Inversion Formula by Application of Laplace & Mellin transforms

U. K. Bajpai[1], Manish Kumar Srivastava[2]

[1] Department of Mathematics, B.S.N.V. I. College, Lucknow

[2]Department of Mathematics, Boy's Anglo Bengali I. College, Lucknow

Abstract: ***In this paper, we have established proof of two theorems on inversion formula by application of Laplace operators & Fox's method. The given integral equation can be reduced in terms of the inverse Mellin transforms of the unknown function and consequently the solution is then readily obtained by repeated applications of Laplace and inverse Laplace operators. The result established in this paper, further acts as the key formula for a large number of integral equations associated with Gauss's hypergeometric functions due to the general character of the H-function.***

Keywords: H-Transformation, Laplace Transforms, Mellin Transforms, Contour, Inversion Formula.

AMS Subject Classification 2010 : 44A05, 44A15

1. H-Transformation based upon L and L^{-1} Operators

In a series of papers Fox [3, 4] has applied L and L^{-1} operators, where L and L^{-1} respectively represent the Laplace and inverse Laplace transform, to obtain the inversion formulae for the integral transforms associated with Bessel and Whittaker functions.

The object of the present paper is to establish the inversion formula for the integral transformation

$$g(t) = \int_0^{\infty} H^{m,n}_{p+n,q+m}\left(\eta t u \,\Big|^{a_j, A_j}_{b_k, B_k}\right) f(u)\, du, \quad t > 0 \tag{1.1}$$

where g(t) is known and f(u) is to be determined in terms of L and L^{-1} operators.

The H-function, introduced by Fox [2] in the notation of Saxena [5], will be defined and represented as follows :

$$H^{m,n}_{p+n,q+m}\left(x\,\Big|^{a_j, A_j}_{b_k, B_k}\right) = \frac{1}{2\pi i}\int_c \chi^{m,n}_{p,q}(s) x^{-s}\, ds \tag{1.2}$$

where

$$\chi^{m,n}_{p,q}(s) = \frac{\prod_{j=1}^{m}\Gamma(b_j + B_j s)\prod_{j=1}^{n}\Gamma(a_j - A_j s)}{\prod_{j=1}^{q}\Gamma(b_{m+j} - B_{m+j}s)\prod_{j=1}^{p}\Gamma(a_{n+j} + A_{n+j}s)} \tag{1.3}$$

where an empty product is to be interpreted as unity and also the following simplified assumptions are made :

(i) a_j, A_j, b_k and B_k are all real for j = 1, 2, ..., (p+n) and k = 1, 2, ..., (q+m).

(ii) A_j's and B_k's are positive for j = 1, 2, ..., (p+n) and k = 1, 2, ..., (q+m).

(iii) All the poles of the integrand in (1.2) are simple.

(iv) Let $s = \sigma + it$ and t being real; then the contour C alongwith which the integral (1.2) is taken, is a straight line parallel to the imaginary axis in the complex s-plane whose equation is $\sigma = \sigma^0$,where σ^0 is constant. The contour is such that all the poles of $\Gamma(b_k + B_k s)$ for k = 1, 2 ..., m lie to the left and those of $\Gamma(a_j - A_j s)$ for j = 1, 2, ..., n to the right of it.

(v) $\sum_{j=1}^{n} A_j + \sum_{j=1}^{m} B_j = \sum_{j=1}^{p} A_{n+j} + \sum_{j=1}^{q} B_{m+j}$

(vi) $\lambda = -\sum_{j=1}^{m} B_j + \sum_{j=1}^{n} A_j - \sum_{j=1}^{q} B_{m+j} + \sum_{j=1}^{p} A_{n+j}$

(vii) $\mu = \sum_{j=1}^{n} a_j + \sum_{j=1}^{m} b_j - \sum_{j=1}^{p} a_{n+j} - \sum_{j=1}^{q} b_{m+j} + \tfrac{1}{2}(p+q-m-n)$

(viii) x > 0

(ix) $\lambda\, \sigma^0 + \mu + 1 > 0$.

From the asymptotic representation of the gamma function, Erdélyi, A. et al [1]

$$\lim_{|t| \to \infty} |\Gamma(\sigma + it)|\,|t|^{1/2-\sigma} \exp\left(\frac{\pi}{2}|t|\right) = (2\pi)^{1/2}, \tag{1.4}$$

it can be readily seen on setting s = σ + it, x = $Re^{i\phi}$ (R > 0, real) that the absolute value of the integrand of (1.2) is comparable with $(\rho/R)^{\sigma^0} |t|^{\lambda\sigma^0+\mu} e^{-\phi(t)}$,

$$\rho = \prod_{j=1}^{n}(A_j)^{A_j}\prod_{j=1}^{m}(B_j)^{-B_j}\prod_{j=1}^{p}(A_{n+j})^{A_{n+j}}\prod_{j=1}^{q}(B_{m+j})^{-B_{m+j}} \tag{1.5}$$

where | t | is large and hence the integral (1.2) converges absolutely, if the conditions (viii) and (ix) are satisfied.

1.1 The Laplace and Mellin Transforms

The Laplace transform of $\phi(x)$ is denoted by $L\{\phi(x)\}$ and defined, Widder [8], as

$$L\{\phi(x)\}=\int_0^\infty e^{-xt}\,\phi(x)\,dx=\psi(t) \tag{1.6}$$

with $\phi(x)$ and $\psi(t)$ related as in (1.6) the inverse Laplace transformer of $\psi(t)$ is written as

$$L^{-1}(\psi(t)\} = \phi(x) \tag{1.7}$$

Usually L will change the variable x to t and L^{-1} will change t to x.

If the Mellin transform of f(x) is denoted by F(s), then

$$F(s) = \int_0^\infty f(x)x^{s-1}dx. \tag{1.8}$$

If (1.8) holds, then the inverse Mellin transform is given by

$$f(x) = \frac{1}{2\pi i}\int_C F(s)x^{-s}\,ds, \tag{1.9}$$

where C is a suitable contour in the complex s-plane.

If F(s) and G(s) are, respectively the Mellin transforms of f(x) and g(x), then the Mellin-Parseval theorem states that

$$\int_0^\infty f(x)g(x)dx = \frac{1}{2\pi i}\int_C F(s)G(1-s)ds, \tag{1.10}$$

where C is suitable contour in the s-plane.

For a discussion of the results (1.8), (1.9) and (1.10) and their conditions of validity the reader is referred to the work of Titchmarsh [6].

Applying (1.9), we see that the Mellin transform of H(x) is equal to
$\chi_{p,q}^{m,n}(s)$.

1.2 Inversion Formulae

Theorem 1.2(1) if (i) $B_i > 0, A_{n+j} > 0$, $½A_{n+j}+a_{n+j}>0$; i=1,2,...,m; j=1,2,....,p; (ii) $f(x)\in L_2(0,\infty)$, (iii) $s^{\chi_1}F(1-s)\in L(½-\infty, ½+i\infty)$ and (iv) $F(1-s)\in(½-i\infty,½+i\infty)$ and (v) $y^{-1/2}f(y)\in L(0,\infty)$, where f(y) is of bounded variation near the point y = x, then the solution of

$$g(t)=\int_0^\infty H_{p,m}^{m,0}[\eta tu\,|_{(b_1,B_1),\ldots,(b_m,B_m)}^{(a_{n+1},A_{n+1}),\ldots,(a_p,A_p)}]f(u)du,(t>0) \tag{1.11}$$

as an integral equation for f(u) is given by

$$f(\eta^{-1}x)=\eta x^{-1}\left\{\prod_{j=1}^{m}\left[\left\{v^{1-b_j}L^{-1}\left[w^{-b_j}\left\{\prod_{j=1}^{p}\left[\left\{v^{a_{n+j}}\left[t^{a_{n+j}-1}g\left(t^{-A_{n+j}}\right)\right]\right.\right.\right.\right.\right.\right.\right.$$
$$\left..\right\}_{v=u}1/A_{n+j}]\}_{u=w}B_j]_{v=u}-1/B_j\}]\}_{u=x^{-1}} \tag{1.12}$$

where

$$\chi_1=\left\{½\left(\sum_{j=1}^{p-1}A_{n+j}-\sum_{j=1}^{m}B_j\right)+\sum_{j=1}^{m}b_j-\sum_{j=1}^{p-1}a_{n+j}+½(p-n-1)\right\}$$

Proof. Bajpai [7] has proved the above theorem earlier.

Theorem 1.2(2)

If (i) $A_i > 0$, $B_{m+j} > 0$, $b_{m+j} - ½B_{m+j} > 0$;

i = 1, 2, ..., n; j = 1, 2, ..., q,

(ii) $f(x)\in L_2(0,\infty)$

(iii) $s^{\chi_2}F(1-s)\in L(½-i\infty,½+i\infty)$,

(iv) $F(1-s)\in L(½-i\infty,½+\infty)$ and

(v) $y^{-1/2}f(y)\in L(0,\infty)$ where f(y) is of bounded variation near the point y=x, then the solution of

$$g(t)=\int_0^\infty H_{q,n}^{n,0}\left[(\eta tu)^{-1}\,|_{1-a_j,A_j}^{1-b_k,B_k}\right]f(u)du,(t>0) \tag{1.13}$$

is given by

$$f(\eta x^{-1})=x\eta^{-1}\left\{\prod_{j=1}^{n}\left[\left\{v^{a_j}L^{-1}\left[w^{a_j-1}\left\{\prod_{j=1}^{q}\left[\left\{v^{1-b_{j+m}}\right.\right.\right.\right.\right.\right.\right.$$
$$\cdot L\left[t^{-b_{m+j}}g(t^{B_{m+j}})\right]\Big\}_{v=u^{1/B_{m+j}}}\Big]\Big\}_{u=w^{A_j}}\Big]_{v=u^{-1/A_j}}\Big\}\Big]\Big\} \tag{1.14}$$

by virtue of the property of the H-function

$$H_{p,q}^{m,n}\left(x\,|_{b_k,B_k}^{a_j,A_j}\right)=H_{q,p}^{n,m}\left(1/x\,|_{1-a_j,A_j}^{1-b_k,B_k}\right) \tag{1.15}$$

where

$$\chi_2=\left\{1/2\left(\sum_{j=1}^{n}A_n\sum_{j=1}^{q-1}B_{m+j}\right)+\sum_{j=1}^{n}a_j-\sum_{j=1}^{q-1}b_{m+j}+1/2(q-n-1)\right\}$$

As a consequence of Theorem 1.2(1) and Theorem 1.2(2) we arrive at Theorem 1.2(3).

Theorem 1.2(3)

If (i) $A_i>0, ½A_{n+j}+a_{n+j}>0, b_{m+j}-½B_{m+j}>0; B_k>0$;
i = 1, 2, ..., (p+n); k = 1, 2, ..., (q+m); j = 1, 2, ..., p
(ii) $f(x)\in L_2(0,\infty)$,
(iii) $S^{k'}F(1-s)\in L_2(1/2-i\infty,1/2+i\infty)$,

(iv) $F(1-s) \in L(1/2 - i\infty, 1/2 + i\infty)$ and

(v) $y^{-1/2} f(y) \in L(0, \infty)$,

where f(y) is of bounded variation near the point y = x, then the solution of (1.1), as an integral equation for f(u) is given by

$$f(\eta^{-1}x) = \eta x^{-1} \left\{ \prod_{j=1}^{n} \left[\left\{ u^{1-a_j} L^{-1} \left[v^{-a_j} \left\{ \prod_{j=1}^{q} \left[\left\{ u^{b_{m+j}} L \left[v^{b_{m+j}-1} \cdot \right.\right.\right.\right.\right.\right.\right.$$

$$\left\{ \prod_{j=1}^{m} \left[\left\{ v^{1-b_j} L^{-1} \left[w^{-b_j} \left\{ \prod_{j=1}^{p} \left[\left\{ v^{a_{n+j}} L \left[t^{a_{n+j}-1} g\left(t^{-A_{n+j}} \right) \right] \right.\right.\right.\right.\right.\right.$$

$$\left. \cdot \; {}_{v=u^{1/A_{n+j}}} \right] \Big\}_{u=w^{B_j}} \Big] {}_{v=u^{-1/B_j}} \Big\} \Big] \Big\} {}_{u=v^{B_{m+j}}} \Big] \Big\}_{u=v^{1/B_{m+j}}} \Big] \Big\}$$

$$\cdot \; {}_{u=v^{-A_j}} \Big] \Big\}_{u=v^{-1/A_j}} \Big] \Big\}_{v=x^{-1}} \tag{1.16}$$

where

$$k' = \left\{ 1/2 \left(\sum_{1}^{n} A_j - \sum_{1}^{m} B_j - \sum_{1}^{q-1} B_{m+j} + \sum_{1}^{p-1} A_{n+j} \right) \right.$$

$$\left. + \sum_{1}^{n} a_j + \sum_{1}^{m} b_j - \sum_{1}^{p-1} a_{n+j} - \sum_{1}^{q-1} b_{m+j} + 1/2(p+q-m-2-2) \right\}$$

2. Acknowledgement

Special thanks to Dr. R.K. Saxena and Dr. P.L. Sethi Professor, J.N.V. University, Jodhpur for their valuable suggestions to improve the paper.

3. Thanks to Referee

A lot of thanks to Referee and Reviewers.

References

[1] Erdélyi, A. et al.: Higher transcendental functions, (1953), Vol. 1 & 2 (Mc-Graw-Hill, New York).
[2] Fox, C.: The G and H-functions as symmetrical Fourier Kernels, Trans. Amer. Math. Soc., 98, (1961), 395-429.
[3] Fox, C.: Solving integral equations by L & L-1 operators, proc. Amer. Math. Soc. 29, (1971), 299-306.
[4] Fox, C.: Applications of Laplace Transforms and their inverse, Proc. Amer. Math. Soc. 35, (1972), 193-200.
[5] Saxena, R.K.: On the formal solution of dual integral equations, Proc. Amer. Math. Soc., 18, (1967a), 1-8.
[6] Titchmarsh, E.C.: An introduction to the theory of Fourier integrals (Oxford), (1937).
[7] Bajpai, U.K.: Use of Integral transforms in inversion formula for H-Transformation, Vijanana Parishad Anusandhan Patrika, Vol. 58, No.2, (2016), 25-31.
[8] Widder, D.V.: The Laplace Transform Princeton Univ. Press, Princeton, New Jersey, (1941).