G_{(\lambda';t)}$,

$\min_{x\in G(\lambda'';t)} \max_{1\le i\le m} f_{i(x)} \le \min_{x\in G(\lambda';t)} \max_{1\le i\le m} f_{i(x)}$.

**Proof**Let's denote $A_i$ and $C_i$ corresponding to $\lambda=\lambda''$ (and $\lambda=\lambda'=\lambda''+\varepsilon$ $(\forall\varepsilon>0)$) by $A''_i$, $C''_i$ , $(i=\overline{1,m})$ and $A'_i$ $C'_i$ respectively.

Then $A'_i \supset A''_i$, $C''_i \supset C'_i$, $(i=\overline{1,m})$.

Hence, $G_{(\lambda'';t)} \supset G_{(\lambda';t)}$,

so $\min_{x\in G(\lambda';t)} \max_{1\le i\le m} f_{i(x)} \ge \min_{x\in G(\lambda'';t)} \max_{1\le i\le m} f_{i(x)}$ ■

**[Corollary 2.2]**

Given $t\in\tilde{T}$, if $\min_{x\in G(\lambda';t)} \max_{1\le i\le m} f_{i(x)} = t_1$ for an $\lambda'\in(0,0.5)$, then $\min_{x\in G(\lambda'';t)} \max_{1\le i\le m} f_{i(x)} = t_1$ for any $\lambda''>0$ $(\lambda''<\lambda')$ too. ■

**[Corollary 2.3]**

Given $t\in\tilde{T}$, if $\min_{x\in G(\lambda';t)} \max_{1\le i\le m} f_{i(x)} = \sum_{i=1}^{m} t_i$ for an $\lambda'\in(0,0.5)$, then also

$\forall\lambda''<0.5\,(\lambda'<\lambda'')$, $\min_{x\in G(\lambda'';t)} \max_{1\le i\le m} f_{i(x)} = \sum_{i=1}^{m} t_i$ . ■

Define followings:

$$\hat{\lambda} = \min\left\{\lambda_1, \frac{t_1-t_m}{2\,(\sum_{i=1}^{m} t_i - t_m)}\right\} \quad (2.3)$$

$$\lambda^{\Delta} = \min\left\{\lambda^*, \frac{t_1}{2\sum_{i=1}^{m} t_i}\right\} \quad (2.4)$$

$$\tilde{\lambda} = \max\left\{\frac{1}{3}, \lambda_{1m}\right\} \quad (2.5)$$

**[Lemma 2.4]**

For any $t=(t_1,t_2,\cdots,t_m)\in\tilde{T}$,

$$\lambda_* \le \frac{t_1-t_m}{2(\sum_{i=1}^{m} t_i - t_m)} \le \lambda^* \quad (2.6)$$

$$\frac{t_1-t_m}{2(\sum_{i=1}^{m} t_i - t_m)} \le \frac{t_1}{2\sum_{i=1}^{m} t_i} \quad (2.7)$$

$$\lambda_{1m} > \lambda^* \quad (2.8)$$

**Proof**

First, prove (2.6). From the definition of $\lambda_*$, $\lambda^*$ for any $t\in\tilde{T}$,

$$\begin{cases} (1-2\lambda_*)t_i \ge t_{i+1}, & i=\overline{1,m-1} \\ (1-2\lambda^*)t_i \le t_{i+1}, & i=\overline{1,m-1} \end{cases} \Rightarrow$$

$$\begin{cases} \sum_{i=1}^{m-1}(1-2\lambda_*)\,t_i \ge \sum_{i=1}^{m-1} t_{i+1} \\ \sum_{i=1}^{m-1}(1-2\lambda^*)t_i \le \sum_{i=1}^{m-1} t_{i+1} \end{cases} \Rightarrow$$

$$\begin{cases} t_1 - t_m \ge 2\lambda_* \sum_{i=1}^{m-1} t_i \\ t_1 - t_m \le 2\lambda^* \sum_{i=1}^{m-1} t_i \end{cases} \Rightarrow$$

$$\begin{cases} t_1 \ge \sum_{i=1}^{m} 2\lambda_* t_i + (1-2\lambda_*)t_m \\ t_1 \le \sum_{i=1}^{m} 2\lambda^* t_i + (1-2\lambda^*)t_m \end{cases}$$

$$\Rightarrow \begin{cases} \lambda_* \le \dfrac{t_1-t_m}{2(\sum_{i=1}^{m} t_i - t_m)} \\ \lambda^* \ge \dfrac{t_1-t_m}{2(\sum_{i=1}^{m} t_i - t_m)} \end{cases}$$

Next, (2.7) is satisfied from the inequality

$$\frac{t_1}{2\sum_{i=1}^{m} t_i} - \frac{t_1-t_m}{2(\sum_{i=1}^{m} t_i - t_m)} = \frac{t_m(\sum_{i=1}^{m} t_i - t_1)}{2\sum_{i=1}^{m} t_i(\sum_{i=1}^{m} t_i - t_m)} > 0 \cdot$$

Finally, prove (2.8).

$$\frac{1}{2}\,(1-\frac{t_{i+1}}{t_i}) < \frac{1}{2}\,(1-\frac{t_m}{t_1}), \quad i=\overline{1,m-1},$$

because $\dfrac{t_{i+1}}{t_i} > \dfrac{t_m}{t_1}$, $i=\overline{1,m-1}$.

Hence, $\dfrac{t_i - t_{i+1}}{2t_i} < \dfrac{t_1-t_m}{2t_1}$, $i=\overline{1,m-1}$,

so $\max_{1\le i\le m-1}\{\lambda_i\} < \lambda_{1m}$. ■

From the inequalities (2.6)-(2.8), the following results are obtained.

$$\frac{t_1}{2\sum_{i=1}^{m} t_i} > \lambda_* \quad (2.9)$$

$$\lambda_{1m} > \frac{t_1-t_m}{2(\sum_{i=1}^{m} t_i - t_m)} \ge \lambda_* \quad (2.10)$$

From the **lemma 2.4**, for all $t\in\tilde{T}$,

$0 < \lambda_* \le \hat{\lambda} \le \lambda^{\Delta} < \tilde{\lambda} < 0.5$ (2.11)

This shows that the interval (0, 0.5) can be always divided into $0 < \lambda_* \le \lambda_1 \le \lambda^* < \tilde{\lambda} < 0.5$ for any $t \in \tilde{T}$.

**[Theorem 2.5]** (Parameter-sufficient foremost optimality condition)

Given $t \in \tilde{T}$, if $\lambda \in (0, \lambda_*]$, then

$\min_{x\in G(\lambda;t)} \max_{1\le i\le m} f_i(x) = \min_{x\in G(\lambda;t)} f_1(x) = t_1$.

**Proof**

From **corollary 2.2** it is sufficient to prove that $\min_{x\in G(\lambda^*;t)} \max_{1\le i\le m} f_i(x) = \min_{x\in G(\lambda^*;t)} f_1(x) = t_1$ at $\lambda = \lambda_*$.

According to (2.1), $(\forall t \in \tilde{T},\ \lambda = \lambda_*) \Rightarrow$ $(\lambda \le \lambda_i,\ i = \overline{1, m-1}) \Leftrightarrow (\Delta_i \ge 0,\ i = \overline{1, m-1}) \Leftrightarrow$ $((1-2\lambda)\, t_i \ge t_{i+1},\ i = \overline{1, m-1}) \Leftrightarrow$ ( $\mu(C_i) \ge \mu(B_{i+1})$, $i = \overline{1, m-1}$ ), where $\mu$ is length of interval.

Hence, selecting some $x_* \in G(\lambda_*;t)$ so that $x_1^* = 0,\ C_i \supset B_{i+1},\ i = \overline{1, m-1}$ we have $f_1(x_*) = t_1,\ f_i(x_*) > f_{i+1}(x_*)$, $i = \overline{1, m-1}$, where $x_1^*$ isthe first component of $x_*$.

Therefore, $\max_{1\le i\le m} f_i(x_*) = f_1(x_*) = t_1$, so

$\max_{1\le i\le m} f_i(x) \ge f_1(x) \ge t_1$ for all $x \in G(\lambda_*;t)$, hence,

$\min_{x\in G(\lambda_*;t)} \max_{1\le i\le m} f_i(x) \ge \min_{x\in G(\lambda_*;t)} f_1(x) \ge t_1$.

From thiswe have

$\min_{x\in G(\lambda_*;t)} \max_{1\le i\le m} f_i(x) = = \min_{x\in G(\lambda_*;t)} f_1(x) = t_1$. ■

**[Theorem 2.6]** (Parameter-sufficient free optimality condition)

Let $t \in \tilde{T}$.

- If $\frac{1}{3} > \lambda_{1m}$, then $\min_{x\in G(\lambda;t)} \max_{1\le i\le m} f_i(x) = \sum_{i=1}^{m} t_i$ for any $\lambda \in [\tilde{\lambda}, 0.5)$.
- If $\frac{1}{3} \le \lambda_{1m}$, then $\min_{x\in G(\lambda;t)} \max_{1\le i\le m} f_i(x) = \sum_{i=1}^{m} t_i$ for any $\lambda \in (\tilde{\lambda}, 0.5)$.

**Proof** According to **corollary 2.3**, it is sufficient to prove $\min_{x\in G(\tilde{\lambda};t)} \max_{1\le i\le m} f_i(x) = \sum_{i=1}^{m} t_i$ at $\lambda = \tilde{\lambda}$ in the first case and $\min_{x\in G(\lambda_{1m}+\varepsilon;t)} \max_{1\le i\le m} f_i(x) = \sum_{i=1}^{m} t_i$ at $\lambda = \lambda_{1m} + \varepsilon$ for all $\varepsilon \in (0, 0.5 - \lambda_{1m})$ in the second case.

In the first case, we have $\tilde{\lambda} = \frac{1}{3}$. Now, let $\lambda = \frac{1}{3}$, then $B_i \cap B_j = \phi,\ (i \ne j),\ i, j = \overline{1, m}$ for all $x \in G(\frac{1}{3};t)$ since $(\lambda = \frac{1}{3}) \Leftrightarrow ((1-2\lambda)\, t_i = \lambda\, t_i,\ i = \overline{1, m}) \Leftrightarrow$ ( $\mu(C_i) = \frac{\mu(A_i)}{2},\ i = 1, \cdots, m$ )

$(\lambda > \lambda_{1m}) \Leftrightarrow ((1-2\lambda) t_1 < t_m) \Leftrightarrow (\mu(C_1) < \mu(B_m))$ (2.12)

Hence, $\mu(\bigcup_{i=1}^{m} B_i) = \sum_{i=1}^{m} \mu(B_i)$ for all $x \in G(\tilde{\lambda};t)$ and then

$\min_{x\in G(\tilde{\lambda};t)} \max_{1\le i\le m} f_i(x) = \sum_{i=1}^{m} t_i$.

Next, in case of $\frac{1}{3} \le \lambda_{1m}$, using the fact that $(\lambda > \frac{1}{3}) \Leftrightarrow$ ( $\mu(C_i) < \frac{\mu(A_i)}{2},\ i = 1, \cdots, m$ ) at $\lambda = \lambda_{1m} + \varepsilon$ for any $\varepsilon \in (0,\ 0.5 - \lambda_{1m})$ and (2.12), we have $B_i \cap B_j = \phi,\ (i \ne j),\ i, j = \overline{1, m}$ for all $x \in G(\lambda_{1m}+\varepsilon;t)$.

Hence, $\mu(\bigcup_{i=1}^{m} B_i) = \sum_{i=1}^{m} \mu(B_i)$ for all $x \in G(\lambda_{1m}+\varepsilon;t)$ and finally

$\min_{x\in G(\lambda_{1m}+\varepsilon;t)} \max_{1\le i\le m} f_i(x) = \sum_{i=1}^{m} t_i$. ■

According to the **theorem 2.5** and the **theorem 2.6,** we know that the intervals $(0, \lambda_*]$ and $[\tilde{\lambda}, 0.5)$ ( $(\tilde{\lambda}, 0.5)$ in case of $\frac{1}{3} \le \lambda_{1m}$ ) are the parametric effective intervals where the optimal values of objective function approach best and worst (free) situations for all $t \in \tilde{T}$, respectively.

**[Theorem 2.7]** (Parameter-sufficient middle optimality condition)

If $\lambda \in (\lambda_1,\ \lambda_{1m}]$ then $t_1 < \min_{x\in G(\lambda;t)} \max_{1\le i\le m} f_i(x) < \sum_{i=1}^{m} t_i$ for all $t \in \tilde{T}$.

**Proof**

Since $(\lambda > \lambda_1) \Leftrightarrow ((1-2\lambda) t_1 < t_2) \Leftrightarrow (\mu(B_2) > \mu(C_1))$,

$\max_{1\le i\le m} f_i(x) > t_1$ for any $x \in G(\lambda;t)$ where $\lambda > \lambda_1$. Hence,

$\min_{x\in G(\lambda;t)} \max_{1\le i\le m} f_i(x) > t_1$ for all $\lambda \in (\lambda_1,\ \lambda_{1m}]$.

In addition, for any $t \in \tilde{T}$ there exists a point $\tilde{x} \in G(\lambda_{1m};t)$ such that

$\max_{1\le i\le m} f_i(\tilde{x}) = \sum_{i=1}^{m} t_i - t_m$. So,

$\min_{x\in G(\lambda_{1m};t)} \max_{1\le i\le m} f_i(x) \le \sum_{i=1}^{m} t_i - t_m < \sum_{i=1}^{m} t_i$.

Therefore, from the **lemma 2.1**, $\min\limits_{x\in G(\lambda;t)}\max\limits_{1\le i\le m} f_i(x) < \sum\limits_{i=1}^{m} t_i$ for all $\lambda \in (\lambda_1,\ \lambda_{1m}]$. ■

**[Definition 2.3]** Let $t$ be in $\tilde{T}$.

If $\exists \hat{I}_{(t)} \subsetneq (0,\ 0.5\ )$ $\min\limits_{x\in G(\lambda;t)}\max\limits_{1\le i\le m} f_i(x) = t_1$ for all $\lambda \in \hat{I}_{(t)}$, then $\hat{I}_{(t)}$ is called $\lambda$-foremost optimization interval, if $\exists \tilde{I}_{(t)} \subsetneq (0,\ 0.5\ )$, $\min\limits_{x\in G(\lambda;t)}\max\limits_{1\le i\le m} f_i(x) = \sum\limits_{i=1}^{m} t_i$ for all $\lambda \in \tilde{I}_{(t)}$, then $\tilde{I}_{(t)}$ is called $\lambda$-free optimization interval, if $\exists \bar{I}_{(t)} \subsetneq (0,\ 0.5\ )$, $t_1 < \min\limits_{x\in G(\lambda;t)}\max\limits_{1\le i\le m} f_i(x) < \sum\limits_{i=1}^{m} t_i$ forall $\lambda \in \bar{I}_{(t)}$, then $\bar{I}_{(t)}$ is $\lambda$-middle optimization interval.

From the **theorems 2.5-2.7** the intervals $(0, \lambda_*]$, $[\tilde{\lambda},\ 0.5)$ ($(\tilde{\lambda}, 0.5)$ in case of $\frac{1}{3} \le \lambda_{1m}$) and $(\lambda_1,\ \lambda_{1m}]$ are $\lambda$-foremost optimization interval, $\lambda$-freeoptimization interval and $\lambda$-middle optimization interval respectively.

From the **lemma 2.1** and **definition 2.3**, it can be easily verified that the following facts are true.

For all $\lambda', \lambda'' \in \hat{I}_{(t)}$ $(\lambda' < \lambda'')$, $\hat{G}_{(\lambda'';t)} \subset \hat{G}_{(\lambda';t)}$,

For all $\lambda', \lambda'' \in \bar{I}_{(t)}$ $(\lambda' < \lambda'')$, $\bar{G}_{(\lambda'';t)} \subset \bar{G}_{(\lambda';t)}$,

For all $\lambda', \lambda'' \in \tilde{I}_{(t)}$ $(\lambda' < \lambda'')$, $\tilde{G}_{(\lambda'';t)} = \tilde{G}_{(\lambda';t)}$.

We denote the supremum of $\lambda$-foremost optimization interval $\sup \hat{I}_{(t)}$ and the supremum of $\lambda$-middle optimization interval sup $\bar{I}_{(t)}$ as $P_{\hat{G}}(t)$ and $P_{\bar{G}}(t)$, respectively.

## 3. Solving the problem according to sufficient optimality condition

A set of $\lambda$-foremost optimal solutions $\hat{G}_{(\lambda;t)}$ defined in **section 2** can be also written as

$$\hat{G}_{(\lambda;t)} = \left\{x \in G_{(\lambda;t)} \middle| (x_1 = 0) \wedge (B_1 = \bigcup_{i=1}^{m} B_i)\right\}.$$

Given $\lambda \in (0,\ \lambda_*]$, denote the set

$$\{x \in \hat{G}_{(\lambda;t)} \,|\, C_i \supset B_{i+1},\quad i = \overline{1, m-1}\} \quad (3.1)$$

by $D_{(\lambda;t)}$. Then obviously $D_{(\lambda;t)} \subset \hat{G}_{(\lambda;t)}$.

Now,we consider thefollowing optimization problem for a given $\lambda \in (0,\ \lambda_*]$ :

$$\begin{cases} \min\limits_{x} f_1(x) \\ f_i(x) > f_{i+1}(x), \quad i = \overline{1, m-1} \quad (3.2) \\ x \in \hat{G}_{(\lambda;t)} \end{cases}$$

According to the **theorem 2.5** and its proving process we can see easily that any $x \in D_{(\lambda;t)}$ is the solution of (3.2) when $0 < \lambda \le \lambda_*$

**[Theorem 3.1]** The following feasible solution $\hat{x} \in \hat{G}_{(\lambda;t)}$ is the solution of (3.2) (the optimal solution of problem (1.1)) and the pareto-optimal solution of problem (1.2) for any $\lambda \in (0, \lambda_*]$:

$$\hat{x}_i = \begin{cases} 0, & i = 1 \\ \sum\limits_{n=1}^{i-1} (\lambda t_n + \Delta_n \sigma_n), & i = \overline{2, m} \end{cases}, \quad (3.3)$$

where $\sigma_n \in [0,\ 1]$, $(n = \overline{1, m-1})$ is arbitrary constant and $\Delta_n = (1-2\lambda)t_n - t_{n+1}, (n = \overline{1, m-1})$ is n-th first order adjoining freedom degree of assignation.

Furthermore, $\min\limits_{x\in G(\lambda;t)}\max\limits_{1\le i\le m} f_i(x) = \min\limits_{x\in G(\lambda;t)} f_1(x) = t_1$.

**Proof** First, prove that $\hat{x}$ is the solution of (3.2) when $0 < \lambda \le \lambda_*$. Obviously,

$$f_i(\hat{x}) - f_{i+1}(\hat{x}) = \sum_{n=1}^{i-1} (\lambda t_n + \Delta_n \sigma_n) + t_i -$$

$$[\sum_{n=1}^{i} (\lambda t_n + \Delta_n \sigma_n) + t_{i+1}] =$$

$$= (1-\lambda)\ t_i - (t_{i+1} + \Delta_i \sigma_i),\ \ i = \overline{1, m-1}\ .$$

And, we have $(1-2\lambda)\ t_i \ge t_{i+1} + \Delta_i \sigma_i$, $i = \overline{1, m-1}$, so $(1-\lambda)\ t_i - (t_{i+1} + \Delta_i \sigma_i) \ge \lambda\ t_i$, $i = \overline{1, m-1}$.

Hence, $f_i(\hat{x}) - f_{i+1}(\hat{x}) \ge \lambda t_i > 0$, $i = \overline{1, m-1}$, so $f_i(\hat{x}) > f_{i+1}(\hat{x})$, $i = \overline{1, m-1}$.

It is clear that $\hat{x} \in \hat{G}_{(\lambda;t)}$, and $f_1(\hat{x}) = t_1$, so $\min\limits_{x\in G(\lambda;t)} f_1(x) = f_1(\hat{x}) = t_1$.

Next, prove that $\hat{x}$ is the praetor-optimal solution of the problem (1.2).

Since

$$f_i(\hat{x}) = \hat{x}_i + t_i = \sum_{n=1}^{i-1} (\lambda t_n + \Delta_n \sigma_n) + t_i \le$$

$$\le \sum_{n=1}^{i-1} (\lambda t_n + \Delta_n) + t_i, \quad i = \overline{2, m},$$

letting $\varepsilon_i(\lambda) = \sum\limits_{n=1}^{i-1} (\lambda t_n + \Delta_n) + t_i$, $i = \overline{2, m}$, $\hat{x}$ is the optimal solution of the following $\varepsilon(\lambda)$-constraint problem.

$$\begin{cases} \min\limits_{x} f_1(x) \\ f_i(x) \le \varepsilon_i(\lambda), \quad i = \overline{2, m}, \\ x \in \hat{G}_{(\lambda;t)} \end{cases}$$

Therefore, $\hat{x}$ is the pareto-optimal solution of problem (1.2) for all $\lambda \in (0, \lambda_*]$. ■

**[Lemma 3.2]** Assume $m > 3$. Then there exists $t_0 \in \tilde{T}$ such that $\mu^* < \lambda_*$ and $\hat{G}(\lambda; t_0) = D(\lambda; t_0)$ for all $\lambda \in (\mu^*, \lambda_*]$.

**Proof** First, prove that there exists some $t_0 \in \tilde{T}$ such that $0 < \mu^* < \lambda_*$.

For this, it is sufficient to prove that there exist some $t_0 = (t_1^o, t_2^o, \cdots, t_m^o) \in \tilde{T}$ such that

$$t_i^o - t_{i+1}^o = d \quad (0 < d(\text{constant}) < \frac{t_m^o}{2(m-3)})$$

, $i = \overline{1, m-1}$ and $\lambda_1 > \mu_{m-2\ m-1\ m}$.

Indeed,

$$\lambda_i = \frac{t_i^o - t_{i+1}^o}{2t_i^o} = \frac{d}{2t_i^o}, \quad (i = \overline{1, m-1})$$ and

$$\mu_{i\ i+1\ m} = \frac{t_i^o - t_{i+1}^o}{2t_i^o + t_m^o} = \frac{d}{2t_i^o + t_m^o}, \quad (i = \overline{1, m-2})$$

are the monotonic increase series with $i$,

so $\lambda_* = \min_{1 \le i \le m-1} \{\lambda_i\} = \lambda_1 = \dfrac{d}{2t_1^o}$ and

$$\mu^* = \max_{1 \le i \le m-2} \{\mu_{i\ i+1\ m}\} = \mu_{m-2\ m-1\ m} = \frac{d}{2t_{m-2}^o + t_m^o}.$$

Moreover,

$$(d < \frac{t_m^o}{2(m-3)}) \to (2(m-3)d < t_m^o) \to$$
$$(2(t_1^o - t_{m-2}^o) < t_m^o) \to (2t_1^o < 2t_{m-2}^o + t_m^o).$$

Therefore, $\lambda_1 > \mu_{m-2\ m-1\ m}$.

Next, prove that $\hat{G}(\lambda; t_0) = D(\lambda; t_0)$ for all $\lambda \in (\mu^*, \lambda_*]$.

From the definition, $D(\lambda; t_0) \subset \hat{G}(\lambda; t_0)$ for all $\lambda \in (\mu^*, \lambda_*]$.

If there exists $\lambda \in (\mu^*, \lambda_*]$ and $\tilde{x} \in \hat{G}(\lambda; t_0)$ such that $\tilde{x} \notin D(\lambda; t_0)$, then $\tilde{x} \in \hat{G}(\lambda; t_0) \setminus D(\lambda; t_0)$. And then, $C_{i_0} \not\supset B_{i_0+1}$ at some $i_0 \in \{1, 2, \cdots, m-1\}$ and $B_1 = \bigcup_{i=1}^{m} B_i$. On the other hand,

$$(\lambda > \mu^*) \to (\lambda > \mu_{i\ i+1\ m}, i = \overline{1, m-2}) \to$$
$$(0 \le \mu(C_i) - \mu(B_{i+1}) < \min_{1 \le i \le m} \left\{ \frac{\mu(A_i)}{2} \right\}, i = \overline{1, m-1})$$

Therefore, either $C_i \supset B_{i+1}$ for all $i \in \{1, 2, \cdots, m-1\}$ or $B_1 \ne \bigcup_{i=1}^{m} B_i$.

This leads to contradiction. ■

**[Lemma 3.3]**

For $t_0$ in the **lemma 3.2** $\min_{x \in G(\lambda_* + \varepsilon; t_0)} \max_{1 \le i \le m} f_i(x) > t_1$, where $\varepsilon$ is arbitrary positive number.

**Proof.**

From the **lemma 3.2,** if $x \in \hat{G}(\lambda_*; t_0)$ then $x \in D(\lambda_*; t_0)$.

However, if $\lambda_{i_0} = \lambda_*$ at some $i_0 \in \{1, 2, \cdots, m-1\}$, then $B_{i_0+1} \not\subset C_{i_0}$ for all $x \in G(\lambda_{i_0} + \varepsilon; t_0)$ $(\forall \varepsilon > 0)$. Furthermore, from the proof of **lemma 3.2** when $\lambda > \mu^*$, $B_1 \ne \bigcup_{i=1}^{m} B_i$ for all $x \in G(\lambda_{i_0} + \varepsilon; t_0)$ and therefore $x \notin \hat{G}(\lambda; t_0)$. It is easy to see $\max_{1 \le i \le m} f_i(x) \ge t_1 + (\lambda_{i_0} + \varepsilon)\, t_m > t_1$, $\forall\ x \in G(\lambda_{i_0} + \varepsilon; t_0)$.

Hence, $\min_{x \in G(\lambda_* + \varepsilon; t_0)} \max_{1 \le i \le m} f_i(x) > t_1$ ■

**[Lemma 3.4]**

If there exist some $\tilde{t} \in \tilde{T}$ such that $\lambda_1 = \lambda_*$, then $\min_{x \in G(\lambda_1 + \varepsilon;\, \tilde{t})} \max_{1 \le i \le m} f_i(x) > t_1$ for all $\varepsilon > 0$.

**Proof**

Indeed, for all $x \in G(\lambda_1 + \varepsilon; \tilde{t})$, $(B_1 \not\supset B_2) \to$
$\to (B_1 \ne \bigcup_{i=1}^{m} B_i) \to (\max_{1 \le i \le m} f_i(x) > t_1)$ because $(\lambda = \lambda_1 + \varepsilon) \to$
$(\lambda > \lambda_1) \to (\Delta_1 < 0)$ for all $\varepsilon > 0$. On the other hand, $\max_{1 \le i \le m} f_i(x) \ge t_1 + (\lambda_1 + \varepsilon)\, t_2 > t_1$ forall $x \in G(\lambda_1 + \varepsilon; \tilde{t})$.

Hence, $\min_{x \in G(\lambda_1 + \varepsilon; \tilde{t})} \max_{1 \le i \le m} f_i(x) > t_1$ ■

According to the **lemma 3.2** and the **theorem 3.1,** we have the following result.

**[Corollary 3.5]**

There is $t_0 \in \tilde{T}$ (for which $\mu^* < \lambda_*$) such that a solution of the problem (1.1) is represented by the expression (3.3) for all $\lambda \in (\mu^*, \lambda_*]$.

We denote the sets of all $t \in \tilde{T}$ satisfying the conditions $\mu^* < \lambda_*$ or $\lambda_1 = \lambda_*$ as $\tilde{T}_{t_0}$, $\tilde{T}_{\tilde{t}}$, respectively. Then, using **lemma 3.3** and **lemma 3.4**, the following result is obtained.

**[Corollary 3.6]**

$\sup \hat{I}(t) = P_{\hat{G}}(t) = \lambda_*$ for all $t \in \tilde{T}_{t_0} \bigcup \tilde{T}_{\tilde{t}}$.

Now, let's find the optimal solution satisfying the parameter-free sufficient optimality condition. Denote the set of all permutations of the set $I = \{1, 2, \cdots, m\}$ by $\Pi_m$ ($|\Pi_m| = m!$) and let $\pi \in \Pi_m$,

$$\pi = \begin{pmatrix} 1 & 2 & \cdots & m \\ \alpha_1 & \alpha_2 & \cdots & \alpha_m \end{pmatrix}.$$

For any $\pi_1, \pi_2 \in \Pi_m$, the relation $\pi_1 \le \pi_2$ is defined as $\eta(\pi_1) \ge \eta(\pi_2)$ and the relation $x' \le x''$ for $x', x'' \in \tilde{G}(\lambda; t)$ by $\rho(x') \le \rho(x'')$, where $\eta(\pi)$ is the reversed number of $\pi$ and $\rho(x) = \sum_{i=1}^{m} x_i$ $(x \in \tilde{G}(\lambda; t))$.

Given any $\lambda \in [\tilde{\lambda}, 0.5)$, we define the mapping

$g:\Pi_m \to \tilde{G}_{(\lambda;t)}$ as follows : $x=g(\pi)(\pi\in\Pi_m)$, $x=(x_{\alpha_1},x_{\alpha_2},\cdots,x_{\alpha_m})^{wo}\in\tilde{G}_{(\lambda;t)}$, where $x_{\alpha_1}=0$, $x_{\alpha_i}=\sum_{n=1}^{i-1}t_{\alpha_n}, (i=\overline{2,m})$ and $(x_{\alpha_1},x_{\alpha_2},\cdots,x_{\alpha_m})^{wo}=(x_1,x_2,\cdots,x_m)$.

Then, g is an isomorphism of $\Pi_m$ onto $\tilde{G}_{(\lambda;t)}$.

**[Lemma 3.7]**

$|\tilde{G}_{(\lambda;t)}|=m!$ for all $\lambda\in[\tilde{\lambda},0.5)$ ($\lambda\in(\tilde{\lambda},0.5)$ in case of $\frac{1}{3}\le\lambda_{1m}$) and the solution of problem (1.1) giving the minimum value to $\rho_{(x)}$ exists uniquely.

**Proof.**

The fact that $|\tilde{G}_{(\lambda;t)}|=m!$ for all $\lambda\in[\tilde{\lambda},0.5)$ ($\lambda\in(\tilde{\lambda},0.5)$ in case of $\frac{1}{3}\le\lambda_{1m}$), is obvious because $\Pi_m$ and $\tilde{G}_{(\lambda;t)}$ are mutually isomorphism.

Besides, $\rho_{(x)}=\sum_{i=1}^m x_i=\sum_{i=1}^m x_{\alpha_i}=\sum_{i=1}^m\sum_{n=1}^{i-1}t_{\alpha_n}=$ $(m-1)t_{\alpha_1}+(m-2)t_{\alpha_2}+\cdots+2t_{\alpha_{m-2}}+t_{\alpha_{m-1}}=$ $\sum_{s=1}^{m-1}(m-s)t_{\alpha_s}\ge\sum_{s=1}^{m-1}(m-s)t_{m-s+1}$ for all $x\in\tilde{G}_{(\lambda;t)}$.

And, settling on $\pi_0=\begin{pmatrix}1 & 2 & \cdots & m\\ m & m-1 & \cdots & 1\end{pmatrix}$, we obtain

$$x_0=g_{(\pi_0)}=(0,\ x_{m-1},x_{m-2},\cdots x_2,x_1)^{wo}$$
$$=(x_1,x_2,\cdots,x_{m-1},0)$$
$$=(\sum_{n=2}^m t_n,\sum_{n=3}^m t_n,\cdots,\ t_m,0)\in\tilde{G}_{(\lambda;t)}.$$

Moreover, $\rho_{(x_0)}$ is the minimum value of $\rho_{(x)}$ since

$$\rho_{(x_0)}=(m-1)t_m+(m-2)t_{m-1}+\cdots+2t_3+t_2=\sum_{s=1}^{m-1}(m-s)t_{m-s+1}.$$

Hence, the minimum of $\rho_{(x)}$ is unique because $g$ is one to one mapping. ■

Now, consider the following problem when $\lambda\in[\tilde{\lambda},0.5)$ ($\lambda\in(\tilde{\lambda},0.5)$ in case of $\frac{1}{3}\le\lambda_{1m}$):

$$\begin{cases}\min_x f_{m(x)}\\ f_{i(x)}>f_{i+1(x)},\quad i=\overline{1,m-1}\\ x\in\tilde{G}_{(\lambda;t)}\end{cases}\quad(3.4)$$

Using the **lemma 3.7**, we obtain the following result.

**[Theorem 3.8]**

The following point $\tilde{x}\in\tilde{G}_{(\lambda;t)}$ is a solution of (3.4) (the solution of problem (1.1)) and the Pareto-optimal solution of problem (1.2) for all $\lambda\in[\tilde{\lambda},0.5)$ ($\lambda\in(\tilde{\lambda},0.5)$ in case of $\frac{1}{3}\le\lambda_{1m}$).

$$\tilde{x}_i=\begin{cases}0, & i=m\\ \sum_{n=i+1}^m t_n, & i=\overline{1,m-1}\end{cases},\quad(3.5)$$

And then $\min_{x\in G(\lambda;t)}\max_{1\le i\le m}f_{i(\tilde{x})}=\min_{x\in G(\lambda;t)}f_{1(\tilde{x})}=\sum_{i=1}^m t_i$.

**Proof.** For $\tilde{x}\in\tilde{G}_{(\lambda;t)}$ defined by (3.5), we have

$$f_{i(\tilde{x})}-f_{i+1(\tilde{x})}=\sum_{n=i+1}^m t_n+t_i-(\sum_{n=i+2}^m t_n+t_{i+1})=\quad,\ i=\overline{1,m-1}.$$
$$=t_{i+1}+\sum_{n=i+2}^m t_n+t_i-\sum_{n=i+2}^m t_n-t_{i+1}=t_i,$$

Hence, $f_{i(\tilde{x})}-f_{i+1(\tilde{x})}>0,\quad i=\overline{1,m-1}$.

And $f_{m(x)}=x_m+t_m\ge t_m$.

Moreover, $f_{m(\tilde{x})}=t_m$ and accordingly $\min_x f_{m(x)}=f_{m(\tilde{x})}=t_m$. Furthermore, the $\tilde{x}$ is an optimal solution of a following lexicographic problem:

$$\begin{cases}\min_x f_{\alpha_1(x)}\\ f_{\alpha_i(x)}=\sum_{n=1}^i t_{\alpha_n},\ (\alpha_i\in I\setminus\bigcup_{k=1}^{i-1}\{\alpha_k\};\ i=\overline{2,m})\\ x\in\tilde{G}_{(\lambda;t)}\end{cases}.$$

Hence, the point $\tilde{x}$ corresponding to $\pi_0=\begin{pmatrix}1 & 2 & \cdots & m\\ m & m-1 & \cdots & 1\end{pmatrix}\in\Pi_m$ is a Pareto-optimal solution of problem (1.2). ■

Given $t\in\tilde{T}$, the algorithm finding the optimal solution of problem (1.1) is as follows:

First, compute the values $\lambda_*$ and $\tilde{\lambda}$.

Second, divide the interval (0, 0.5) into three parts, namely $(0,\lambda_*],(\lambda_*,\tilde{\lambda})$ and $[\tilde{\lambda},0.5)$ ($(\tilde{\lambda},0.5)$ in case of $\frac{1}{3}\le\lambda_{1m}$).

Third, find the solutions of the problem (1.1) corresponding to the values of $\lambda$. If $\lambda\in(0,\lambda_*]$ then (3.3) gives a solution, if $\lambda\in[\tilde{\lambda},\ 0.5)$ ($\lambda\in(\tilde{\lambda},0.5)$ in case of $\frac{1}{3}\le\lambda_{1m}$) then (3.5) gives a solution.

## 4. The relaxation of parameter-sufficient optimality conditions

**[Lemma 4.1] Let** $t',\ t''\in\tilde{T}$.

If $t'=\alpha t''$ ($\alpha\in(0,+\infty)$),

then $\lambda'_{ij}=\lambda''_{ij}$ and $\mu'_{ij\ell}=\mu''_{ij\ell}$. ■

Now, define following set dependent on $t$.

$$\Lambda=\{\omega=(\lambda_1,\lambda_2,\cdots,\lambda_{m-1}) \;\big|\; \lambda_* \le \lambda_i \le \lambda^*,\ i=\overline{1,m-1}\ \} \quad (4.1)$$

Moreover, we define a mapping $\varphi:\tilde{T}\to\Lambda$ as follows:

$$\omega=(\lambda_1,\lambda_2,\cdots,\lambda_{m-1})=\varphi(t),$$

$$\lambda_i=\lambda_i(t)=\frac{t_i-t_{i+1}}{2\,t_i},\ i=\overline{1,m-1}.$$

Let's consider a family of sets $\{\tilde{T}_\omega\}_{\omega\in\Lambda}$, where $\tilde{T}_\omega=\varphi^{-1}_{(\omega)}\ (\omega\in\Lambda)$.

**[Lemma 4.2]** $\tilde{T}=\sum_{\omega\in\Lambda}\tilde{T}_\omega$ and $\tilde{T}_\omega$ is self-similar set, i.e. $t'=\alpha\, t''$ for all $t',t''\in\tilde{T}_\omega\ (\exists\alpha\in(0,+\infty))$.■

Let $J=\{1,2,\cdots,m-1\}$, $\gamma=\begin{pmatrix}1 & 2 & \cdots & m-1\\ k_1 & k_2 & \cdots & k_{m-1}\end{pmatrix}\in\Pi_{m-1},\omega\in\Lambda$.

We define a mapping $\phi:\omega\in\Lambda\to\gamma\in\Pi_{m-1}$ :

$$\lambda_i^{(k_i)} \square \underset{1\le j\le m-1}{mid}(k_i)\{\lambda_j\},\ k_i\in J\setminus\bigcup_{s=1}^{i-1}\{k_s\},\ i=\overline{1,m-1}, \quad (4.2)$$

where $mid(k_i)$: $k_i$-th middle index ($k_i$-th smaller number) operator considering even the repeated number and $\left|J\setminus\bigcup_{s=1}^{i-1}\{k_s\}\right|=m-i,\ i=\overline{1,m-1}$.

For example, $\underset{1\le j\le 3}{mid}(2)\{0.1,0.3,0.2\}=\lambda_3^{(2)}=0.2$, $\underset{1\le j\le 3}{mid}(2)\{0.1,0.3,0.3\}=\lambda_2^{(2)}=0.3$ and $\underset{1\le j\le 3}{mid}(3)\{0.1,0.3,0.3\}=\lambda_3^{(3)}=0.3$.

According to the above definition, $\underset{1\le j\le m-1}{mid}(1)\{\lambda_j\}=\min_{1\le j\le m-1}\{\lambda_j\}$ and $\underset{1\le j\le m-1}{mid}(m-1)\{\lambda_j\}=\max_{1\le j\le m-1}\{\lambda_j\}$.

Consider a family of sets $\{\Lambda_\gamma\}_{\gamma\in\Pi_{m-1}}$ where $\Lambda_\gamma=\phi^{-1}_{(\gamma)}\ (\gamma\in\Pi_{m-1})$,

$$\Lambda_\gamma=\{(\lambda_1^{(k_1)},\lambda_2^{(k_2)},\cdots,\lambda_{m-1}^{(k_{m-1})})\in\Lambda \;\big|\; \lambda_i^{(k_i)}= \underset{1\le j\le m-1}{mid}(k_i)\{\lambda_j\},k_i\in J\setminus\bigcup_{s=1}^{i-1}\{k_s\},\ i=\overline{1,m-1}\ \} \quad (4.3)$$

Let's denote the lexicographic order of $\gamma_1,\gamma_2$ as $\gamma_1\le\gamma_2$ and define the order $\omega_1\le\omega_2$ as $\omega_1\in\Lambda_{\gamma_1}$ and $\omega_2\in\Lambda_{\gamma_2}$.

Then the mapping $\phi:\Lambda\to\Pi_{m-1}$ is an order homomorphism and $\Pi_{m-1}$ is homomorphism to $\Lambda$.

The order $\omega'\le\omega''$ for all $\omega',\omega''\in\Lambda_\gamma$ means $\pi_{\gamma(\omega')}\le\pi_{\gamma(\omega'')}$, where $\pi_{\gamma(\omega)}=\sum_{i=1}^{m-1}\lambda_i^{(k_i)}$ (4.4)

Moreover, the order $\Lambda_{\gamma_1}\le\Lambda_{\gamma_2}$ for $\Lambda_{\gamma_1},\Lambda_{\gamma_2}\in\{\Lambda_\gamma\}_{\gamma\in\Pi_{m-1}}$ means $\gamma_1\le\gamma_2$.

**[Lemma 4.3]** $\Lambda=\sum_{\gamma\in\Pi_{m-1}}\Lambda_\gamma$ and $\Lambda$ it is a linearly ordered set. ■

Now, let's denote the composition mapping of $\varphi:\tilde{T}\to\Lambda$ and $\phi:\Lambda\to\Pi_{m-1}$ by $h=\phi\circ\varphi:\tilde{T}\to\Pi_{m-1}$ and let $\tilde{T}_{\omega_{(\gamma)}}=h^{-1}(\gamma)=(\phi\circ\varphi)^{-1}(\gamma),\ (\omega\in\Lambda_\gamma,\gamma\in\Pi_{m-1})$.

From the **lemma 4.2** and **lemma4.3 we have the** following result.

**[Theorem 4.4]** $\tilde{T}=\sum_{\gamma\in\Pi_{m-1}}\sum_{\omega\in\Lambda_\gamma}\tilde{T}_\omega,\ (\tilde{T}_{\omega_{(\gamma)}}=\sum_{\omega\in\Lambda_\gamma}\tilde{T}_\omega)$ and it is linearly ordered set with the following order in the following meaning. Let $t',t''\in\tilde{T}$. If $t'\in\tilde{T}_{\omega'},t''\in\tilde{T}_{\omega''},\omega'\ne\omega''$ then $t'\le t''$ means $\varphi(t')\le\varphi(t'')\ (\omega'\le\omega'')$ either, if $t',t''\in\tilde{T}_\omega$ then $t'\le t''$ means $t'_i\le t''_i,\ i=\overline{1,m}$. ■

Let $J_1=J\setminus\{1\}$ and denote the set of all permutation

$$\gamma_0=\phi_{(\omega)}=\begin{pmatrix}1 & 2 & \cdots & m-1\\ 1 & k_2 & \cdots & k_{m-1}\end{pmatrix},$$

$$k_i\in\begin{cases}J_1\ , & i=2\\ J_1\setminus\bigcup_{s=2}^{i-1}\{k_s\}, & i=\overline{3,m-1}\end{cases} \text{ as } \Pi_{m-1}^{(\gamma_0)}.$$

Then, $\Pi_{m-1}^{(\gamma_0)}\subset\Pi_{m-1}$.

**[Lemma 4.5]** $\sup\hat{I}_{(t)}=\lambda_*$ for all $t\in\tilde{T}_{\omega(\gamma)}$ where $\gamma\in\Pi_{m-1}^{(\gamma_0)}$. ■

Let $\gamma_e=\begin{pmatrix}1 & 2 & \cdots & m-1\\ 1 & 2 & \cdots & m-1\end{pmatrix}$, and denote

$\Lambda_{\gamma_e}=\phi^{-1}(\gamma_e)$ $\omega_{0(\gamma_e)}=\arg\min_{\omega\in\Lambda_{\gamma_e}}\pi_{\gamma_e(\omega)}=\arg\min_{\omega\in\Lambda_{\gamma_e}}\sum_{i=1}^{m-1}\lambda_i^{(i)}$ and $\tilde{T}_{\omega_0(\gamma_e)}=\varphi^{-1}_{(\omega_{0(\gamma_e)})}$.

**[Corollary 4.6]** $\sup\hat{I}_{(t)}=\lambda_*=P_{\hat{G}}(t)$ for all $t\in\tilde{T}_{\omega_{0(\gamma_e)}}$, and $\lambda_1=\lambda_*=P_{\hat{G}}(t)$.

Now, we consider the following problem.

$$P_0:\max(\lambda-\lambda_*) \quad (4.5)$$

$$\min_{x\in G(\lambda;t)}\max_{1\le i\le m} f_i(x)=t_1,\ \lambda\in[\lambda_*,\lambda_1] \quad (4.6)$$

If $\lambda_{opt}^{(P_0)}$ is the solution of $P_0$, then, obviously, $\lambda_{opt}^{(P_0)}=P_{\hat{G}}(t)$.

Now, let $\Lambda_q=\{\omega\in\Lambda \,|\, k_1=q\},\ q=\overline{1,m-1}$, then $\Lambda=\sum_{q\in J}\Lambda_q$ for the set $\{\Lambda_q\}_{q\in J}$

And according to the mapping $\omega=\varphi(t)$, $(\forall t\in\tilde{T})$ there exist three components of the vector $\omega\in\Lambda_q$ that equal $\lambda_*$, $\lambda_1^{(q)}$ and $\lambda^*$, respectively, and if $\lambda_1^{(q)}=\lambda_*\ (q=1)$, then $\lambda_{opt}^{(P_0)}=P_{\hat{G}}(t)=\lambda_*$ from the **lemma 4.5.**

Now, consider the method searching the solution

$\lambda_{opt}^{(P_0)} = P_{\hat{G}}(t)$ of problem $P_0$ when $\lambda_1^{(q)} \neq \lambda_* \ (q \neq 1)$, $\lambda_i^{(k_i)} \neq \lambda_j^{(k_j)} \ (i \neq j),\ i, j = \overline{1, m-1}$.

We know that, if $t_i > t_j$ for all $i, j \in \{2,3,\cdots,m-1\} \times \{3,4,\cdots,m\}$ such that $i < j$, then $\Delta_{1i} < \Delta_{1j}$ and vice versa. Furthermore, from the **definition 2.2**, obviously, $\underset{\lambda_1^{(q)}}{index}^{+}\omega = m-q$, $\underset{\lambda_1^{(q)}}{index}^{-}\omega = q-1$.

Therefore, the points for searching $P_{\hat{G}}(t)$ have to be chosen so that the number of exclusion points of $\lambda_1^{(q)}$ for components of $\omega$ belonging to $[\lambda_*, \lambda_1^{(q)})$ may be no more than ($q-1$), and along the direction in which the first order freedom degrees of assignation $\Delta_{1i}\ (i = \overline{2, m})$ increase ($\Delta_{1i} \uparrow$) (or $t_i, i = \overline{1, m}$ decrease).

**[Lemma 4.7]**

For all $\omega \in \sum_{q \in J} \Lambda_q : \omega = \varphi(t), (t \in \tilde{T})$, the solution $\lambda_{opt}^{(P_0)} = P_{\hat{G}}(t)$ of problem $P_0$ is the minimum value among the (q-1) exclusion points for $\lambda_1^{(q)}$ in the direction ( $\underset{\lambda_1^{(q)}}{index}^{-}\omega \downarrow \wedge \Delta_{1i} \uparrow$ ) and the corresponding second assignation ratios, where ( $\underset{\lambda_1^{(q)}}{index}^{-}\omega \downarrow \wedge \Delta_{1i} \uparrow$ ) is the direction along which $\underset{\lambda_1^{(q)}}{index}^{-}\omega$ decrease and $\Delta_{1i}$ increase.

**Proof**

First, prove that the solution $\lambda_{opt}^{(P_0)} = P_{\hat{G}}(t)$ of problem $P_0$ exists among the exclusion points for $\lambda_1^{(q)}$ belonging to the interval $[\lambda_*, \lambda_1^{(q)}]$ and the corresponding second assignation ratios.

Let's contradict the above conclusion, then, obviously,when $\exists\, q_0 \in M, \lambda_{opt}^{(P_0)} = P_{\hat{G}}(t) \in (\lambda_{\alpha_{q_0-1}}^{(q_0-1)},\ \lambda_{\alpha_{q_0}}^{(q_0)})$, accordingto the above assumption, $\exists\, i_p \in \{1, 2, \cdots, \alpha_{q_0-p}-1\}$, $\varepsilon_{i_p i_{p+1} \alpha_{q_0-p}+1} \geq 0$, $p = \overline{1, q_0-1}$ and $\Delta_j \geq 0$ for all $j \in \{q_0, q_0+1, \cdots, q\}$, where $M \square \{1, 2, \cdots, q\}$. Hence, from the **definition 2.1**, $\lambda_{opt}^{(P_0)} \leq \mu_{i_p i_{p+1} \alpha_{q_0-p}+1}$, $p = \overline{1, q_0-1}$ and $\lambda_{opt}^{(P_0)} \leq \lambda_j$ forall $j \in \{q_0, q_0+1, \cdots, q\}$.

Now, let $\delta = \min_{p,j}\{\mu_{i_p i_{p+1} \alpha_{q_0-p}+1},\ \lambda_j\}$,then, obviously,the condition (4.6) is satisfied also for $\lambda = P_{\hat{G}}(t) + \delta$. This means that $P_{\hat{G}}(t)$ is not the solution of problem $P_0$. Now, because the value of $\lambda$ decreases along the direction ( $\underset{\lambda_1^{(q)}}{index}^{-}\omega \downarrow \wedge \Delta_{1i} \uparrow$ ),the conclusion of the theorem is true. ■

Now, by using order relations

$0 < \lambda_{\alpha_1}^{(1)} < \lambda_{\alpha_2}^{(2)} < \cdots < \lambda_{\alpha_{q-1}}^{(q-1)} < \lambda_{\alpha_q}^{(q)} < \lambda_{\alpha_{q+1}}^{(q+1)} < \cdots$, $\cdots < \lambda_{\alpha_{m-2}}^{(m-2)} < \lambda_{\alpha_{m-1}}^{(m-1)}$

We can divide the interval $(0,\ \lambda^*]$ as follows:

$$(0,\ \lambda^*] = \left(\bigcup_{S=1}^{q} A_S\right) \cup \left(\bigcup_{\tau=1}^{m-q-1} B_\tau\right),$$

where $A_S = (\lambda_{\alpha_{S-1}}^{(s-1)},\ \lambda_{\alpha_S}^{(s)}],\ s = \overline{1, q}$; $B_\tau = (\lambda_{\alpha_{q+\tau-1}}^{(q+\tau-1)},\ \lambda_{\alpha_{q+\tau}}^{(q+\tau)}],\ \tau = \overline{1,\ m-q-1}$ and $\lambda_{\alpha_0}^{(0)} = 0,\ \lambda_{\alpha_1}^{(1)} = \lambda_*,\ \lambda_{\alpha_q}^{(q)} = \lambda_1^{(q)}$.

Denote $K \square \left\{\partial^+ A_S\right\}_{s \in M}$ ,where $\partial^+ A_S \square \lambda_{\alpha_S}^{(s)}$, $\alpha_S \in J \setminus \bigcup_{\nu=1}^{s-1}\{\alpha_\nu\},\ s = \overline{1,\ q}$,

$M_S \square M \setminus \bigcup_{\nu=1}^{q-S+1}\{s+\nu-1\},\ s = \overline{2,\ q}; J_{\alpha_{s_k}} \square J_1 \setminus \bigcup_{\nu=1}^{k}\{\alpha_{s_\nu}\},\ s_k \in M_q$

Now, we break the problem $P_0$ into the multistage problems as follows.

The **first**stage problem is as follows:

$$P_1: \begin{array}{l} \max(\lambda - \hat{\lambda}_1) \\ \min\limits_{x} \max\limits_{1 \leq i \leq \alpha_{s_1}+1} f_i(x) = t_1,\ \lambda \in [\hat{\lambda}_1,\ \lambda_1^{(q)}] \end{array},$$

where $s_1$ is that: $s_1 \in M_q,\ \alpha_{s_1} \in J_1,\ \alpha_{s_1} = \min_{S \in M_q}\{\alpha_S\}$, and $\hat{\lambda}_1 \square \lambda_{\alpha_{s_1}}^{(s_1)}$.

Let $\lambda_{opt}^{(P_1)}$ be a solution of the problem $P_1$ and consider the following **second** stage problem.

$$P_2: \begin{array}{l} \max(\lambda - \hat{\lambda}_2) \\ \min\limits_{x} \max\limits_{1 \leq i \leq \alpha_{s_2}+1} f_i(x) = t_1,\ \lambda \in [\hat{\lambda}_2,\ \lambda_{opt}^{(P_1)}] \end{array}.$$

Now, assume $\exists \hat{s}_1 \in \bigcup_{\nu=1}^{q-S_1+1}\{s_1+\nu-1\}$,

$$\lambda_{opt}^{(P_1)} \in \begin{cases} A_{\hat{s}_1}, & \hat{s}_1 \neq s_1 \\ \{\partial^+ A_{s_1}\}, & \hat{s}_1 = s_1 \end{cases}.$$

Then $\exists s_2 \in \hat{M}_1$, $\exists \alpha_{s_2} \in J_{\alpha_{s_1}}$, $\alpha_{s_2} = \min_{s \in \hat{M}_1}\{\alpha_s\}$ ,where

$\hat{M}_1 \square M \setminus \left(\bigcup_{n=1}^{q-\hat{S}_1+1}\{\hat{s}_1+n-1\} \bigcup \{s_1\}\right)$ and $\hat{\lambda}_2 \square \lambda_{\alpha_{s_2}}^{(s_2)}$.

Similarly, let $\lambda_{opt}^{(P_2)}$ be a solution of problem $P_2$ and consider the following **third** stage problem.

$$P_3: \begin{array}{l} \max(\lambda - \hat{\lambda}_3) \\ \min\limits_{x} \max\limits_{1 \leq i \leq \alpha_{s_3}+1} f_i(x) = t_1,\ \lambda \in [\hat{\lambda}_3,\ \lambda_{opt}^{(P_2)}] \end{array}.$$

Assume $\exists \hat{s}_2 \in \bigcup_{\nu=1}^{q-S_2+1}\{s_2+\nu-1\}$ , $\lambda_{opt}^{(P_2)} \in \begin{cases} A_{\hat{S}_2}, & \hat{s}_2 \neq s_2 \\ \{\partial^+ A_{S_2}\}, & \hat{s}_2 = s_2 \end{cases}$,

then

$\exists s_3 \in \hat{M}_2$ , $\exists \alpha_{S_3} \in J_{\alpha_{S_2}}, \alpha_{S_3} = \min_{s\in\hat{M}_2}\{\alpha_S\}$, where

$\hat{M}_2 \triangleq M \setminus \left(\bigcup_{n=1}^{q-\hat{S}_2+1}\{\hat{s}_2+n-1\}\bigcup\{s_1, s_2\}\right)$ and $\hat{\lambda}_3 \triangleq \lambda_{\alpha_{S_3}}^{(s_3)}$.

Generally, let $\lambda_{opt}^{(P_{k-1})}$ be a solution of problem $P_{k-1}$ and consider the following **k-th** stage problem.

$$P_k: \begin{matrix}\max(\lambda-\hat{\lambda}_k) \\ \min_x \max_{1\le i\le \alpha_{s_i}+1} f_i(x)=t_1,\ \lambda\in[\hat\lambda_k, \lambda_{opt}^{(P_{k-1})}]\end{matrix}.$$

Assume $\exists\hat{s}_{k-1}\in\bigcup_{\nu=1}^{q-S_{k-1}+1}\{s_{k-1}+\nu-1\}$, $\lambda_{opt}^{(P_{k-1})}\in\begin{cases}A_{\hat{S}_{k-1}}, & \hat{s}_{k-1}\neq s_{k-1},\\ \{\partial^+A_{S_{k-1}}\}, & \hat{s}_{k-1}=s_{k-1}\end{cases}$

then $\exists s_k\in\hat{M}_{k-1}$ , $\exists\alpha_{S_k}\in J_{\alpha_{S_{k-1}}}$, $\alpha_{S_k}=\min_{S\in\hat{M}_{k-1}}\{\alpha_S\}$, where

$\hat{M}_{k-1}\triangleq M\setminus\left((\bigcup_{n=1}^{q-\hat{S}_{k-1}+1}\{\hat{s}_{k-1}+n-1\})\bigcup(\bigcup_{r=1}^{k-1}\{s_r\})\right)$ and $\hat\lambda_k\triangleq\lambda_{\alpha_{S_k}}^{(s_k)}$.

Finally, let $\lambda_{opt}^{(P_{N-1})}$ be a solution of problem $P_{N-1}$, and consider the following **N-th** stage problem.

$$P_N: \begin{matrix}\max(\lambda-\hat{\lambda}_N) \\ \min_x \max_{1\le i\le \alpha_{S_N}+1} f_i(x)=t_1,\ \lambda\in[\hat\lambda_N, \lambda_{opt}^{(P_{N-1})}]\end{matrix}.$$

Assume

$\exists\hat{s}_{N-1}\in\bigcup_{\nu=1}^{q-S_{N-1}+1}\{s_{N-1}+\nu-1\}$ , $\lambda_{opt}^{(P_{N-1})}\in\begin{cases}A_{\hat{S}_{N-1}}, & \hat{s}_{N-1}\neq s_{N-1},\\ \{\partial^+A_{S_{N-1}}\}, & \hat{s}_{N-1}=s_{N-1}\end{cases}$

$\exists s_N\in\hat{M}_{N-1}$ , $(\hat{M}_{N-1}\setminus\{s_N\}=\varnothing)$,

then $\exists\alpha_{S_N}\in J_{\alpha_{S_{N-1}}}$ , $\alpha_{S_N}=\min_{s\in\hat{M}_{N-1}}\{\alpha_s\}$ , where

$\hat{M}_{N-1}\triangleq M\setminus\left((\bigcup_{n=1}^{q-\hat{S}_{N-1}+1}\{\hat{s}_{N-1}+n-1\})\bigcup(\bigcup_{r=1}^{N-1}\{s_r\})\right)$ and $\hat\lambda_N\triangleq\lambda_{\alpha_{S_N}}^{(s_N)}$.

**[Theorem 4.8]**

The value $\lambda_{opt}^{(P_k)}$ defined by

$$\lambda_{opt}^{(P_k)}=\max\bigcup_{j=1}^{\alpha_{S_k}-1}\{\beta_{kj}\,\mu_{j\,j+1\,\alpha_{S_k}+1}\},\quad k=\overline{1,N}\ (4.7)$$

is a solution of the partial problem $P_k$ $(k=\overline{1,N})$, where $\beta_{kj}$ is given as follows;

① If $\mu_{j\,j+1\,\alpha_{S_\nu}+1}\notin\bigcup_{t=1}^{k-1}\{\lambda_{opt}^{(P_t)}\}$, $j=\overline{1,\ \alpha_{S_k}-1}$ for all $\nu\in\{1,2,\cdots,k-1\}$, then $\beta_{kj}\triangleq\delta_{kj}$.

② If $\exists\tilde\nu\in\{1,2,\cdots,k-1\}$, $\exists j_0\in\{1,2,\cdots,\alpha_{S_k}-1\}$, $\mu_{j_0 j_0+1\alpha_{S_{\tilde\nu}}+1}\in\bigcup_{t=1}^{k-1}\{\lambda_{opt}^{(P_t)}\}$, then $\beta_{kj_0}=0$ and $(\alpha_{S_{\tilde\nu}}+1=\alpha_{S_k})\Leftrightarrow(\beta_{k\alpha_{S_{\tilde\nu}}}\triangleq\alpha_{k\alpha_{S_{\tilde\nu}}})$, $(\alpha_{S_{\tilde\nu}}+1\neq\alpha_{S_k})\Leftrightarrow(\beta_{k\alpha_{S_{\tilde\nu}}}\triangleq\gamma_{k\alpha_{S_{\tilde\nu}}})$, where

$$\delta_{kj}\triangleq\begin{cases}\dfrac{\lambda_{\alpha_{s_k}}^{(s_k)}}{\mu_{j\,j+1\,\alpha_{s_k}+1}}, & \mu_{j\,j+1\,\alpha_{s_k}+1}\le\lambda_{\alpha_{s_k}}^{(s_k)}\\ 1, & \lambda_{\alpha_{s_k}}^{(s_k)}<\mu_{j\,j+1\,\alpha_{s_k}+1}<\lambda_{opt}^{(P_{k-1})}\\ \dfrac{\lambda_{opt}^{(P_{k-1})}}{\mu_{j\,j+1\,\alpha_{s_k}+1}}, & \mu_{j\,j+1\,\alpha_{s_k}+1}\ge\lambda_{opt}^{(P_{k-1})}\end{cases}\quad(4.8)$$

$$\alpha_{k\alpha_{s_{\tilde\nu}}}\triangleq\begin{cases}\dfrac{\lambda_{\alpha_{s_k}}^{(s_k)}}{\mu_{\alpha_{s_{\tilde\nu}}\alpha_{s_k}\alpha_{s_k}+1}}, & \lambda_{\alpha_{s_{\tilde\nu}}\alpha_{s_k}+1}\le\lambda_{\alpha_{s_k}}^{(s_k)}\\ \dfrac{\lambda_{\alpha_{s_{\tilde\nu}}\alpha_{s_k}+1}}{\mu_{\alpha_{s_{\tilde\nu}}\alpha_{s_k}\alpha_{s_k}+1}}, & \lambda_{\alpha_{s_k}}^{(s_k)}<\lambda_{\alpha_{s_{\tilde\nu}}\alpha_{s_k}+1}<\lambda_{opt}^{(P_{k-1})}\\ \dfrac{\lambda_{opt}^{(P_{k-1})}}{\mu_{\alpha_{s_{\tilde\nu}}\alpha_{s_k}\alpha_{s_k}+1}}, & \lambda_{\alpha_{s_{\tilde\nu}}\alpha_{s_k}+1}\ge\lambda_{opt}^{(P_{k-1})}\end{cases}\quad(4.9)$$

$$\gamma_{k\alpha_{s_{\tilde\nu}}}\triangleq\begin{cases}\dfrac{\lambda_{\alpha_{s_k}}^{(s_k)}}{\mu_{\alpha_{s_{\tilde\nu}}\alpha_{s_{\tilde\nu}}+1\ \alpha_{s_k}+1}}, & \mu_{\alpha_{s_{\tilde\nu}}\alpha_{s_{\tilde\nu}}+2\alpha_{s_k}+1}\le\lambda_{\alpha_{s_k}}^{(s_k)}\\ \dfrac{\mu_{\alpha_{s_{\tilde\nu}}\alpha_{s_{\tilde\nu}}+2\alpha_{s_k}+1}}{\mu_{\alpha_{s_{\tilde\nu}}\alpha_{s_{\tilde\nu}}+1\ \alpha_{s_k}+1}}, & \lambda_{\alpha_{s_k}}^{(s_k)}<\mu_{\alpha_{s_{\tilde\nu}}\alpha_{s_{\tilde\nu}}+2\alpha_{s_k}+1}<\lambda_{opt}^{(P_{k-1})}\\ \dfrac{\lambda_{opt}^{(P_{k-1})}}{\mu_{\alpha_{s_{\tilde\nu}}\alpha_{s_{\tilde\nu}}+1\ \alpha_{s_k}+1}}, & \mu_{\alpha_{s_{\tilde\nu}}\alpha_{s_{\tilde\nu}}+2\alpha_{s_k}+1}\ge\lambda_{opt}^{(P_{k-1})}\end{cases}\quad(4.10)$$

**Proof** It is easily proved by using (4.8) − (4.10) and the **lemma 4.7** and the optimality principle for the problem $P_0$ (optimality principle for dynamical programming). ■

**[Remark 4.8-1]**

If $\exists\ k_0\in\{1,2,\cdots,N\}$, $\lambda_{opt}^{(P_{k_0})}\in A_1$,then $\lambda_{opt}^{(P_0)}=\lambda_{\alpha_1}^{(1)}=\lambda_*$.

**[Remark 4.8-2]**

From the **lemma 4.1** and the **lemma 4.7,** $\lambda_{opt}^{(P_0)}=P_{\hat{G}}(t')=P_{\hat{G}}(t'')$ for all $t', t''\in\tilde{T}$ such that $t'=\alpha t''$, $(\forall\alpha\in(0,+\infty))$.

Now, we describe a numerical example finding the solution $P_{\hat{G}}(t_0)$ of the problem $P_0$ and the corresponding solution $x_0\in\hat{G}(P_{\hat{G}}(t_0);t_0)$ of the problem (1.1) with the given data $t_0\in\tilde{T}$.

Let $t_0=(23, 17, 13, 12, 10, 8, 7)\in\tilde{T}$ (m=7). Then the vector $\omega_0=(\frac{3}{23},\frac{2}{17},\frac{1}{26},\frac{1}{12},\frac{1}{10},\frac{1}{16})\in\Lambda$ and the permutation $\gamma_0=\begin{pmatrix}1&2&3&4&5&6\\6&5&1&3&4&2\end{pmatrix}\in\Pi_6$ are determined according to some mappings $\omega_0=\varphi(t_0)$ and $\gamma_0=\phi_{(\omega_0)}$ ,respectively.

Hence, $\omega_0=(\lambda_1^{(6)},\lambda_2^{(5)},\lambda_3^{(1)},\lambda_4^{(3)},\lambda_5^{(4)},\lambda_6^{(2)})\in\Lambda_6$ and $\lambda_*=\lambda_3^{(1)}=\dfrac{1}{26}$, $\lambda_1=\lambda^*=\lambda_1^{(6)}=\dfrac{3}{23}$.

Now the problem $P_0$ is formulated as follows.

$$P_0: \begin{array}{l} \max\,(\lambda - \lambda_3^{(1)}) \\ \min\limits_{x \in G(\lambda;t_0)} \max\limits_{1 \le i \le 7} f_{i(x)} = t_1, \ \lambda \in [\lambda_3^{(1)}, \lambda_1^{(6)}] \end{array}$$

The first stage problem is

$$P_1: \begin{array}{l} \max\,(\lambda - \lambda_2^{(5)}) \\ \min\limits_{x} \max\limits_{1 \le i \le 3} f_{i(x)} = t_1, \ \lambda \in [\lambda_2^{(5)}, \lambda_1^{(6)}] \end{array}.$$

Since $s_1 = 5, \alpha_{s_1} = 2$ and $\mu_{12\alpha_{s_1}+1} = \mu_{123} = \dfrac{6}{59}$, we have $\mu_{12\alpha_{s_1}+1} < \lambda_2^{(5)} = \dfrac{2}{17}$.

Therefore, according to (4.8) $\beta_{11} = \dfrac{\lambda_2^{(5)}}{\mu_{12\alpha_{s_1}+1}}$ and using (4.7),

$$\lambda_{opt}^{(P_1)} = \lambda_2^{(5)} = \frac{2}{17}.$$

The second stage problem is

$$P_2: \begin{array}{l} \max\,(\lambda - \lambda_3^{(1)}) \\ \min\limits_{x} \max\limits_{1 \le i \le 4} f_{i(x)} = t_1, \ \lambda \in [\lambda_3^{(1)}, \lambda_{opt}^{(P_1)}] \end{array}.$$

Then $\lambda_3^{(1)} < \mu_{23\alpha_{s_2}+1} < \mu_{12\alpha_{s_2}+1} < \lambda_{opt}^{(P_1)}$,

since $s_2 = 1, \alpha_{s_2} = 3$ and $\mu_{12\alpha_{s_2}+1} = \mu_{124} = \dfrac{3}{29}$,

$\mu_{23\alpha_{s_2}+1} = \mu_{234} = \dfrac{2}{23}$, $\lambda_3^{(1)} = \dfrac{1}{26}$, and $\lambda_{opt}^{(P_1)} = \dfrac{2}{17}$.

Hence, according to (4.8) $\beta_{21} = \beta_{22} = 1$ and from (4.7) we obtain

$$\lambda_{opt}^{(P_2)} = \max\left\{\mu_{12\alpha_{s_2}+1}, \mu_{23\alpha_{s_2}+1}\right\} = \mu_{12\alpha_{s_2}+1} = \mu_{124} = \frac{3}{29}.$$

Also, considering the problem

$$P_3: \begin{array}{l} \max\,(\lambda - \lambda_4^{(3)}) \\ \min\limits_{x} \max\limits_{1 \le i \le 5} f_{i(x)} = t_1, \ \lambda \in [\lambda_4^{(3)}, \lambda_{opt}^{(P_2)}] \end{array},$$

we have $\lambda_4^{(3)} < \mu_{23\alpha_{s_3}+1} < \lambda_{opt}^{(P_2)} < \lambda_{35}$ since $s_3 = 3, \alpha_{s_3} = 4$

and $\mu_{23\alpha_{s_3}+1} = \mu_{235} = \dfrac{1}{11}$, $\lambda_{35} = \dfrac{3}{26}$, $\lambda_4^{(3)} = \dfrac{1}{12}$ and $\lambda_{opt}^{(P_2)} = \dfrac{3}{29}$.

Therefore, according to (4.8) and (4.9),

$$\beta_{31} = 0, \ \beta_{32} = 1 \text{ and } \beta_{33} = \frac{\lambda_{opt}^{(P_2)}}{\mu_{34\alpha_{s_3}+1}}.$$

And then, using (4.8), we obtain

$$\lambda_{opt}^{(P_3)} = \max\left\{\mu_{23\alpha_{s_3}+1}, \lambda_{opt}^{(P_2)}\right\} = \lambda_{opt}^{(P_2)} = \ .$$

$$= \mu_{12\alpha_{s_2}+1} = \mu_{124} = \frac{3}{29}$$

The fourth problem is

$$P_4: \begin{array}{l} \max\,(\lambda - \lambda_5^{(4)}) \\ \min\limits_{x} \max\limits_{1 \le i \le 6} f_{i(x)} = t_1, \ \lambda \in [\lambda_5^{(4)}, \lambda_{opt}^{(P_3)}] \end{array}.$$

Since $s_4 = 4, \alpha_{s_4} = 5$ and $\mu_{23\alpha_{s_4}+1} = \mu_{236} = \dfrac{2}{21}$,

$\mu_{35\alpha_{s_4}+1} = \mu_{356} = \dfrac{3}{34}$, $\lambda_5^{(4)} = \dfrac{1}{10}$ and $\lambda_{opt}^{(P_3)} = \dfrac{3}{29}$, we have $\mu_{35\alpha_{s_4}+1} < \mu_{23\alpha_{s_4}+1} < \lambda_5^{(4)} < \lambda_{opt}^{(P_3)}$.

Hence, from (4.8)—(4.10), $\beta_{41} = 0$, $\beta_{42} = \dfrac{\lambda_5^{(4)}}{\mu_{23\alpha_{s_4}+1}}$, $\beta_{43} = \dfrac{\lambda_5^{(4)}}{\mu_{34\alpha_{s_4}+1}}$ and using (4.7)

we have $\lambda_{opt}^{(P_4)} = \max\left\{\lambda_5^{(4)}\right\} = \lambda_5^{(4)} = \dfrac{1}{10}$.

The last stage problem is

$$P_5: \begin{array}{l} \max\,(\lambda - \lambda_6^{(2)}) \\ \min\limits_{x} \max\limits_{1 \le i \le 7} f_{i(x)} = t_1, \ \lambda \in [\lambda_6^{(2)}, \lambda_{opt}^{(P_4)}] \end{array}.$$

Since $s_5 = 2, \alpha_{s_5} = 6$ and $\mu_{23\alpha_{s_5}+1} = \mu_{237} = \dfrac{4}{41}$,

$\mu_{35\alpha_{s_5}+1} = \mu_{357} = \dfrac{1}{11}$, $\mu_{56\alpha_{s_5}+1} = \mu_{567} = \dfrac{2}{27}$, $\mu_{36\alpha_{s_5}+1} = \mu_{367} = \dfrac{5}{33}$,

$\lambda_6^{(2)} = \dfrac{1}{16}$ and $\lambda_{opt}^{(P_4)} = \lambda_5^{(4)} = \dfrac{1}{10}$ we have $\lambda_6^{(2)} < \mu_{56\alpha_{s_5}+1} < \mu_{35\alpha_{s_5}+1} < \mu_{23\alpha_{s_5}+1} < \lambda_{opt}^{(P_4)} < \mu_{36\alpha_{s_5}+1}$.

Hence, from (4.8)—(4.10), $\beta_{51} = 0$, $\beta_{52} = 1$, $\beta_{53} = \dfrac{\lambda_{opt}^{(P_4)}}{\mu_{34\alpha_{s_5}+1}}$ and $\beta_{55} = 1$.

The searching process of solution is finished because $\hat{M}_4 \backslash \{s_5\} = \varnothing$;

and we conclude that $\lambda_{opt}^{(P_0)} = \lambda_{opt}^{(P_5)} = \dfrac{1}{10} = 0.1$.

The distribution of first order adjoining assignation and exclusion points in the corresponding subintervals partitioned along the searching line, and the search directions are tabulated as follows.

TABLE 1

| $\Delta_{1i} \uparrow$ / index$^{-}\omega \downarrow$ | $t_1$ | $\lambda_1^{(6)}$ | $t_2$ | $\lambda_2^{(5)}$ | $t_3$ | $\lambda_3^{(1)}$ | $t_4$ | $\lambda_4^{(3)}$ | $t_5$ | $\lambda_5^{(4)}$ | $t_6$ | $\lambda_6^{(2)}$ | $t_7$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | 23 | 3/23 | 17 | 2/17 | 13 | 1/26 | 12 | 1/12 | 10 | 1/10 | 8 | 1/16 | 7 |
| (0,1/26] | | + | | + | | + | | + | | + | | + | |
| (1/26,1/16] | | + | | + | | − | | + | | + | | + | |
| (1/16,1/12] | | + | | + | | − | | + | | + | | − | |
| (1/12.1/10] | | + | | + | | − | | − | | ⊕ | | ⊖ | |
| (1/10,2/17] | | + | | ⊕ | | ⊖ | | ⊖ | | ⊖ | | − | |
| (2/17,3/23] | | ⊕ | | ⊖ | | − | | − | | − | | − | |

The signs "+" and " - " of **table 1** mean individually the state of assignation and exclusion of $\lambda_i^{(k_i)}$ in the corresponding subintervals partitioned along the search directions respectively and"⊕", and"⊖"denote the search nodes and "$\rightarrow$" indicate the search procedure and directions.

On the other hand, according to **remark 4.8-2** we are sure that

$$\exists\, t'_0 \in \tilde{T}_{\omega_0} : t'_0 = \alpha\, t_0, (\alpha = 0.5)\ ,$$

$$\lambda^{(P_0)}_{opt} = P_{\hat{G}}(t_0) = P_{\hat{G}}(t'_0) = \frac{1}{10}.$$

Now, we show the performance schedule of the work for stuff for work-time data $t'_0 = (11.5, 8.5, 6.5, 6, 5, 4, 3.5) \in \tilde{T}_{\omega_0}$ under $\lambda = P_{\hat{G}}(t'_0) = \frac{1}{10} = 0.1$ in followingGantt chart.

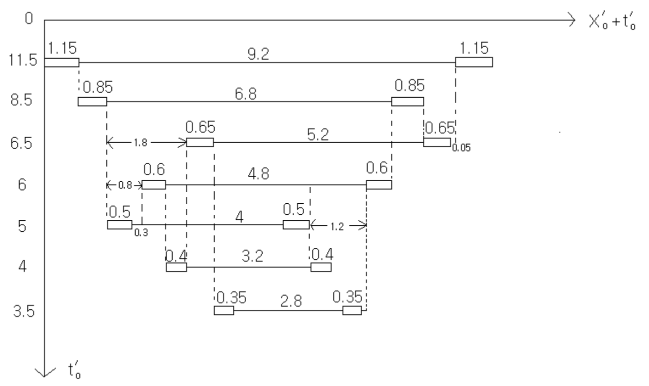


figure 1.

From the **figure 1** we have $x'_0 = \arg \min_{x' \in G(0.1; t'_0)} \max_{1 \le i \le 7} f_i(x') =$ $(0, 1.15, 3.8, 2.8, 2, 3.4, 4.45)$, $x_0 = \arg \min_{x \in G(0.1; t_0)} \max_{1 \le i \le 7} f_i(x) = 2x'_0 =$ $(0, 2.3, 7.6, 5.6, 4, 6.8, 8.9)$.

Now we consider a necessary and sufficient condition under which the solution of problem (1.1)-(1.2) is $\lambda$-free optimal solution.

**[Theorem 4.9]** ($\lambda$-free optimality necessary and sufficient condition)

Let $t \in \tilde{T}$. The point $x \in G_{(\lambda; t)}$ is $\lambda$-free optimal solution of the problem (1.1)-(1.2) if and only if

$$\left[ \left( \mu_{(C_i)} \ge \frac{\mu_{(A_j)}}{2} \right) \wedge \left( \mu_{(C_j)} \ge \frac{\mu_{(A_i)}}{2} \right), (i < j), \; i = \overline{1, m-1};\ j = \overline{2, m} \right] \wedge$$

$$\left( \max_{1 \le i \le m} \mu_{(C_i)} < \min_{1 \le i \le m} \mu_{(B_i)} \right) \quad (4.11)$$

**Proof**

$(\rightarrow)$ It is sufficient to prove that

$$\left[ \exists\, i_0, j_0 \in \{1, 2, \cdots, m\}, (i_0 < j_0), \left( \mu_{(C_{i_0})} \ge \frac{\mu_{(A_{j_0})}}{2} \right) \wedge \left( \mu_{(C_{j_0})} \ge \frac{\mu_{(A_{i_0})}}{2} \right) \right] \vee$$

$$\left( \max_{1 \le i \le m} \mu_{(C_i)} \ge \min_{1 \le i \le m} \mu_{(B_i)} \right) \Rightarrow \left( \min_{x \in G(\lambda; t)} \max_{1 \le i \le m} f_i(x) < \sum_{i=1}^{m} t_i \right)$$

(4.12)

By using

$$\mu_{(C_{i_0})} = (1 - 2\lambda) t_{i_0}, \left( \mu_{(C_{j_0})} = (1 - 2\lambda) t_{j_0} \right), \frac{\mu_{(A_{i_0})}}{2} = \lambda t_{i_0}, \left( \frac{\mu_{(A_{j_0})}}{2} = \lambda t_{j_0} \right)$$

$\max_{1 \le i \le m} \mu_{(C_i)} = (1 - 2\lambda) t_1$ and $\min_{1 \le i \le m} \mu_{(B_i)} = t_m$, we see (4.12) is equivalent to the following fact.

$$\left[ \left( \lambda \le \min\{ \frac{t_{i_0}}{t_{j_0} + 2t_{i_0}}, \frac{t_{j_0}}{t_{i_0} + 2t_{j_0}} \} \right) \vee \left( \lambda \le \lambda_{1m} \right) \right] \Rightarrow \quad (4.13)$$

$$\left( \min_{x \in G(\lambda; t)} \max_{1 \le i \le m} f_i(x) < \sum_{i=1}^{m} t_i \right)$$

Considering $\frac{t_{i_0}}{t_{j_0} + 2t_{i_0}} > \frac{1}{3}, \frac{t_{j_0}}{t_{i_0} + 2t_{j_0}} < \frac{1}{3}$ and letting $\lambda_0 = \min\{ \frac{t_{i_0}}{t_{j_0} + 2t_{i_0}}, \frac{t_{j_0}}{t_{i_0} + 2t_{j_0}} \} = \frac{t_{j_0}}{t_{i_0} + 2t_{j_0}}$, the (4.13) can be written as follows.

$$\left[ (\lambda \le \lambda_0) \vee (\lambda \le \lambda_{1m}) \right] \Rightarrow \left( \min_{x \in G(\lambda; t)} \max_{1 \le i \le m} f_i(x) < \sum_{i=1}^{m} t_i \right).$$

Then, we can choose some $x_0 \in G_{(\lambda_0; t)}$ so that

$$\max_{1 \le i \le m} f_i(x_0) = \sum_{i=1}^{m} t_i - (1 - \lambda_0) t_{j_0}.$$

Hence, $\min_{x \in G(\lambda_0; t)} \max_{1 \le i \le m} f_i(x) \le \sum_{i=1}^{m} t_i - (1 - \lambda_0) t_{j_0}$. Thus, from the **lemma 2.1** wehave $\min_{x \in G(\lambda; t)} \max_{1 \le i \le m} f_i(x) \le \sum_{i=1}^{m} t_i - (1 - \lambda_0)\, t_{j_0} < \sum_{i=1}^{m} t_i$ for all $\lambda \in (0, \lambda_0]$.

Similarly, we can choose some $\tilde{x} \in G_{(\lambda_{1m}; t)}$ so that

$$\max_{1 \le i \le m} f_i(\tilde{x}) = \sum_{i=1}^{m} t_i - t_m.$$

Hence, $\min_{x \in G(\lambda_{1m}; t^\circ)} \max_{1 \le i \le m} f_i(x) \le \sum_{i=1}^{m} t_i - t_m$.

From the **lemma 2.1**,

$\min_{x \in G(\lambda; t)} \max_{1 \le i \le m} f_i(x) \le \sum_{i=1}^{m} t_i - t_m < \sum_{i=1}^{m} t_i$ for all $\lambda \in (0, \lambda_{1m}]$.

$(\rightarrow)$ The condition (4.11) can be written as follows.

$$\left[ \left( \lambda > \min\{ \frac{t_i}{t_j + 2t_i}, \frac{t_j}{t_i + 2t_j} \} \right) \wedge \left( \lambda > \lambda_{1m} \right) \right]$$

$$\Leftrightarrow \left( \lambda > \max\{ \frac{t_j}{t_i + 2t_j}, \lambda_{1m} \} \right)$$

for all $(i, j) \in \{1, 2, \cdots, m-1\} \times \{2, 3, \cdots m\}$ such that $i < j$.

Now, let $\lambda_{ij}^{o} = \max\{\frac{t_j}{t_i + 2t_j},\ \lambda_{1m}\}$, then $B_i \bigcap B_j = \varnothing$ for all $x \in G_{(\lambda;t)}$ when $\lambda > \lambda_{ij}^{o}$.

Hence, $\mu(\bigcup_{i=1}^{m} B_i) = \sum_{i=1}^{m} \mu_{(B_i)}$ for all $x \in G_{(\lambda;t)}$ when $\lambda > \max \bigcup_{i=1}^{m-1} \bigcup_{j=i+1}^{m} \{\lambda_{ij}^{o}\}$.

Thus, letting $\lambda^{o} = \max \bigcup_{i=1}^{m-1} \bigcup_{j=i+1}^{m} \{\lambda_{ij}^{o}\}$ and considering **corollary 2.3,** we obtain $\min_{x \in G(\lambda;t)} \max_{1 \le i \le m} f_{i}(x) = \sum_{i=1}^{m} t_i$ for all $\lambda \in (\lambda^{o}, 0.5)$. ■

Now, we consider the searching method of critical parameter $\lambda^{o} = P_{\tilde{G}}(t)$ according to $t \in \tilde{T}$.

**[Lemma 4.10]**

For any $t \in \tilde{T}$ such that $t_1 \ge 3t_m$, we have $\lambda^{o} = P_{\tilde{G}}(t) = \lambda_{1m}$.

**Proof**

The condition (4.11) is equivalent to the following fact.

$$\lambda > \max \bigcup_{i=1}^{m-1} \bigcup_{j=i+1}^{m} \{\lambda_{ij}^{o}\},\ \forall\ (i,j) \in \{1,2,\cdots,m-1\} \times \{2,3,\cdots m\}:\ i < j$$

(4.14)

And, since $(t_1 \ge 3\,t_m) \Leftrightarrow \left(\lambda_{1m} \ge \frac{1}{3}\right)$ and $\frac{t_j}{t_i + 2t_j} < \frac{1}{3}$ for all $i, j \in \{1, 2, \cdots, m-1\} \times \{2, 3, \cdots m\}$ such that $i < j$, we have $\lambda_{ij}^{o} = \max\{\frac{t_j}{t_i + 2t_j},\ \lambda_{1m}\} = \lambda_{1m}$.

Therefore, $\lambda^{o} = \max \bigcup_{i=1}^{m-1} \bigcup_{j=i+1}^{m} \{\lambda_{ij}^{o}\} = \lambda_{1m}$ ■

**[Lemma 4.11]**

Let $t \in \tilde{T}$, $t_1 < 3t_m$. If $\bar{\lambda} \le \lambda_{1m}$ (or $\bar{\lambda} > \lambda_{1m}$), then $\lambda^{o} = P_{\tilde{G}}(t) = \lambda_{1m}$ (or $\lambda^{o} = P_{\tilde{G}}(t) = \bar{\lambda}$), where $\bar{\lambda} = \max \bigcup_{i=1}^{m-1} \bigcup_{j=i+1}^{m} \left\{\frac{t_j}{t_i + 2t_j}\right\}$.

**Proof** We use (4.14) equivalent to the condition (4.11). Since, if $\bar{\lambda} \le \lambda_{1m}$, then $\lambda_{ij}^{o} = \max\{\frac{t_j}{t_i + 2t_j},\ \lambda_{1m}\} = \lambda_{1m}$, wehave $\lambda^{o} = \max \bigcup_{i=1}^{m-1} \bigcup_{j=i+1}^{m} \{\lambda_{ij}^{o}\} = \lambda_{1m}$.

And, if $\bar{\lambda} = \max \bigcup_{i=1}^{m-1} \bigcup_{j=i+1}^{m} \left\{\frac{t_j}{t_i + 2t_j}\right\} > \lambda_{1m}$, then

$$\max \bigcup_{i=1}^{m-1} \bigcup_{j=i+1}^{m} \left\{\frac{t_j}{t_i + 2t_j}\right\} = \max \bigcup_{i=1}^{m-1} \bigcup_{j=i+1}^{m} \left\{\max\{\frac{t_j}{t_i + 2t_j},\ \lambda_{1m}\}\right\} = \max \bigcup_{i=1}^{m-1} \bigcup_{j=i+1}^{m} \{\lambda_{ij}^{o}\}$$

Hence, $\lambda^{o} = \max \bigcup_{i=1}^{m-1} \bigcup_{j=i+1}^{m} \left\{\frac{t_j}{t_i + 2\,t_j}\right\} = \bar{\lambda}$. ■

The algorithm finding According the critical parameter $\lambda^{o} = P_{\tilde{G}}(t)$ ($\forall t \in \tilde{T}$) is as follows:

Step 1. Find $\lambda_{1m}$ and $\bar{\lambda} = \max \bigcup_{i=1}^{m-1} \bigcup_{j=i+1}^{m} \left\{\frac{t_j}{t_i + 2\,t_j}\right\}$ according to the given date $t \in \tilde{T}$.

Step2. If $t_1 \ge 3t_m$ then let $P_{\tilde{G}}(t) = \lambda_{1m}$ .If $t_1 < 3t_m$ then go to step 3.

Step3. If $\bar{\lambda} \le \lambda_{1m}$ then $P_{\tilde{G}}(t) = \lambda_{1m}$, if $\bar{\lambda} > \lambda_{1m}$ then $P_{\tilde{G}}(t) = \bar{\lambda}$.

For example,let $\tilde{t}_0 = (50, 46, 40, 35) \in \tilde{T}$. Then

$\lambda_{14} = \frac{50 - 35}{2 \times 50} = 0.15$ and $\bar{\lambda} = \max \bigcup_{i=1}^{3} \bigcup_{j=i+1}^{4} \left\{\frac{t_j}{t_i + 2t_j}\right\} =$

$$= \max\left\{\frac{46}{50 + 2\times 46}, \frac{40}{50 + 2\times 40}, \frac{35}{50 + 2\times 35}, \frac{40}{46 + 2\times 40}, \frac{35}{46 + 2\times 35}, \frac{35}{40 + 2\times 35}\right\} = \frac{23}{71} \approx 0.324.$$

And, we see $t_1 < 3t_4$ since $t_1 = 50$ and $3t_4 = 105$.

Obviously, $\lambda_{14} < \bar{\lambda}$ ,hence $P_{\tilde{G}}(\tilde{t}_0) = \bar{\lambda} = \frac{23}{71} \approx 0.324$.

Finally, the relaxed $\lambda$- free optimization interval for a data $\tilde{t}_0 = (50, 46, 40, 35) \in \tilde{T}$ is $(\frac{23}{71},\ 0.5)$.

## 5. Conclusions

In this paper, we proposed a new form of parametric discrete min-max problem and studied the parameter sufficient optimality conditions for it and developed solving methods.

It is our view that the solved contents have the actual significance.

The results of the paper are useful to calculate the reasonable distributions of the jobs so that the total necessary time is minimize when thestuffs for workhave the structures and constraints enacted strictly according to the variable supply limit of resources.

Moreover, it is our expectation that those will contribute to the development of theory and method for the discrete parametric programming.

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