T-Pure Fuzzy Submodules

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Abstract: The main aim of this paper is to extend and study the notion of (ordinary) T-pure submodule into T-pure fuzzy submodule and T-pure ideal into T-pure fuzzy ideal. This lead us to introduced and study other notions such as T-pure fuzzy submodule and T-pure fuzzy ideal

Keywords: Fuzzy module, submodules

1. Introduction

Let $X$ be a fuzzy module of an $R$-module $M$, we denoted by $X-F(M)$, it is well known that $A$ is fuzzy submodules of $X$ denoted by $F-S(X)$ is called T-pure fuzzy submodule of $X$. if for each fuzzy ideal $I$ of $R$ such that $I^2X \cap A = I^2A$. And then fuzzy ideal of a ring $R$ denoted by $F-I(R)$ is called T-pure fuzzy ideal of $R$ if for each fuzzy ideal $J^2$ of $R$ $J^2 \cap I = J^2I$

In this paper, we fuzzify these concepts T-pure fuzzy submodule and T-pure fuzzy ideal, moreover we generalize many properties of T-pure fuzzy submodule and T-pure fuzzy ideal

This paper consists of two part. In part one, various basis properties about T-pure fuzzy submodule are discussed. part two included T-pure fuzzy ideal and basic properties about this concept

1.1 T-pure Fuzzy submodule

In this section we introduce the concept of T-pure fuzzy submodule by of provided some properties of these concepts.

Definition (1.1): Let $X-F(M)$ let $A$ be a $F-S(X)$. A is called a T-pure fuzzy submodule if for each fuzzy ideal $K$ of $R$, $KX \cap A = KA$. [4]

Proposition (1.2): Let $X-F(M)$ and let $A$ be fuzzy sub modules of $X$. Then $A$ is a T-pure $F-S(X)$ if and only if $A_t$ is a T-pure submodules of $X_t$, $\forall \ t \in (0,1)$

Definition (1.3): Let $X-F(M)$ and let $A$ be a $F-S(X)$. A is called a T-pure fuzzy submodule if for each fuzzy ideal $I$ of $R$ such that $I^2X \cap A = I^2A$.

Proposition (1.4): Let $X-F(M)$ and let $A$ be fuzzy sub modules of $X$. Then $A$ is a T-pure fuzzy submodule of $X$ if and only if $A_t$ is T-pure fuzzy submodules of $X_t$, $\forall \ t \in (0,1)$

Proof: Let $I$ be an ideal of ring $R$
Define $I^2 : R \to [0,1] by I^2(x) = \begin{cases} t & if \ x \in J \\ 0 & otherwise \end{cases}$ $\forall \ t \in (0,1)$
And let $N \ll M$
Define $A : M \to [0,1] by A(x) = \begin{cases} t & if \ x \in N \\ 0 & otherwise \end{cases}$ $\forall \ t \in (0,1)$
It is clear that $I^2$ is $F-I(R)$ and $A$ is $F-S(X)$.
Now, $A_t = N, I^2_t = J , X_t = M$

Thus $A_t$ is T-pure submodules of $X_t$, $\forall \ t \in (0,1)$.
Conversely Let $I^2$ be $F-I(R)$ and $A$ be a $F-S(X)$.
If $A$ is T-pure fuzzy submodule of $X$.
Then $I^2X_t \cap A_t = I^2A_t$
Hence $(I^2X \cap A)_t = (I^2A)_t$

Therefore $A$ is T-pure fuzzy submodule of $X$.

Remarks and Examples (1.5):
1- Let $X-F(M)$ and let $A$ be a pure $F-S(X)$, Then $A$ is T-pure $F-S(X)$.

Proof:
It is clear
The converse not true by
Example: Let $M=Z_4$ as Z-module and $N=2Z_4$
Define $X : M \to [0,1]$ by $X(x) = \begin{cases} 1 & if \ x \in M \\ 0 & otherwise \end{cases}$ $\forall \ t \in (0,1)$
Define $A : M \to [0,1] by A(x) = \begin{cases} 1 & if \ x \in N \\ 0 & otherwise \end{cases}$ $\forall \ t \in (0,1)$
It is clear that $X$ is $F(M)$, $A$ is $F-S(X)$ and $X_t=M, A_t=N$
$A_t$ is T-pure submodules of $X_t$, by[7]
Thus $A$ is T-pure fuzzy submodule of $X$ by (Proposition 1.4)
But $A$ is not pure fuzzy submodule of $X$ since if $I_t=2Z_4$ where $I:R \to [0,1]$
Such that $I(x)=1 if x \in Z_4$ and $I(x)=0 if x \notin Z_4$
Now $2Z_4 \cap 2Z_4=\{0,2\}$ but $2Z_2 \cap 2Z_2=\{0,2\}=\{0\}$
Thus $A_t$ is not pure submodules of $X_t$
Therefore $A$ is not pure fuzzy submodule of $X$. [4]
2- Let $X$-F($M$) . It is clear that the fuzzy singleton ($o_1$) and $X$ are always T-pure fuzzy submodule of $X$. \( \forall \in (0,1) \)

3-In the fuzzy module $Z$ as $Z$-module. The only T-pure fuzzy submodule are fuzzy singleton ($o_1$) and

Proof:
Let $X: Z\rightarrow [0,1]$ by $X(x) = \begin{cases} 1 \text{ if } x \in Z \\ 0 \text{ otherwise} \end{cases}$

Define $O_1: X\rightarrow [0,1]$ by $O_1(x) = \begin{cases} 0 \text{ if } x = 0 \\ 1 \text{ otherwise} \end{cases}$

If there exists a fuzzy submodule $A: nZ\rightarrow [0,1]$ by $A(x) = \begin{cases} 0 \text{ if } x \in nZ \\ 1 \text{ otherwise} \end{cases}$

Let $I(x) = (n)^2 \rightarrow [0,1]$ by $I(x) = \begin{cases} 0 \text{ if } x \in n \^2 \\ 1 \text{ otherwise} \end{cases}$

It is clear that $I = (n)^2$ and $I$ is a fuzzy ideal if $n^2 = n^2$ and $1_e(n)^2 \cap n^2 = (n)^2 \cap n^2 = 0$

Since $n^2 \cap n^2 = 0$ Thus $A_1$ is not T-pure by [7] $X_e = Z, O_1 = 0$ only two T-pure submodule of $Z$-module $\forall \in (0,1)$ by (Proposition 1.4)

4- Let $X$ be a fuzzy module of an $Z$-module $Q$. and let $A$ be a non-empty FC-S($X$) . then $A$ is not T-pure F-S($X$).

Proof:
Define $X: Q\rightarrow [0,1]$ by $X(x) = \begin{cases} 1 \text{ if } x \in Q \\ 0 \text{ otherwise} \end{cases}$

Define $A: Q\rightarrow [0,1]$ by $A(x) = \begin{cases} 1 \text{ if } x \in N \\ 0 \text{ otherwise} \end{cases}$

where $N$ is submodule of $Q, X_e = Q$ and $A_e = N$

N is not T-pure fuzzy submodule of $Q$ by [7]

Then $A$ is not T-pure F-S($X$) by (Proposition 1.4)

5- Let $X$-F($M$) . let $A$ be a T-pure F-S($X$) such that $A \subseteq B$ where $B$ is F-S($X$) , then $B$ is not T-pure F-S($X$) for example.

Example: Let $M=Z$

Let $X: M\rightarrow [0,1]$ by $X(x) = \begin{cases} 1 \text{ if } x \in M \\ 0 \text{ otherwise} \end{cases}$

Let $A: Z\rightarrow [0,1]$ by $A(x) = \begin{cases} 0 \text{ if } x \in Z \\ 1 \text{ otherwise} \end{cases}$

If $B: Z\rightarrow [0,1]$ by $B(x) = \begin{cases} 0 \text{ if } x \in 2Z \\ 1 \text{ otherwise} \end{cases}$

It is clear that $A$ and $B$ are F-S($X$), Now $A_e = Z$ and $B_e = 2Z$

It is not T-pure submodules [7] $B$ is not T-pure fuzzy submodule by (Proposition 1.4)

Proposition (1.6):
Let $X$-F($M$), and $A$ and $B$ are two F-S($X$). if $A$ is T-pure F-S($X$), $B \subseteq A$. and $B$ is T-pure F-S($A$), then $B$ is T-pure fuzzy submodule of $X$.

Proof:
Since $A$ be a T-pure F-S($X$) then $I^2 \cap A = I^2 A$ ... (1)

where $I$ is F-I($R$) and since $B$ is a T-pure F-S($A$) then

$I^2 A \cap B = I^2 B$ ... (2)

Now, we get $I^2 A \cap B = I^2 A \cap B$ ... (2)

Hence $I^2 \cap (A \cap B) = I^2 (A \cap B)$ since $B \subseteq A$.

Therefore $B$ is T-pure fuzzy submodule of $X$.

Proposition (1.7):
Let $X$-F($M$) . and let $C$ be a T-pure F-S($X$). if $B$ is a F-S($X$) containing $A$, then $A$ is T-pure F-S($B$).

Proof:
Let $I^2$ be a F-I($R$) and let $C$ be a T-pure F-S($X$).

Hence $I^2 \cap C = I^2 C$.

Now, $I^2 B \cap C = (I^2 B \cap I^2 X) \cap C$ since $C \subseteq B \subseteq X$

$I^2 B \cap I^2 C$

$I^2 C$

Thus $C$ is T-pure fuzzy submodule of a fuzzy submodule $B$.

“Definition (1.8):”
Let $X, Y$ -F($M_1, M_2$) respectively. Define $X \oplus Y: M_1 \oplus M_2 \rightarrow [0,1]$ by $(X \oplus Y)(a,b) = \min[I(a), Y(b)]$ for all $(a,b) \in M_1 \oplus M_2$. $X \oplus Y$ is called a fuzzy external direct sum of $X$ and $Y$. [9]

Lemma (1.9): Let $N_1$ and $N_2$ be two submodule of $M_1 \oplus M_2$ and $S(M_2)$ if $N_1 \oplus N_2$ is T-pure submodule of $M_1 \oplus M_2$, then $N_1$ and $N_2$ are T-pure submodule in $M_1$ and $M_2$.

Proof:
$T \ni I^2 M_1 \cap N_1 = I^2 N_1$ and $I^2 M_2 \cap N_2 = I^2 N_2$ for each ideal $I^2$ of $R$.

Since $N_1 \oplus N_2$ is T-pure in $M_1 \oplus M_2$ we get:

$I^2 (M_1 \oplus M_2) \cap (N_1 \oplus N_2) = I^2 (N_1 \oplus N_2)$

$I^2 M_1 \oplus I^2 M_2 \cap (N_1 \oplus N_2) = I^2 N_1 \oplus I^2 N_2$

Hence $I^2 M_1 \cap N_1 = I^2 N_1$ and $I^2 M_2 \cap N_2 = I^2 N_2$

Thus $N_1$ and $N_2$ are T-pure.

Proposition (1.10):
Let $X_1$-F($M_1$) and $X_2$-F($M_2$). If $A$ and $B$ are two F-S($X_1$) and F-S($X_2$) respectively then $A$ and $B$ are T-pure fuzzy submodule of $X_1$ and $X_2$ if and only if $A \oplus B$ is T-pure fuzzy submodule of $X_1 \oplus X_2$.

Proof:
$A \oplus B = (A \oplus B) \cap (X_1 \oplus X_2) = (X_1 \oplus X_2)$.

Therefore $(A \oplus B)$ is T-pure fuzzy submodule of $X_1 \oplus X_2$ by [7] Thus $A \oplus B$ is T-pure fuzzy submodule of $X_1 \oplus X_2$.

(Proposition 1.4)

$\Leftrightarrow$ let $A \oplus B$ is T-pure fuzzy submodule of $X_1 \oplus X_2$.

To show that $A$ and $B$ are T-pure fuzzy submodule of $X_1$ and $X_2$ respectively.

By [4, lemma (2.2.4)] and (Proposition 1.4) we get :-

$(A \oplus B) = A \oplus B$ is T-pure in module $(X_1 \oplus X_2)$. \( \forall t \leq 0 \) by [4]

Therefore $A \oplus B$ is T-pure submodule of $(X_1 \oplus X_2)$.

(Proposition 1.4)

Proposition (1.11):
Let $H$ be a direct summand of a fuzzy module $X$. then $H$ is T-pure fuzzy submodule of $X$.

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Then since $A$ submodule
Thus for $S(Y)$
Therefore $H$ is T-pure fuzzy submodule of $X$.

Proposition (1.12):
let $f:X \to Y$ be epimorphism of $X_1$, $F(M_1)$ and $X_2$, $F(M_2)$ respectively, let $B$ be a fuzzy submodule of $X$ and $Y$ is $f$-invariant, if $B$ is a T-pure F-S(X), then $f(B)$ is T-pure F-S(Y).

Proposition (1.13):
let $f:X \to Y$ be epimorphism of $X_1$, $F(M_1)$ and $X_2$, $F(M_2)$ respectively, such that every submodule of $X$ is $f$-invariant, if $C$ is T-pure F-S(Y), then $f^{-1}(C)$ is T-pure fuzzy submodule of $X$.

Proposition (1.14):
Let $f: X \to Y$ be an epimorphism of $X$ and $Y$ is a T-pure fuzzy submodule of $X$ then $f^{-1}(C)$ is T-pure fuzzy submodule of $X$.

Definition (2.1):
An fuzzy ideal I of a ring $R$ is called T-pure F-I(R) if for each fuzzy ideal $J^2$ of $R$ $J^2 \cap I = J^2 I$

Definition (2.2):
If every F-I(R) is T-pure fuzzy ideal, then we say $R$ is T-regular fuzzy ring.

Proposition (2.3):
Let $I$ be a fuzzy ideal of $R$ then $I$ is T-pure if and only if $I$ is a T-pure ideal of $R$.

Proof:
($\implies$) Let $I$ is T-pure F-I(R) $T$, $p$ $I$, $I$ is a T-pure ideal of $R$ $\forall$ $t \in (0,1]$

Let $J^2$ be an ideal of $R$
Define $K^2: J^2 \to [0,1]$ by $K^2(x) = (t \leftarrow 0 \text{ if } x \in J^2 \forall t \in (0,1]]$

It is clear that $K^2$ is F-I(R) and $K^2_1 = J^2$
$T$, $p$ $J^2 \cap I = J^2 I$
$J^2 \cap I = K^2_1 \cap I$
$= (K^2 \cap I)_t$

$=(K^2_1 \cap I)_t$ since $I$ is T-pure ideal

Therefore $K^2_1 \cap I = (K^2 \cap I)_t$

Thus $K^2$ is fuzzy ideal of $R$ $T$, $p$ $K^2 \cap I = K^2 I$

Let $K^2$ is fuzzy ideal of $R$ $T$, $p$ $K^2 \cap I = K^2 I$

Remarks and Examples (2.4):
1- Let $X(F(M))$, $R$ is regular fuzzy ring then $R$ is $T$-regular fuzzy ring.

Proof:
It is clear that $I$ is T-pure fuzzy ideal of $R$.
Example: Let $M=\mathbb{Z}_4$ as $Z$-module and $N=\{0,2\}$

Define $X: M\rightarrow [0,1]$ by $X(x) = \begin{cases} 1 & \text{if } x \in M \\ 0 & \text{otherwise} \end{cases}$

Define $I: R\rightarrow [0,1]$ by $A(x) = \begin{cases} t & \text{if } x \in S \\ 0 & \text{otherwise} \end{cases}$ for each $x \in [0,1]$.

It is clear that $I$ is fuzzy ideal for $R$ and $X_1 = M, J_1 = N$.

Then $X_1$ is $T$-regular ring by [7].

Hence $X$ is $T$-regular fuzzy ring since every ideal of $M$ is $T$-pure by (Proposition1.4) every ideal is fuzzy $T$-pure.

But $X_1$ is not regular since ideal $\{0,2\}$ is not pure [7].

Thus $I$ is not fuzzy pure ideal by [4].

Then $X$ is not regular fuzzy ring.

2- Let $X-F(M)$, if $R$ is a ring then the fuzzy singleton $\{0\}$ and a ring $R$ are always $T$-pure $F-I(R)$.

3- Let $X-F(M)$, if $R$ is a field, then $X$ is $T$-regular ring.

Proof:

Since every field has only one submodule 0 then by (2), $X$ is $T$-regular ring.

The converse of (3) is true if we give the condition, $R$ is a fuzzy integral domain.

Proposition (2.5):

Let $R_1$ and $R_2$ be two rings and $g$ any epimorphism function from $R_1$ to $R_2$. If $C$ is a $T$-pure $F-I(R_1)$, then $g(C)$ is a $T$-pure $F-I(R_2)$.

Proof:

Let $f^2$ be a $F-I(R_2)$. To prove $g(C) \cap f^2 = g(C) \cap f^2 \cap g(I_2)$ [5].

But $g(I_2) = F-I(R_1)$ by [5].

And $g(C \cap f^2) = F-I(R_1)$ so that

$g(C) \cap g(I_2) = g(C \cap f^2)$ [3].

Hence $g(I_2) = F-I(R_1)$ by [2].

$= g(C) \cap g(I_2)$ [5].

$= g(C) \cap f^2$ [by 2].

Proposition (2.6):

Let $R_1$ and $R_2$ be two rings and $g$ any epimorphism function from $R_1$ to $R_2$, then every ideal of $R_1$ is $T$-invariant. If $C$ is $T$-pure fuzzy ideal of $R_2$. Then $f^{-1}(C)$ is $T$-pure $F-I(R_1)$.

Proof:

Let $C$ be a $F-I(R_2)$, then $f^{-1}(C)$ is $F-I(R_1)$ by [6].

Let $f^2$ be a $F-I(R_2)$. To prove $f^{-1}(C) \cap f^2 = f^{-1}(C) \cap f^2$.

And by $f^{-1}(C) \cap f^2 = f^{-1}(C) \cap f^2$ by [3].

Then $g(C) \cap g(I_2) = g(C) \cap f^2$ [3].

Hence

$= g(C) \cap g(I_2)$ [5].

$= g(C) \cap f^2$ [by 2].

Therefore $f^{-1}(C)$ is $T$-pure $F-I(R_1)$.

Proposition (2.7):

Let $K$ be a $F-I(R_1)$ and let $J$ be a $F-I(R_2)$, then $K \oplus J$ is $T$-pure fuzzy ideal of $R_1 \oplus R_2$ if and only if $K$ and $J$ are $T$-pure fuzzy ideal in $R_1$ and $R_2$ respectively.

Proof:

$(\Rightarrow)$ Let $K \oplus J$ is $T$-pure fuzzy ideal. To prove $K$ and $J$ are $T$-pure fuzzy ideal.

Let $A^2$ and $B^2$ be two fuzzy ideal of $R_1$ and $R_2$ respectively. Then $A^2 \oplus B^2$ is $T$-pure fuzzy ideal of $R_1 \oplus R_2$ see [4].

Hence $(K \oplus J) \cap (A^2 \oplus B^2) = (K \oplus J) \cap (A^2 \oplus B^2)$ since $(K \oplus J)$ is $T$-pure fuzzy ideal.

And by $(K \oplus J) \cap (A^2 \oplus B^2) = (K \cap A^2) \oplus (J \cap B^2)$ see [4].

$(\Leftarrow)$ Let $K$ and $J$ are $T$-pure fuzzy ideal of $R_1$ and $R_2$.

Then $K \oplus J$ and $J$ are $T$-pure fuzzy ideal of $R_1$ and $R_2$.

Hence $A^2 \oplus B^2$ is $T$-pure fuzzy ideal in $R_1 \oplus R_2$ see [8].

T.p $(K \oplus J) \cap (A^2 \oplus B^2) = (K \oplus J) \cap (A^2 \oplus B^2)$.

And by $(K \oplus J) \cap (A^2 \oplus B^2) = (K \cap A^2) \oplus (J \cap B^2)$ see [4].

$(K \oplus J) \cap (A^2 \oplus B^2) = (K \oplus J) \cap (A^2 \oplus B^2)$ since $K$ and $J$ are $T$-pure fuzzy ideal of $R_1 \oplus R_2$.

Thus $K \oplus J$ is $T$-pure fuzzy ideal in $R_1 \oplus R_2$.

Proposition 2.8:

Let $I$ and $J$ are $T$-pure fuzzy ideal of $R_1 \oplus R_2$.

Then $I \cap J$ is $T$-pure fuzzy ideal of $R_1 \oplus R_2$.

Proof:

T.p $I \cap J$ is $T$-pure fuzzy ideal of $R_1 \oplus R_2$.

To show that for each fuzzy ideal $K$ of $R$ (I \cap J) \cap K is $T$-pure fuzzy ideal of $R_1 \oplus R_2$.

Now, $\forall t \in [0,1] (I \cap J)K^2 = (I \cap J)tK^2$ by [10].

$= (I \cap J)K^2$ by [10].

$= (I \cap J)K^2$ since $I$ is $T$-pure (by level).

$= I \cap J(K^2)$ by [10].

$= I \cap J(K^2)$ since $J$ is $T$-pure.

$= I \cap J(K^2)$ by [10].

Therefore $(I \cap J)K^2 = (I \cap J)K^2$.

Thus $I \cap J$ is $T$-pure fuzzy ideal of $R_1 \oplus R_2$.

Lemma 2.9:

Let $(j_n, t \in N)$ be an ascending chain of $T$-pure fuzzy ideal $R$.

Then $C^2 \subseteq \bigcup_{j \in I} C^2[j_n]$.

Proof:

$c^2 \subseteq c^2[i \in \mathbb{N}]$ implies that $\cup_{j \in I} c^2[j_n] \subseteq c^2[i \in \mathbb{N}]$.

But $c^2[i \in \mathbb{N}] \subseteq c^2$ and $c^2[i \in \mathbb{N}] \subseteq c^2$ since $c^2[i \in \mathbb{N}]$ is fuzzy ideal.

Therefore $c^2[i \in \mathbb{N}] \subseteq c^2 \cap c^2[j_n]$.

$= \cup_{j \in I} c^2[j_n]$ by [4].

Thus $c^2 \subseteq \bigcup_{j \in I} c^2[j_n]$ since $j$ is $T$-pure fuzzy ideal ideal of $R$.

Therefore $c^2 \subseteq \bigcup_{j \in I} c^2[j_n]$. 

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Proposition 2.10:
If \( \{J_i, i \in N\} \) be an ascending chain of \( T \)-pure \( F \)-I(\( R \)), then \( \bigcup_{i \in N} J_i \) is \( T \)-pure \( F \)-I(\( R \)).

Proof:
We must show that for each fuzzy ideal \( C^2 \) of \( R \)
\[
\begin{align*}
C^2 \cap \left[ \bigcup_{i \in N} J_i \right] &= C^2 \left[ \bigcup_{i \in N} J_i \right] \\
&= \bigcup_{i \in \mathbb{I}} [C^2 \cap J_i] \quad \text{by \[4\]} \\
&= \bigcup_{i \in \mathbb{I}} \left[ C^2 J_i \right] \quad \text{since} \ J_i, \text{\( T \)-pure \( F \)-I(\( R \))} \]
\[= C^2 \left[ \bigcup_{i \in N} J_i \right] \quad \text{by \lemma(2.9)}
\]
Hence \( \bigcup_{i \in N} J_i \) is \( T \)-pure \( F \)-I(\( R \)).

References