On the Non-Homogeneous Quintic with Five Unknowns $\left(x^2-y^2\right)\left(9x^2+9y^2-16xy\right)=21\left(X^2-Y^2\right)z^3$

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Abstract: ***Five different methods of the non-zero integral solutions of the Non homogeneous Quintic Diophantine equation with five unknowns*** $\left(x^2-y^2\right)\left(9x^2+9y^2-16xy\right)=21\left(X^2-Y^2\right)z^3$ ***are determined. Some interesting relations among the special numbers and the solutions are exposed.***

Keywords: The non homogeneous Diophantine equation, Quintic equation with five unknowns. Integral solutions, special numbers, a few interesting relation, 2010 Mathematics subject Classification: 11D09

Notations

- Tm,n - Polygonal number of rank n with size m
- P_n^m - Pyramidal number of rank n with size m
- gn - Gnomonic number of rank n
- Prn - Pronic number of rank n
- Ct16,n- Centered hexadecagonal pyramidal number of rank n
- OHn - Octahedral number of rank n
- SOn - Stella octangular number of rank n
- RDn - Rhombic dodecagonal number of rank n
- Hon - Hauy Octahedral number of rank n

1. Introduction

The number theory is the Queen of Mathematics. In particular, the Diophantine equations have a blend of attracted interesting problems. For an extensive review of variety of problems, one may refer to [1,9] the problem of finding all integer solutions of a Diophantine equation with three or more Variables and degree at least three, in general presents a good deal of difficulties. There is vast general theory of homogeneous quadratic equations with three variables [1-4,9]. Cubic equations in two variables fall in to the theory of elliptic curves which is a very developed theory but still an important topic of current research [5-7]. A lot is known about equations in two variables in higher degrees. For equations with more than three variables and degree at least three very little is known. It is worth to note that undesirability appears in equations, even perhaps at degree four with fairly small coefficients. In [1.8 and 9] a few higher order equations are considered for integral solutions .In this communication a seventh degree non-homogeneous equation with five variables represented by $\left(x^2-y^2\right)\left(9x^2+9y^2-16xy\right)=21\left(X^2-Y^2\right)z^3$ is considered and in particular a few interesting relations among the solutions are presented.

2. Description of Method

Consider the Quintic Diophantine equation

$$\left(x^2-y^2\right)\left(9x^2+9y^2-16xy\right)=21\left(X^2-Y^2\right)z^3 \quad (1)$$

Introduce of the linear transformation

$$x=u+v,\ y=u-v, X=2u+v,\ Y=2u-v \quad (2)$$

In (1) leads to

$$u^2+17v^2=21z^3 \quad (3)$$

Now we solve (3) through various choices and the different methods of solutions of (1)
are obtained as follows.

2.1 Method: I

Assume

$$z=z(a,b)=a^2+17b^2 \quad (4)$$

Where a & b are non zero distinct integers
Write 13 as

$$21=\left(2+i\sqrt{17}\right)\left(2-i\sqrt{17}\right) \quad (5)$$

Using, (4) and (5) in (3) and applying the method of factorization. Define

$$(u+i\sqrt{17}v)=\left(2+i\sqrt{17}\right)\left(a+i\sqrt{17}\,b\right)^2$$

Equating real and imaginary parts, we get

$$u=u(a,b)=2a^3-102ab^2-51a^2b+289b^3$$

$$v=v(a,b)=a^3-51ab^2+6a^2b-34b^3$$

Hence in view of (2) the corresponding solutions of (1) are given by

$$x=x(a,b)=3a^3-153ab^2-45a^2b+255b^3$$

$$y=y(a,b)=a^3-51ab^2-57a^2b+323b^3$$

$$X=X(a,b)=5a^3-255ab^2-96a^2b+544b^3$$

$$Y=Y(a,b)=3a^3-153ab^2-108a^2b+612b^3$$

$$z=z(a,b)=a^2+17b^2$$

Observations:-

1. $3X(a,1)-5Y(a,1)-252T_{4,a} \equiv 7 \pmod{204}$
2. $x(a,1)-3y(a,1)-126T_{4,a} \equiv 0 \pmod{2}$
3. $3z(3a,3a)$ *a Nasty number*
4. $x(a+1,1)+y(a+1,1)-RD_a-84T_{4,a}-G_{200a} \equiv 0 \pmod{2}$
5. $3X[a(a+1),1)]-5Y[a(a+1),1)]-252(P_a)^2 \equiv 0 \pmod{2}$
6. $2[x(a+1),1)-y(a+1),1)]-3HO_a-42T_{4,a}+G_{76a} \equiv 0 \pmod{2}$

2.2 Method: II

Instead of (5) Write 21 as

$$21=\frac{1}{25}\left(10+i5\sqrt{17}\right)\left(10-i5\sqrt{17}\right) \quad (6)$$

Following the procedure similar to Method-I, the corresponding non-zero distinct integral solutions of (1) are found to be

$$x=x(a,b)=375a^3-19125ab^2-5625a^2b+31875b^3$$
$$y=y(a,b)=125a^3-6375ab^2-7125a^2b+40375b^3$$
$$X=X(a,b)=875a^3-44625ab^2-18375a^2b+104125b^3$$
$$Y=Y(a,b)=625a^3-31875ab^2-4125a^2b+23375b^3$$
$$z=z(a,b)=25a^2+425b^2$$

Observations:-

1. $\frac{1}{10}z(2a,2a)$ *is a Nasty number.*
2. $x(a,1)-3y(a,1)-15750T_{4,a} \equiv 0 \pmod{2}$
3. $625X\left[(a(2a^2-1),1)\right]-875Y\left[(a(2a^2-1),1)\right]+7875000\,[SO_a]^2 \equiv 0 \pmod{2}$
4. $z[a+1,a+1]-450T_{4,a}-G_{45a} \equiv 11 \pmod{41}$
5. $625X[a(a+1),1]-875Y[a(a+1),1]+(P_a)^2 \equiv 7 \pmod{408}$
6. $x[(2a-1),1]-3y[(2a-1),1]-15750(G_a)^2 \equiv 0 \pmod{2}$

2.3 Method: III

Rewrite (3) as $\quad u^2+17v^2=21z^3 * 1$

Write 1 as

$$1=\frac{1}{81}\left(8+i\sqrt{17}\right)\left(8-i\sqrt{17}\right)$$

Following the procedure similar to Method-I, the corresponding non-zero distinct integral solutions of (1) are found to be

$$x=x(a,b)=15309a^3-780759ab^2-45927a^2b+260253b^3$$
$$y=y(a,b)=11907a^3-607257ab^2-127575a^2b+722925b^3$$
$$X=X(a,b)=28917a^3-1474767ab^2-132678a^2b+751842b^3$$
$$Y=Y(a,b)=25515a^3-1301265ab^2-214326a^2b+1214514b^3$$
$$z=z(a,b)=81a^2+1377b^2$$

Observations:-

1. $\frac{1}{72}Z(2a,2a)$ *is a Perfect Square*
2. $\frac{1}{18}Z(1,1)$ *is a Cubic integer*
3. $\frac{1}{183709}[x(1,1)-y(1,1)]$ *is a Cubic integer*

2.4 Method: IV

Instead of (5) Write 21 as

$$21=\frac{1}{49}\left(14+i7\sqrt{17}\right)\left(14-i7\sqrt{17}\right)$$

Following the procedure similar to Method-I, the corresponding non-zero distinct integral solutions of (1) are found to be

$$x=x(a,b)=1029a^3-52479ab^2-15435a^2b+87465b^3$$
$$y=y(a,b)=343a^3-17493ab^2-19551a^2b+110789b^3$$
$$X=X(a,b)=1715a^3-87465ab^2-32928a^2b+18659b^3$$
$$Y=Y(a,b)=1029a^3-52479ab^2-37044a^2b+209916b^3$$
$$z=z(a,b)=25a^2+425b^2$$

Observations:

1. $343x(a,1)-1029(a,1)-14823774T4,a \equiv 17 \pmod{4941258}$
2. $\frac{1}{5713}[X(1,1)+2Y(1,1)]$ *is a Perfect Square*
3. $1029X[a(a+1),1]-1715Y[a(a+1),1]-29647548(P_a)^2 \equiv 349 \pmod{976521}$
4. $\frac{1}{4116}[x(1,1)+y(1,1)]$ *is a woodall number.*
5. $\frac{1}{162}z[9a,9a]$ *is a perfect square*

2.5 Method: V

Instead of (5) Write 21 as

$$21=\frac{1}{64}\left[\left(16+i8\sqrt{17}\right)\left(16-i8\sqrt{17}\right)\right]$$

Following the procedure similar to Method-I, the corresponding non-zero distinct integral solutions of (1) are found to be

$$x=x(a,b)=1536a^3-78336ab^2-23040a^2b+130560b^3$$
$$y=y(a,b)=512a^3-26112ab^2-29184a^2b+165376b^3$$
$$X=X(a,b)=2560a^3-130560ab^2-49152a^2b+278528b^3$$
$$Y=Y(a,b)=1536a^3-78336ab^2-55296a^2b+313344b^3$$

Observations:

1. $\frac{1}{524288}[512x(a,1)-1536y(a,1)-33030144T_{4,a}]-21$ *is a nasty number*
2. $1536X[a(a+1),1]-2560Y[a(a+1),1]-66060288(P_a)^2 \equiv 0 \pmod{2}$
3. $\frac{1}{18}z[2a,2a]$ *is a Perfect square*
4. $1536X(a,1)-2560Y(a,1). \equiv 0 \pmod{2}$

3. Conclusion

In linear transformation (2), the variables X and Y may also be represented by X=2uv+1, Y=2uv-1.Applying the procedure similar to that of Method I-V choices of integral solutions to (1) are obtained. To conclude, one may search for other method of solutions and their corresponding properties.

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