The Generalization of Lowndes' Operators in Cos hx

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Abstract: ***In this paper, I have defined Generalized Lowndes' operators and Hankel operator involving Cos h x, some product relations of the operators and inversion of the operators have been discussed. A formal solution of dual integral equation associated by Cos hx have been discussed.***

Keywords: Lowndes' operator, Hyperbolic function, Bessel function, Hankel operator, Erdélyi-Kober operators.

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1. Introduction

In this paper I introduce a generalized Hankel operator and generalized Erdélyi-Kober operators of hyperbolic function and deduce some relations between them. The operators are then applied to obtain solutions to some dual integral equations which have applications in diffraction theory. The operators and methods followed essentially due to Lowndes [3]. The analysis throughout is formal.

Burlak[1] has shown that if - 1 < β < 0, then the solutions of the integral equations.

$$\int_0^x \psi(y)(x^2-y^2)^{\frac{1}{2}\beta} J_\beta\left\{k\sqrt{(x^2-y^2)}\right\}dy = h(x) \tag{1.1}$$

and

$$\int_x^\infty \psi(y)(y^2-x^2)^{\frac{1}{2}\beta} J_\beta\left\{k\sqrt{(y^2-x^2)}\right\}dy = m(x) \tag{1.2}$$

are given by :

$$\psi(x) = k\frac{d}{dx}\int_0^x yh(y)(x^2-y^2)^{-\frac{1}{2}(\beta+1)} I_{-(\beta+1)} \cdot\left\{k\sqrt{(x^2-y^2}\right\}dy \tag{1.3}$$

and

$$\psi(x) = -k\frac{d}{dx}\int_x^\infty ym(y)(y^2-x^2)^{-\frac{1}{2}(\beta+1)} I_{-(\beta+1)} \cdot\left\{k\sqrt{(y^2-x^2}\right\}dy \tag{1.4}$$

respectively. If J_β the Bessel function of the first kind, is replaced by I_β, the modified Bessel function of the first kind, in equations (1.1) and (1.2). Then $I_{-(\beta+1)}$ must be replaced by $J_{-(\beta+1)}$ in equations (1.3) and (1.4). The condition $-1<\beta<0$ was not stated by Burlak but was pointed out in a more recent paper by Srivastav[5].

We introduce the generalized hankel transform of the form

$$S\begin{pmatrix}\cosh a, \cosh k, \cosh k\\ \eta, \quad \alpha, \quad \sigma\end{pmatrix} f(x) = 2^{\alpha}(\cosh x)^{2\sigma-\alpha}(\cosh^2 x-\cosh^2\alpha)$$
$$\times\int_x^\infty(\cosh u)^{1-2\alpha-\sigma}(\cosh^2 u-\cosh^2 k)^{\sigma} f(u).J_{2\eta+\alpha}\{\chi_1\}du, \tag{1.5}$$

where

$$\chi_1 = \left\{\left[\left(\cosh^2 x-\cosh^2 a\right)\left(\cosh^2 u-\cosh^2 b\right)\right]\right\}^{1/2}$$

which is related to the modified Hankel operator $S_{\eta,\alpha}$ defined in Sneddon[4] and Lowndes' operators defined in Lowndes'[3].

Applying the Hankel inversion theorem Sneddon[4] to the equation

$$S\begin{pmatrix}0, \cosh k, \cosh k\\ \eta, \quad \alpha, \quad \sigma\end{pmatrix} f(x) = g(x)$$

We find that

$$f(u) = S\begin{pmatrix}\cosh k, & 0, & 0\\ \eta+\alpha, & -\alpha, & \sigma\end{pmatrix} g(u)$$

and hence an inversion theorem for the generalized operator of the Hankel transform can be written in the form

$$S^{-1}\begin{pmatrix}0, \cosh k, \cosh k\\ \eta, \quad \alpha, \quad \sigma\end{pmatrix} = S\begin{pmatrix}\cosh k, & 0, & 0\\ \eta+\alpha, & -\alpha, & \sigma\end{pmatrix} \tag{1.6}$$

when k = 0, I see that the above equation becomes

$$S\begin{pmatrix}0 & 0 & 0\\ \eta, & \alpha & \sigma\end{pmatrix} = S\begin{pmatrix}0, & 0, & 0\\ \eta+\alpha, & 0, & 0\end{pmatrix} = S\eta,\alpha \tag{1.7}$$

The generalized Lowndes' operators $\Im(\eta,\alpha)$ and $\Re(\eta,\alpha)$ by the formulae

$$\Im_k(\eta,\alpha) f(x) = 2^{\alpha}(\cosh x)^{-2\eta-2\alpha}(\cosh k)^{1-\alpha}\int_0^x (\cosh u)^{1+2\eta}$$
$$\times(\cosh^2 x - \cosh^2 u)^{½(\alpha-1)}$$
$$J_{\alpha-1}\left\{\cosh k\sqrt{(\cosh^2 x - \cosh^2 u)}\right\} f(u)du, \quad (1.8)$$

$$\Re_k(\eta,\alpha) f(x) = 2^{\alpha}(\cosh x)^{2\eta}(\cosh k)^{1-\alpha}\int_x^{\infty}(\cosh u)^{1-2\eta-2\alpha}$$
$$\times(\cosh^2 u - \cosh^2 x)^{½(\alpha-1)}$$
$$J_{\alpha-1}\left\{k\sqrt{(\cosh^2 u - \cosh^2 x)}\right\} f(u)du, \quad (1.9)$$

where $\alpha > 0, \eta > -1/2$, and the operators $\Im_{ik}(\eta,\alpha)$ and $\Re_{ik}(\eta,\alpha)$ by the above equations when $J_{\alpha-1}$ is replaced by $I_{\alpha-1}$.

Similar formulae have been briefly discussed by Srivastav[5]. I shall make use of some of the basic properties of the operators.

If I let k tend to zero I see that these operators are related to the Erdélyi-Kober operators Sneddon[4] by

$$\Im_0(\eta,\alpha) = I_{\eta,\alpha}, \Re_0(\eta,\alpha) = K_{\eta,\alpha} \quad (1.10)$$

Letting α tend to zero in equation (1.10) I have the identity operators

$$\Im_0(\eta,\alpha) = I_{\eta,0} = I, \Re_0(\eta,0) = K_{\eta,0} = 1 \quad (1.11)$$

From the definition (1.8) and (1.9) if follows immediately that

$$\Im_k(\eta,\alpha)\cosh^2 x\, f(x) = \cosh^2 x\, \Im_k(\eta+\beta,\alpha) f(x)$$
$$\Re_k(\eta,\alpha)\cosh^2 x\, f(x) = \cosh^2 x\, \Re_k(\eta-\beta,\alpha) f(x) \quad (1.12)$$

Writing down the expressions for $\Im_{ik}(\eta+\alpha,\beta)$ and $\Im_k(\eta,\alpha)$ I find that

$$\Im_{ik}(\eta+\alpha,\beta)\Im_k(\eta,\alpha) f(x) = 2^{\alpha+\beta}(\cosh x)^{-2(\eta+\alpha+\beta)}(\cosh k)^{2-\alpha-\beta}$$
$$\int_0^x \chi(u) I_{\beta-1}(\chi_2) f(u)du \int_0^u \chi(y) J_{a-1}(\chi_3) f(y)dy$$

Where

$$\chi(u) = \cosh u(\cosh^2 x - \cos^2 u)^{½(\beta-1)}$$
$$\chi_2 = \cosh k(\cosh^2 x - \cos^2 u)^{½}$$
$$\chi(y) = (\cosh y)^{1+2\eta}(\cosh^2 u - \cos^2 y)^{½(\alpha-1)}$$
$$\chi_3 = \cosh k(\cosh^2 u - \cos^2 y)^{½}$$

Interchanging the order of the integrations and evaluating the inner integral Sneddon[4] I get.

$$\Im_{ik}(\eta+\alpha,\beta)\Im_k(\eta,\alpha) f(x)$$
$$= \frac{2}{\Gamma(\alpha+\beta)}(\cosh x)^{-2(\eta+\alpha+\beta)}\int_0^x(\cosh y)^{1+2\eta}$$
$$\cdot(\cosh^2 x - \cosh^2 y)^{\alpha+\beta-1} f(y)dy$$
$$= \Im_0(\eta+\alpha,\beta) f(x) = I_{\eta,\alpha+\beta} f(x)$$

so I have the product rule

$$\Im_{ik}(\eta+\alpha,\beta)\Im_{ik}(\eta,\alpha) = I_{\eta,\alpha+\beta} \quad (1.13)$$

In similar way I can derive the formulae

$$\Im_k(\eta+\alpha,\beta)\Im_{ik}(\eta,\alpha) = I_{\eta,\alpha+\beta} \quad (1.14)$$

and

$$\Im_{ik}(\eta,\alpha)\Re_k(\eta+\alpha,\beta) = \Re_k(\eta,\alpha)\Re_{ik}(\eta+\alpha,\beta) = K_{\eta,\alpha+\beta} \quad (1.15)$$

The above results indicate the manner in which I should define the operators $\Im_k(\eta,\alpha)$ and $\Re_k(\eta,\alpha)$ for $\alpha < 0$. From equations (1.11) and (1.13) I have

$$\Im_{ik}(\eta+\alpha,-\alpha)\Im_k(\eta,\alpha) = I \quad (1.16)$$

which suggests that if $\alpha < 0$ I define $\Im_k(\eta,\alpha) f$ to be the solution of the integral equation.

$$\Im_{ik}(\eta+\alpha,-\alpha) g(x) = 2^{-\sigma}(\cosh x)^{-2\eta}(\cosh k)^{1+\sigma}$$
$$\times\int_0^x(\cosh u)^{1+2\eta+2\alpha}(\cosh^2 x - \cosh^2 u)^{-½(\alpha+1)} I_{-(\alpha+1)}$$
$$\left\{\cosh k\sqrt{(\cosh^2 x - \cosh^2 u)}\right\} g(u)du = f(x) \quad (1.17)$$

Using the results (1.1) and (1.2) if follows that $\Im_k(\eta,\alpha) f$ is given by the equation

$$\Im_k(\eta,\alpha) f(x) = 2^{\alpha}(\cosh x)^{-1-2\eta-2\alpha}(\cosh k)^{-\alpha}\frac{d}{dx}\int_0^x(\cosh u)^{1+2\eta}$$
$$(\cosh^2 x - \cosh^2 u)^{½\alpha} J_{\alpha}\left\{\cosh k\sqrt{(\cosh^2 x - \cosh^2 u)}\right\} f(u)du$$
$$= (\cosh x)^{-1-2\eta-2\alpha} D_x\left\{(\cosh x)^{3+2\eta+2\alpha}\Im_k(\eta,\alpha+1) f(x)\right\} \quad (1.18)$$

when $-1 < \alpha < 0$ and where have written $D_x = \frac{1}{2}\frac{d}{dx}x^{-1}$.

Similarly from equations (1.11) and (1.15) I see that if I define $g = \Re_k(\eta,\alpha) f$ to be the solution of integral equation $\Re_{ik}(\eta+\alpha,-\alpha)g = f$, and use the results (1.2) and (1.4) then

$$\Re_k(\eta,\alpha) f(x) = 2^{\alpha}(\cosh x)^{-2\eta-1}(\cosh k)^{-\alpha}\frac{d}{dx}\int_x^{\infty}(\cosh u)^{1-2\eta-2\alpha}$$
$$\times(\cosh^2 u - \cosh^2 u)^{½\alpha} J_\alpha\left\{\cosh k\sqrt{(\cosh^2 u - \cosh^2 x)}\right\} f(u)du$$

$$=(\cosh x)^{-2\eta-1} D_x\left\{(\cosh x)^{3-2\eta}\Re_k(\eta-1,\alpha+1) f(x)\right\} \tag{1.19}$$

where $-1<\alpha<0$ using a similar method to that employed in Sneddon[4], I can show that when $\alpha<0$ general expressions for the operators are

$$\Im_k(\eta,\alpha) f(x) = (\cosh x)^{-1-2\eta-2\alpha}$$

$$D_x^m\left\{(\cosh x)^{2m+1+2\alpha+2\eta}\Im_k(\eta,\alpha+m) f(x)\right\} \tag{1.20}$$

and

$$\Re_k(\eta,\alpha) f(x) = (-1)^m(\cosh x)^{2\eta-1}$$

$$D_x^m\left\{(\cosh x)^{2m+1-2\eta}\Re_k(\eta-m,\alpha+m) f(x)\right\}$$

where $-m<\alpha<0$ and m is a positive integer.

Now that I have defined the operators for negative and I see that equations (1.11), (1.13), (1.14) and (1.15), can be interpreted as yielding the inverse operators.

$$\Im_{ik}^{1}(\eta,\alpha) = \Im_k(\eta+\alpha,-\alpha), \qquad \Im_k^{-1}(\eta,\alpha) = \Im_{ik}(\eta+\alpha,-\alpha) \tag{1.22}$$

$$\Re_{ik}^{-1}(\eta,\alpha) = \Re_k(\eta+\alpha,-\alpha), \qquad \Re_k^{-1}(\eta,\alpha) = \Re_{ik}(\eta+\alpha,-\alpha) \tag{1.23}$$

Finally, it is an easy matter to show that

$$\int_0^{\infty}\cosh x\, f(x)\Im_k(\eta,\alpha)g(x)d(x)$$

$$=\int_0^{\infty}\cosh x\, g(x)\Re_k(\eta,\alpha) f(x)d(x)$$

2. Relations between the Generalized Hankel and the Generalized Erdélyi-Kober Operators

From the definitions (1.5) and (1.8) I have that

$$\Im_{ik}(\eta+\alpha,\beta)\begin{pmatrix}0, & 0, & \cosh k\\ \eta, & \alpha, & \sigma\end{pmatrix} f(x)$$

$$=2^{\alpha+\beta}(\cosh x)^{-2\eta-2\alpha-2\beta}(\cosh k)^{1-\beta}$$

$$\int_0^{\infty}(\cosh u)^{1+\eta+\alpha}(\cosh^2 x - \cosh^2 u)^{½(\beta-1)}$$

$$\cdot I_{\beta-1}\left\{\cosh k\sqrt{\cosh^2 x - \cosh^2 u}\right\}du$$

$$\int_k^{\infty}(\cosh y)^{1-2\sigma-\alpha}(\cosh^2 y - \cosh^2 k)^{\sigma}$$

$$\cdot J_{2\eta+\alpha}(\cosh u \cosh y)\, f(y)dy$$

interchanging the order of the integrations and evaluating the inner integral using the result Sneddon[5]. I get

$$\Im_{ik}(\eta+\alpha,\beta)\begin{pmatrix}0, & 0, & \cosh k\\ \eta, & \alpha, & \sigma\end{pmatrix} f(x) = 2^{\alpha+\beta}(\cosh x)^{-\alpha-\beta}$$

$$\int_k^{\infty}(\cosh y)^{1+2\eta-2\sigma}(\cosh^2 y - \cosh^2 k)^{\sigma-\lambda}$$

$$\cdot J_{2\lambda}\sqrt{\cosh x(\cosh^2 y - \cosh^2 k)}\, f(y)dy$$

$$= S\begin{pmatrix}0, & \cosh k, & \cosh k\\ \eta, & \alpha+\beta, & \sigma-\lambda\end{pmatrix} f(x) \tag{2.1}$$

where $2\lambda = 2\eta+\alpha+\beta$

In this way I have established the relation

$$\Im_{ik}(\eta+\alpha,\beta)S\begin{pmatrix}0, & 0, & \cosh k\\ \eta, & \alpha, & \sigma\end{pmatrix} = S\begin{pmatrix}0, & \cosh k, & \cosh k\\ \eta, & \alpha+\beta, & \sigma-\lambda\end{pmatrix} \tag{2.2}$$

Using the result Sneddon[4] I can, by a similar method show that

$$\Re_k(\eta,x)S\begin{pmatrix}0, & 0, & \cosh k\\ \eta+\alpha, & \beta, & \sigma\end{pmatrix}S\begin{pmatrix}0, & \cosh k, & \cosh k\\ \eta, & \alpha+\beta, & \sigma+\lambda\end{pmatrix}$$

$$2\lambda = 2\eta+\alpha+\beta \tag{2.3}$$

By a similar process I can also establish the relations

$$S\begin{pmatrix}0, & 0, & \cosh k\\ \eta+\alpha, & \beta, & \sigma\end{pmatrix}S\begin{pmatrix}\cosh k, & 0, & 0\\ \eta, & \alpha, & \sigma-\eta+½\alpha\end{pmatrix} = \Im_k(\eta,\alpha+\beta) \tag{2.4}$$

and

$$S\begin{pmatrix}0, & 0, & 0\\ \eta, & \alpha, & 0\end{pmatrix}S\begin{pmatrix}\cosh k, & 0, & 0\\ \eta+\alpha, & \beta, & \eta+\alpha+½\beta\end{pmatrix} = \Re_{ik}(\eta,\alpha+\beta) \tag{2.5}$$

3. Solution of the Dual Integral Equations

The dual integral equations

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$$\int_k^\infty \chi^*(u)J_\mu(\cosh x\cosh u)\psi(u)du = F_1(x), 0\le x\le 1, k\ge 0 \tag{3.1}$$

$$\int_k^\infty \psi(u)J_\nu(\cosh x \cosh u) = G_2(x),\ x>1 \tag{3.2}$$

Where $\chi^*(u) = (\cosh u)^{-\mu-\nu}(\cosh^2 u - \cosh^2 k)^\beta$

where $F_1(x)$ and $G_2(x)$ are prescribed functions, have been solved by Burlak[1] using a generalization of the method introduced by Sneddon[4] and developed by Copson[2] for solving the case $\mu=\nu, k=0$.

Following Sneddon I use the notation $I_1=\{x: 0\le x<1\}, I_2=\{x: x>1\}$ and write any function $f(x), x\ge 0,$ as

$$f(x) = f_1(x) + f_2(x)$$

where

$$f_1(x) = \begin{cases} f(x), & x\in I_1, \\ 0, & x\in I_2, \end{cases} \quad f_2(x) = \begin{cases} 0, & x\in I_1, \\ f(x), & x\in I_2, \end{cases}$$

If I make the substitutions

$$\Psi(u) = (\cosh u)^{1+\nu}\phi(u),\ f(x) = 2^{\mu-2\beta}(\cosh x)^{2\beta-\mu}F(x)$$

$$g(x) = 2^{-\nu}(\cosh x)^{\nu}(x)G(x) \tag{3.3}$$

We see that equations (3.1) and (3.2) can be written in the operator from

$$S\begin{pmatrix} 0, & 0, & \cosh k \\ \beta, & \mu-2\beta, & \beta \end{pmatrix}\phi(x) = f(x) \tag{3.4}$$

$$S\begin{pmatrix} 0, & 0, & 0 \\ \nu, & -\nu, & 0 \end{pmatrix}\phi(x) = g(x) \tag{3.5}$$

where $\phi(x), f_2(x)$ and $g_1(x)$ are unknown and the functions

$$f_1(x) = 2^{\mu-2\beta}(\cosh x)^{2\beta-\mu}F_1(x), g_2(x) = 2^{-\nu}(\cosh x)^{\nu}G_2(x) \tag{3.6}$$

are given

Solution

It follows from the results (2.2) and (2.3) that

$$\Im_{ik}(\mu-\beta,\beta-\mu)S\begin{pmatrix} 0, & 0, & \cosh k \\ b, & \mu-2\beta, & \beta \end{pmatrix}$$

$$= \Re_k(\beta,\nu-\beta)S\begin{pmatrix} 0, & 0, & 0 \\ \nu, & -\nu, & 0 \end{pmatrix} = S\begin{pmatrix} 0, & \cosh k, & \cosh k \\ \beta, & -\beta, & ½\beta \end{pmatrix} \tag{3.7}$$

and hence I can write equations (3.4) and (3.5) in the form

$$S\begin{pmatrix} 0, & \cosh k, & \cosh k \\ \beta, & -\beta, & ½\beta \end{pmatrix}\varphi(x) = h(x) \tag{3.8}$$

where

$$h_1(x) = \Im_{ik}(\mu-\beta,\beta-\mu)f_1(x), h_2(x) = \Re_k(\beta,\nu-\beta)g_2(x) \tag{3.9}$$

are known functions.

Applying the inversion formula (1.7) to equation (3.8) I see that

$$\phi(u) = S^{-1}\begin{pmatrix} 0, & \cosh k, & \cosh k \\ \beta, & -\beta, & ½\beta \end{pmatrix}h(u) = S\begin{pmatrix} \cosh k, & 0, & 0 \\ 0, & \beta, & ½\beta \end{pmatrix}h(u) \tag{3.10}$$

Reverting to the original variables (3.3) and making use of the formulae (1.12), I find that the solution of the dual integral equations can be written as

$$\psi(u) = \cosh^{1+\nu} u\phi(u) = (\cosh u)^{1+\nu}S\begin{pmatrix} \cosh k, & 0, & 0 \\ 0, & \beta, & ½\beta \end{pmatrix}h(u) \tag{3.11}$$

which is

$$\psi(u) = 2^\beta(\cosh u)^{1+\nu}(\cosh^2 u - \cosh^2 k)^{-½\beta}$$

$$\cdot\left\{\int_0^1 h_1(x) + \int_1^\infty h_2(x) + \right\}(\cosh x)^{1-\beta}$$

$$J_\beta\left\{\cosh x\sqrt{\cosh^2 u - \cosh^2 k}\right\}dx \tag{3.12}$$

where

$$\left.\begin{aligned} h_1(x) &= 2^{\mu-2\beta}(\cosh x)^{2\beta-\mu}\Im_{ik}(½\mu,\beta-\mu)F_1(x) \\ h_2(x) &= 2^{-\nu}(\cosh x)^{\nu}\Re_k(\beta-½\nu,\nu-\beta)G_2(x) \end{aligned}\right\} \tag{3.13}$$

we consider four cases

(i) when $\nu>\beta>\mu>-1$

Using the definitions (1.8) and (1.9) I see that equations (3.13) are

$$h_1(x) = 2^{-\beta}(\cosh k)^{1+\mu-\beta}\int_0^\infty(\cosh t)^{1+\mu}(\cosh^2 x - \cosh^2 t)^{½(\beta-\mu-1)}$$

$$\times I_{\beta-\mu-1}\left\{\cosh k\sqrt{\cosh^2 x - \cosh^2 t}\right\}F_1(t)dt, \tag{3.14}$$

$$h_2(x) = 2^{-\beta}(\cosh k)^{1+\beta-\nu}\cosh^{2\beta} x\int_x^\infty(\cosh t)^{1+\nu}(\cosh^2 t - \cosh^2 x)^{½(\nu-\beta-1)}$$

$$\times I_{\nu-\beta-1}\left\{\cosh k\sqrt{\cosh^2 t - \cosh^2 x}\right\}G_2(t)dt, \tag{3.15}$$

which together with equation (3.12) furnish a solution to the dual integral equations.

(ii) when $1+\beta > \mu > \beta > \nu > \beta - 1$

Applying the definitions (1.18) and (1.19) I find that equations (3.13) become

$$h_1(x) = 2^{-\beta}(\cosh x)^{-1}(\cosh k)^{\mu-\beta}\frac{d}{dx}\int_x^\infty (\cosh t)^{1+\mu}(\cosh^2 x - \cosh^2 t)^{\frac{1}{2}(\beta-\mu)}$$
$$\times J_{\beta-\mu}\left\{\cosh k\sqrt{\cosh^2 x - \cosh^2 t}\right\}F_1(t)dt, \qquad (3.16)$$

$$h_2(x) = 2^{-\beta}(\cosh x)^{2\beta-1}(\cosh k)^{\beta-\nu}\frac{d}{dx}\int_x^\infty (\cosh t)^{1-\nu}(\cosh^2 t - \cosh^2 x)^{\frac{1}{2}(\nu-\beta)}$$
$$\times J_{\nu-\beta}\left\{\cosh k\sqrt{\cosh^2 t - \cosh^2 x}\right\}G_2(t)dt, \qquad (3.17)$$

The solution to the dual integral equations given by equations (3.12), (3.16) and (3.17) is in complete agreement with that obtained by Burlak.

(iii) when $\nu = \mu, \beta - 1 < \mu < \beta$

In this case $\psi(u)$ is given by equation (3.12) with $\nu = \mu, h_1(x)$ by equation (3.16) and $h_2(x)$ by equation (3.17) with $\nu = \mu$.

(iv) when $\nu = \mu, \beta < \mu < \beta + 1$

Here $\psi(u)$ is given by equation (3.12) with $\nu = \mu, h_1(x)$ by equation (3.16) and $h_2(x)$ by equation (3.15) with $\nu = \mu$.

It is perhaps of interest to note that if in the solution (iii) and (iv), I write $\beta = \mu - \alpha$ and let k tend to zero. I obtain solutions to the equations:

$$\int_0^\infty (\cosh u)^{-2\alpha}\psi(u)J_\mu(\cosh x - \cosh u)du = F_1(x), 0 \le x < 1$$
$$\int_0^\infty \psi(u)J_\mu(\cosh x - \cosh u)du = F_2(x), x < 1 \qquad (3.18)$$

valid for $-1 < \alpha < 0$ and $0 < \alpha < 1$ respectively, which are in agreement with those given in Sneddon[4].

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