ψ-Primary Submodules

Nuhad S. AL-Mothafar[1], Adwia J. Abdil .Al-Khalik[2]

[1]Department of Mathematics, College of Science, University of Baghdad

[2]Department of Mathematics, College of Science, Al-Mustansiriya University

Abstract: ***Let R be a commutative ring with identity and M be a unitary R-module. Let $\delta(M)$ be the set of all submodules of M, and ψ: $\delta(M) \longrightarrow \delta(M) \cup \{\phi\}$ be a function. We say that a proper submodule P of M is ψ-primary if for each $r \in R$ and $x \in M$, if $rx \in P$, then either $x \in P + \psi(P)$ or $r^n M \subseteq P + \psi(P)$ for some $n \in Z_+$. Some of the properties of this concept will be investigated. Some characterizations of ψ-primesubmodules will be given, and we show that under some assumptions primesubmodules and ψ-primarysubmodules are coincide.***

Keywords: Prime submodule, Primary submodules, ϕ-primesubmodules

1. Introduction

Throughout this paper, $R$is a commutative ring with identity and M isan unitary R-module. Aproper ideal $P$of a ring R is primary if for all element $a, b \in R$, $ab \in P$ implies either $a \in P$ or $b^n \in P$ for some n $\in Z_+$, [1].In the theory of rings, primary ideals play important roles. One of the natural generalizations of primary ideals which have attracted the interest of several authors in the last two decades is the notion of primarysubmodule. These have led to more information on the structure of the R-module M. For an ideal I of R and a submodule N of M, let$\sqrt{I}$ denote the radical of I, and $[N \underset{R}{:} M] = \{r \in R : rM \subseteq N\}$ which is clearly anideal of R.A proper submodule P of$M$ is called a primarysubmodule if $r \in R$ and $x \in M$ with $rx \in P$ implies that $r^n \in [P:M]$for some n $\in Z_+$ or $x \in P$, [2].A proper ideal$I$ of R is said to be primeideal if $a.b \in I$ implies that either $a \in I$ or $b \in I$, [1].A proper submodule N of M is said to be prime submodule of M if $r \in R$ and $x \in M$with$rx \in N$ gives that $r \in [N:M]$ or $x \in N$, [3].Khaksari and Jafariextended the notion of prime submodule to ϕ-prime. Let M be an R-module andδ(M) be the set of all submodules of M and ϕ: $\delta(M) \longrightarrow \delta(M) \cup \{\phi\}$ be a function. A proper submodule P of M is said to be ϕ-prime if $r \in R$ and $x \in M$, $rx \in P \backslash \phi(P)$ implies that r ∈ [P:M] or $x \in P$ [4]. In this paper, we define and study the notion of ψ-primarysubmodules.Let $\delta(M)$ be the set of all submodules of M and ψ: $\delta(M) \longrightarrow \delta(M) \cup \{\phi\}$ be a function. A proper submodule P of M is said to be ψ-prime if for each$r \in R$and $x \in M$, if $rx \in P$, then either $x \in P + \psi(P)$ or $r^n M \subseteq P + \psi(P)$for some n ∈ Z_+.

2. Basic Properties of ψ-PrimarySubmodules

First we give the following definition.

Definition (2.1):
Let$M$be an R-module and $\delta(M)$ be the set of all submodulesof$M$.Letψ: $\delta(M) \longrightarrow \delta(M) \cup \{\phi\}$ be a function. A proper submodule N of M is said to be ψ-primary if for each $r \in R$and $x \in M$, if$rx \in N$, then $x \in N + \psi(N)$ or $r^n M \subseteq N + \psi(N)$ for some n ∈ Z_+.

Remarks and Examples (2.2)

(1) It is clear that every primary submodule of an R-module M is ψ-primarysubmodule of M, but the convers is not true in general for example:Let $M = Z_{p^\infty}$ as aZ-module, $N = <\frac{1}{pi} + Z>$where p is a prime number.Then N is not primary submodule of M, since$pk.(1/p^{i+k} + Z) = 1/pi + Z \in N$ for some $k \in Z+$.But$(1/p^{i+k} + Z) \notin N$ and $pk \notin [N: Z_{p^\infty}] = 0$. But N is ψ-primary submodule of M.

Proof: Letψ: $\delta(Z_{p^\infty}) \longrightarrow \delta(Z_{p^\infty}) \cup \{\phi\}$, where $\psi(N) = Z_{p^\infty}$, $\forall N \subseteq M$, then for each $r \in Z$, $x \in Z_{p^\infty}$, if $rx \in N$, then $x \in N + \psi(N) = Z_{p^\infty}$.Therefore N is a ψ-primary submodule ofZ_{p^∞}.

(2) If $\psi(N) \subseteq N$or $\psi(N) = 0$, then every ψ-primarysubmodule of M is a primarysubmodule.

(3) Let N, W be two submodules of an R- module $M$and $N \subseteq W$. If N is ψ-primarysubmodule of M andψ$(N) \subseteq \psi'(N)$, whereψ$'$: $\delta(W) \longrightarrow \delta(W) \cup \{\phi\}$ and ψ: $\delta(M) \longrightarrow \delta(M) \cup \{\phi\}$, then N is ψ$'$-primarysubmodule of W.

Proof: Let$r \in R, m \in W$such that $rm \in N$. Since N is ψ-primarysubmodule of M, so either$m \in N + \psi(N)$ or $r^n M \subseteq N + \psi(N)$for some n ∈ Z_+. But $\psi(N) \subseteq \psi'(N)$
, so either $m \in N + \psi'(N)$ or $r^n M \subseteq N + \psi'(N)$for some n ∈ Z_+.Hence either $m \in N + \psi'(N)$ or $r^n W \subseteq N + \psi'(N)$ for some n ∈ Z_+.Therefore N is ψ'-primary submodule of .

(4) Given two function $\psi, \psi': \delta(M) \longrightarrow \delta(M) \cup \{\phi\}$.We define $\psi \leq \psi'$ if $\psi(N) \subseteq \psi'(N)$for each $N \in \delta(M)$.If N is a ψ-primarysubmodule of M implies N isψ$'$-primarysubmodule of.

Proof: Let $r \in R, m \in M$ such that $rm \in N$. Since N is ψ-primarysubmodule of, so either $m \in N + \psi(N)$ or $r^n M \subseteq N + \psi(N)$for some n ∈ Z_+.But $\psi(N) \subseteq \psi'(N)$.so either $m \in N + \psi'(N)$ or $r^n M \subseteq N + \psi'(N)$ for some n ∈ Z_+. Therefore $N$is ψ$'$-primarysubmodule of M.

(5) Let N and W be two submodules of an R –module M such that$N \cong W$.IfNisψ-primarysubmodule of; it is not necessary that W is ψ-primarysubmodule of M as the following example explains:

Consider the $Z-$ module Z, thesubmodule$2Z$ is ψ-primarysubmodule of Z (since it is primary) but $2Z \cong 30Z$and $30Z$ is not ψ-primarysubmodule of Z.Since if $\psi(N) = N, \forall N \subseteq M$ and $6.5 = 30 \in 30Z$ but $5 \notin 30Z + 30Z = 30Z$ and $6Z \nsubseteq 30Z + 30Z = 30Z$.

(6) $I$is a ψ-primary ideal of R if and only if I is a ψ-primary submodule of R.

(7) Let $M = Z_{12}$ as a Z – module and$N = \langle\bar{6}\rangle$.$N$is notψ-primarysubmodule of M.

Proof: Let $\psi: \delta(Z12) \longrightarrow \delta(Z12) \cup \{\phi\}$, where $\psi(N) = N + \langle\bar{6}\rangle, \forall N \subseteq Z_{12}$. Now, $2.\bar{3} = \bar{6} \in N$, but $\bar{3} \notin N + \psi(N) = N$and $2^n \notin [N + \psi(N): Z_{12}] = [N: Z_{12}] = [\langle\bar{6}\rangle:_Z Z_{12}] = 6Z_{12}$for each n ∈ Z_+.

(8) The only ψ-primarysubmodule of a simple module is {0}.Therefore $(\bar{0})$ of a simple Z-module Zp (p is prime) is ψ-primarysubmodule.

(9) Let $M = Z \oplus Z$ as a Z-module, $N = 2Z \oplus (0), N$ is not ψ-primarysubmodule of M.

Proof: Let $\psi: \delta(M) \longrightarrow \delta(M) \cup \{\phi\}$ such that $\psi(N) = N, \forall N \subseteq M$. Now, $2(1,0) \in N$, but $(1,0) \notin N + \psi(N)$ and $2^n \notin [2Z \oplus (0): (Z \oplus Z)] = (0)$ for each n ∈ Z_+.

Now, if N is a primarysubmodule, then sometimes Niscalled $P-$primarysubmodule, where $P = \sqrt{[N: M]}$, [5].

For a ψ-primary, we called P- ψ-primarysubmodule, where $P = \sqrt{[N + \psi(M): M]}$

The following theorem gives some characterizations for ψ-primarysubmodules.

Theorem (2.3):

Let Nbe a propersubmodule of an R- module$M$ and$P = \sqrt{[N + \psi(M): M]}$

Then, the following statement are equivalent:

1. N isψ-primarysubmoduleof M.
2. For every submodule$K$ of M and for every an ideal I of R such that $IK \subseteq N$, implies that either $K \subseteq N + \psi(N)$or $I \subseteq P = \sqrt{[N + \psi(M): M]}$

Proof:(1) $\longrightarrow$(2):Let$IK \subseteq N$, where$I$is an ideal of R and K is a submodule of M. Suppose$K \nsubseteq N + \psi(N)$, then there exists $k \in K$ such that $k \notin N + \psi(N)$. It is clear that for each $y \in I$, thus$yk \in N$.But $N$is ψ-primarysubmodule of $M$and$k \notin N + \psi(N)$, hence $y \in P = \sqrt{[N + \psi(M): M]}$. Therefore $I \subseteq P$.

(2) $\longrightarrow$(1): Let$r \in R, m \in M$ suchthat $rm \in N$.Then$\langle r \rangle \langle m \rangle \subseteq N$. So either$\langle m \rangle \subseteq N + \psi(N)$ or $\langle r \rangle \subseteq P = \sqrt{[N + \psi(M): M]}$by (2); i.e., either$m \in N + \psi(N)$ or $r \in P = \sqrt{[N + \psi(M): M]}$Therefore N is ψ-primarysubmodule of .

We can give the following result.

Proposition (2.4):

Let N be a propersubmodule of an R- module M.If$\sqrt{[N + \psi(M): M]} = \sqrt{[N + \psi(M): K]}$for each submodule K of M such that $K \supsetneq N + \psi(N)$, then N is ψ-primarysubmodule of M.

Proof: submodule of M. Let $r \in R, m \in M$ such that $rm \in N$ and suppose $m \notin N + \psi(N)$. Let $K = N + \psi(N) + \langle m \rangle$.Thus $K \supsetneq N + \psi(N), m \in K$ andso $r \in [N: K] \subseteq [N + \psi(N): K] \subseteq \sqrt{[N + \psi(N): K]} = \sqrt{[N + \psi(M): M]}$. It follows that$r \in \sqrt{[N + \psi(M): M]}$and hence N is ψ-primary.

However, we can give another corollary of proposition (2.4).But first we state and prove the following lemma which is needed.

Lemma (2.5)

Let N be a proper submodule of an R- module M. If$\sqrt{[N + \psi(M): M]} = \sqrt{[N + \psi(M): c]}$ for each $c \in M \setminus N + \psi(N)$, then$\sqrt{[N + \psi(M): M]}$=$\sqrt{[N + \psi(M): K]}$for each submodule$K$ of M such that$K \supsetneq N + \psi(N)$.

Proof: Since $K \subseteq M$so$\sqrt{[N + \psi(M): M]} \subseteq \sqrt{[N + \psi(M): K]}$.Let $r \in \sqrt{[N + \psi(M): K]}$, hence $r^n K \subseteq N + \psi(N)$for some n $\in Z_+$.But $N + \psi(N) \subsetneq K$, implies that there exists $x \in K$ and$x \notin N + \psi(N)$.Hence $r^n x \in N + \psi(N)$ for some n $\in Z_+$and then $r \in \sqrt{[N + \psi(M): x]} = \sqrt{[N + \psi(M): M]}$, which impliesthat$\sqrt{[N + \psi(M): K} \subseteq \sqrt{[N + \psi(M): M]}$. Therefore$\sqrt{[N + \psi(M): M]} = \sqrt{[N + \psi(M): K]}$ for each submodule $K$of M such that $K \supsetneq N + \psi(N)$.

Corollary (2.6):

Let N bea proper submodule of an R- module M. If$\sqrt{[N + \psi(M): M]} = \sqrt{[N + \psi(M): c]}$ for each $c \in M \setminus N + \psi(N)$, then N is ψ-primary submodule of M.

Now, the following proposition shows that under the condition $\psi(N) \subseteq N$ for all submodule$N$ of M.the convers of proposition (2.4) is true.

Proposition(2.7):

If N is aψ-primarysubmodule of an $R-$ module $M$and $\psi(N) \subseteq N$, then$\sqrt{[N + \psi(M): M]} = \sqrt{[N + \psi(M): K]}$for each submodule$K$ of M such that$K \supsetneq N + \psi(N)$.

Proof:

Since N is a ψ-primarysubmodule of M and$\psi(N) \subseteq N$, so by (remark 2.2, (5))N is a primary submodule. Hence$\sqrt{[N : M]} = \sqrt{[N : K]}$, for each submodule K of M such that $K \supsetneq N$, [6]. Since $\psi(N) \subseteq N$, then$\sqrt{[N + \psi(M): M]} = \sqrt{[N + \psi(M): K]}$ for each submodule K of M such that $K \supsetneq N + \psi(N)$.

It is well Know if N is a primarysubmodule of an $R-$ module, then$[N: M]$ is a primary ideal of R, see [6].But for a ψ-primary we have:

Remark(2.8):
If N is ψ-primarysubmodule of M, then it is not necessarily that$[N: M]$ is aψ- primary ideal of R.

Now, the following proposition shows that under the condition $\psi(N) \subseteq N$ for all submodule $N$of M.the above statement is true.

Proposition(2.9):
If N is a ψ-primarysubmodule of an $R-$ module $M$and $\psi(N) \subseteq N$, then [N: M] is a ψ- primary ideal of R.

Proof:
Since N is a ψ-primarysubmodule of an $R-$ module $M$and $\psi(N) \subseteq N$, so N is a primarysubmodule by (2.2, 2), then [N: M] is a primary ideal of R and hence is a ψ- primary ideal of R.

Remark (2.10):
If $[N: M]$ is ψ- primary ideal of R, then it is not necessarily that N is ψ-primarysubmodule of M, for example: Let $M = Z \oplus Z$ as a Z-module, $N = 2Z \oplus (0)$, N is not ψ-primarysubmodule of M, by (2.2, 8).But $[N: M] = [2Z \oplus (0) : Z \oplus Z] = 0$ is aprimary ideal of Z and hence is ψ- primary ideal of Z.

Now, we shall give characterization of ψ-primarysubmoules, but first recall the following: Let R be any ring.A subset S of R is called multiplicatively closed if $1 \in S$ and $ab \in S$ for every $a, b \in S$. We Know that every proper ideal$P$ in$R$is prime if and only if R-P is multiplicatively closed sub set of R, [1]. And if $N$is a submodule of an R-module M and S is multiplicatively closed sub set of R, then $N(S) = \{x \in M: \exists\, t \in S, such\ that\ tx \in N\}$be a submodule of $M$and $N \subseteq N(S)$.

Proposition (2.11):
Let N be a proper submodule of an $R-$ module M.If$\sqrt{[N + \psi(M): M]}$ is a prime ideal of R and $N(S) \subseteq N + \psi(N)$ for each multiplicatively closed sub set of R such that $S \cap \sqrt{[N + \psi(M): M]} = \phi$, then N is ψ-primarysubmodule of M.

Proof: Let $r \in R, m \in M$ such that $rm \in N$ and suppose $m \notin N + \psi(N)$, $r \notin \sqrt{[N + \psi(M): M]}$.Claim the set $S = \{1, r, r^2, \dots\dots\dots\}$, this is multiplicatively closed sub set of R and it is clear that $S \cap \sqrt{[N + \psi(M): M]} = \phi$, since$\sqrt{[N + \psi(M): M]}$ is a prime ideal of R. But $m \notin N + \psi(N)$ implies that $m \notin N(S)$ and so $r\,m \notin N$ which is a contradiction.Therefore either $m \in N + \psi(N)$ or $r \in \sqrt{[N + \psi(M): M]}$and hence N is ψ-primary submoduleo f .
Conversely, if N is ψ-primarysubmodule of M, to prove $N(S) \subseteq N + \psi(N)$. Let $\in N(S)$, so there exists $t \in S$ such that $tx \in N$. But N is ψ-primarysubmodule of M, so either $x \in N + \psi(N)$ or $t \in \sqrt{[N + \psi(M): M]}$.But $t \in \sqrt{[N + \psi(M): M]}$implies that $S \cap \sqrt{[N + \psi(M): M]} = \phi$which is a contradiction.Thus, $x \in N + \psi(N)$ and hence $N(S) \subseteq N + \psi(N)$..

Proposition (2.12):
If$\sqrt{[N + \psi(M): M]}$ is maximal ideal of R, then N is ψ-primarysubmodule of M.

Proof: Let $r \in R, m \in M$ such that $rm \in N$.If $r \notin \sqrt{[N + \psi(M): M]}$, then $R = < r > + \sqrt{[N + \psi(M): M]}$.Therefore there exist $s \in R$ and $k \in \sqrt{[N + \psi(M): M]}$such that $1 = s\,r + k$ and so $m = srm + k\,m \in N + \psi(N)$ for some n $\in Z_+$. Therefore N is ψ-primarysubmodule of M.

Proposition (2.13):
Let N be a proper submodule of an R- module M such that $[K: M] \nsubseteq [N + \psi(N): M]$ for eachsubmodule$K$ of M and containing $N + \psi(N)$ properly.If $[N + \psi(N): M]$ is a primaryideal of R, then N is ψ-primarysubmodule of M.

Proof: Suppose$[N + \psi(N): M]$ is a primary ideal of R, to prove N is ψ-primarysubmodule of M. Let $r \in R, m \in M$ such that $rm \in N$and suppose$m \notin N + \psi(N)$.Let$K = N + \psi(N) + < m >$, it is clear that $N + \psi(N) \subsetneq K$, and so $[K: M] \nsubseteq [N + \psi(N): M]$.Then there exists $s \in [K: M]$ and $s \notin [N + \psi(N): M]$.Thus, $sM \subseteq K$ and $s\,M \nsubseteq N + \psi(N)$.But$sM \subseteq K$ implies, $r\,s\,M \subseteq r\,K = r(N + \psi(N) + < m >) \subseteq N + \psi(N)$and $rs \in [N + \psi(N): M]$. Since $[N + \psi(N): M]$ is a primary ideal of $R$and $s \notin [N + \psi(N): M]$, so $r^n \in [N + \psi(N): M]$ for some n $\in Z_+$. Therefore N is ψ-primarysubmodule of M.

Recall that an R- module M is called mulitplication module if for every submodule$N$ of M, there exists an ideal I of R such that $IM = N$, equivalently; for every submodule N of M, $N = [N: M]M$, see[7].

Corollary (2.14):
Let N be a proper submodule of a mulitplication R- module M. Then N is ψ-primarysubmodule of M if $[N + \psi(N): M]$ is a primary ideal of R.

Proof: Suppose $[N + \psi(N): M]$ is a primary ideal of R, to prove N is ψ-primarysubmodule of M. Let $r \in R, m \in M$ such that $rm \in N$ and suppose $m \notin N + \psi(N)$. Let $K = N + \psi(N) + < m >$, it is clear that $N + \psi(N) \subsetneq K$. Since M is multiplication, so $[K: M] \nsubseteq [N + \psi(N): M]$by[9, remark (2-15), chapter one].Then there exists $s \in [K: M]$ and $s \notin [N + \psi(N): M]$.Thus, $sM \subseteq K$and $sM \nsubseteq N + \psi(N)$.But, $sM \subseteq K$implies, $r\,s\,M \subseteq r\,K = r(N + \psi(N) + < m >) \subseteq N + \psi(N)$ and $rs \in [N + \psi(N): M]$. Since $[N + \psi(N): M]$ is a primary ideal of $R$and $s \notin [N + \psi(N): M]$, so $r^n \in [N + \psi(N): M]$for some n $\in Z_+$. Therefore N is ψ-primarysubmodule of .

As anther consequence of (2.13), we have the following result:

Corollary (2.15):

Let N be a proper submodule of a cyclic R- module M. Then N is ψ-primarysubmodule of M if $[N + \psi(N): M]$ is a primary ideal of R.

Proof:
Since$M$ is cyclic, then M is a multiplication. Hence the result follows immediately from corollary (2.14).

Recall that an R – module M is said to be a bounded module if there exists an element x ∈ M such that $ann_R M = ann_R(x)$, where $arr_R M = \{r \in R: rm = 0, \forall m \in M\}$, [8].And an R – module is said to fully stable if each submodule is stable, where a submodule N of an R – module M is said to be stable if $f(N) \subseteq N$ for each $f \in Hom(N, M)$, [9].

Corollary (2.16):
Let N be a proper submodule of a bounded fully stable R-module M. Then N is ψ-primary submodule of M if $[N + \psi(N): M]$ is a primary ideal of R.

Proof:
Since M is a bounded fully stable R- module M, so$M$ is a cyclic by [10]. Hence the result follows immediately from corollary (2.14).

proposition (2.17):
Let P be an ideal of a ring R and let M be an R – module. Then a proper submodule N of M is a P - ψ - Primary if and only if

1. $\subseteq\sqrt{[N + \psi(M): M]}$, and
2. $\notin N$, for all $c \in R \setminus P, m \in M \setminus N + \psi(N)$.

Proof:
Suppose N is a P - ψ - Primary. To prove that (1) and (2) are hold. It is clear that $P = \sqrt{[N + \psi(M): M]}$.Therefore $\subseteq\sqrt{[N + \psi(M): M]}$.

Now if $c \in R \setminus P$and $m \in M \setminus N + \psi(N)$, then $c \notin \sqrt{[N + \psi(M): M]}$ and $m \notin N + \psi(N)$, hence $cm \notin N$. Conversely, let $c \in R$ and $m \in M$ such that $m \notin N + \psi(N)$ and $c \notin \sqrt{[N + \psi(M): M]}$.Since $\subseteq\sqrt{[N + \psi(M): M]}$, then $m \in M \setminus N + \psi(N)$ and $c \notin P$.Therefore, $c \in R \setminus P$. Hence $cm \notin N$, which implies that N is a P - ψ - Primary.

proposition (2.18):
Let M be an R-module and N, L be two submodulesof M.If K be a P- ψ-primarysubmodule of M such that $N \cap L \subseteq K$, then $L \subseteq K + \psi(K)$ or $[N: M] \subseteq P = \sqrt{[K + \psi(M): M]}$

Proof:
Suppose $[N: M] \nsubseteq \sqrt{[K + \psi(M): M]} = P$, so there exists $s \in [N: M]$ and $s \notin P = \sqrt{[K + \psi(M): M]}$. Let $t \in L$, then $st \in L \cap N$ and so $st \in K$. But K is ψ-primarysubmodule of M and $s \notin \sqrt{[K + \psi(M): M]}$. Therefore $t \in K + \psi(K)$, thus $L \subseteq K + \psi(K)$.

Corollary (2.19):
Let A an ideal of R and N be a submodule of ..If K be a P-ψ-primarysubmodule of M such that $AM \cap N \subseteq K$, then either $AM \subseteq K + \psi(K)$ or $N \subseteq K + \psi(K)$.

proposition (2.20):
Let M be an R-module and N be a submoduleof M.If $P = [N + \psi(N): M]$ is a primary ideal of R, then$\sqrt{[N + \psi(M): M]} = \sqrt{[N + \psi(M): rM]}, \forall r \notin \sqrt{[N + \psi(M): M]}$.

Proof: Since $rM \subseteq M$, so$\sqrt{[N + \psi(M): M]} \subseteq \sqrt{[N + \psi(M): rM]}$.Let $a \in \sqrt{[N + \psi(M): rM]}$. Hence$a^n \in [N + \psi(N): M]$ for some $n \in Z_+$, and so $a^n r M \subseteq N + \psi(N)$ which means that $a^n r \in [N + \psi(N): M]$. But$[N + \psi(N): M]$is a primary of $R$and $r \notin [N + \psi(N): M]$, so $(a^{nm}) \in [N + \psi(N): M]$for some. Thus, $a \in \sqrt{[N + \psi(M): M]}$. Therefore, $\sqrt{[N + \psi(M): rM]} \subseteq \sqrt{[N + \psi(M): M]}$and hence$\sqrt{[N + \psi(M): M]} = \sqrt{[N + \psi(M): rM]}$.
Now, we can give the following proposition:

Proposition (2.21):
Let N be a submodule of an R – module $M$and $P = \sqrt{[N + \psi(M): M]}$. If the ideal $\sqrt{[N + \psi(N): <e>]} = P$, for each $e \in M, e \notin N + \psi(N)$, then N is a ψ- primary submodule of M.

Proof: Let $r \in R$, $x \in M$ such that $rx \in N$ and suppose $x \notin N + \psi(N)$. Thus, $r \in [N + \psi N: <x> \subseteq N + \psi N: <x>$. But $[N + \psi N: <x>] = P$, so $r \in P$. Therefore N is a ψ - primary submodule of M.

Note that, the intersection of two ψ-primarysubmodules of an R – module M need not be aψ-primary submodule of M, for examples:

(1) The Z-module Z_6 has two ψ-primarysubmodules, $N_1 = <\bar{2}>$ and $N_2 = <\bar{3}>$but $N_1 \cap N_2 = <\bar{0}>$is not a ψ-primarysubmodule of N_6. Since if$r = 3, x = \bar{2}$and$\psi(N) = N \ \forall N \subseteq M$, then $rx = 3.\bar{2} = \bar{0} \in <\bar{0}> + \psi<\bar{0}> = <\bar{0}>$.But $\bar{2} \notin <\bar{0}> + \psi<\bar{0}> = <\bar{0}>$and $3 \notin [<\bar{0}> + \psi<\bar{0}>: Z_6] = <\bar{0}>$.

(2) The Z-module Z_{12} has two ψ-primarysubmodules, $N_1 = <\bar{2}>$ and $N_2 = <\bar{3}>$.But $N_1 \cap N_2 = <\bar{6}>$ is not a ψ-primarysubmodule of Z_{12} as we have seen in (2.2, (7)).

However, we have the following proposition:

Proposition (2.22):
Let $K$is a ψ-primary of an R-module M and let $N < M$such that $\psi(K) \subseteq K$. Then either $N \subseteq K$ or $K \cap N$ is aψ'-primary in N, where $\psi': \delta(N) \longrightarrow \delta(N) \cup \{\phi\}$ and $\psi: \delta(M) \longrightarrow \delta(M) \cup \{\phi\}$.

Proof: Suppose that $N \nsubseteq K$, then $K \cap N$ is a proper submodule in N. Let $r \in R$, $m \in N$such that $rm \in K \cap N$.Suppose $m \notin (K \cap N) + \psi'(K \cap N)$, where

ψ': $\delta(N) \longrightarrow \delta(N) \cup \{\phi\}$ be a function, then $m \notin K$. We must show that $r^n N \subseteq (K \cap N) + \psi'(K \cap N)$ for some $\in Z_+$.

Since K is aψ-primarysubmodule of M and $m \notin K + \psi(K)$, this implies that $r^n M \subseteq K + \psi(K) = K$ for some $n \in Z_+$and so$r^n N \subseteq K \subseteq K \cap N \subseteq K \cap N + \psi'(K \cap N)$ for some $n \in Z_+$.Therefore $K \cap N$ is a ψ'-primary in N.

proposition (2.23):
Let $\phi: M \longrightarrow M'$ be anhomomorphism. If $N$is ψ'-primarysubmodule of an R-module M', such that $\phi(M) \nsubseteq N$ and $\psi(\phi^{-1}(N)) = \phi^{-1}(\psi'(N))$, then$\phi^{-1}(N)$ is ψ-primarysubmodule of M, where ψ': $\delta(M') \longrightarrow \delta(M') \cup \{\phi\}$ and ψ: $\delta(M) \longrightarrow \delta(M) \cup \{\phi\}$.

Proof: First, we must show that $\phi^{-1}(N)$ is a proper submodule of M. Suppose that $\phi^{-1}(N) = M$, then $\phi(M) \subseteq N$, which a contradiction to the assumption. Let $r \in R, m \in M$ such that $rm \in \phi^{-1}(N)$.Then $r\phi(m) \in N$and as N is ψ' - primarysubmodule of an R-module M', then either $\phi(m) \in N + \psi'(N)$or $r^n M' \subseteq N + \psi'(N)$for some $\in Z_+$.If $\phi(m) \in N + \psi'(N)$, then $m \in \phi^{-1}(N) + \phi^{-1}(\psi'(N))$and hence $m \in \phi^{-1}(N) + \psi\left(\phi^{-1}(N)\right)$. If $r^n M' \subseteq N + \psi'(N)$, then $r^n \phi(M) \subseteq N + \psi'(N)$ since$\phi(M) \subseteq M'$. This implies $r^n M \subseteq \phi^{-1}(N) + \phi^{-1}(\psi'(N)) = \phi^{-1}(N) + \psi(\phi^{-1}(N))$ for some $n \in Z_+$.Therefore$\phi^{-1}(N)$ is ψ-primarysubmodule of M.

Theorem (2.24):
Let $f: M \longrightarrow M'$ be an epimorphism and let $N < M$ such that $ker\, f \leq N$. If N is aψ-primarysubmodule of a module M and $\psi'(f(N)) = f(\psi(N))$, then $f(N)$ is aψ'-primarysubmodule of a module of M', where ψ': $\delta(M') \longrightarrow \delta(M') \cup \{\phi\}$ and ψ: $\delta(M) \longrightarrow \delta(M) \cup \{\phi\}$.

Proof: First, we must show that $f(N)$ is a proper submodule of a module M'. Suppose$f(N) = M'$. But f is an epimorphism, thus $f(N) = f(M)$ and hence $M = N + ker\, f$. This implies that $M = N$. A contradiction.

Now, let $r m' \in f(N)$, where $r \in R$ and $m' \in M', m' = f(m)$ for some $m \in M$since f is anepimorphism. Then $r f(m) \in f(N)$, so $f(r m) = f(n)$ for some $n \in N$ and hence $f(r m) - f(n) = 0$.Thus we get that $rm - n \in ker\, f \subseteq N$ which implies that $rm \in N$.But N is a ψ-primary, so either $m \in N + \psi(N)$ or $r^n M \subseteq N + \psi(N)$ for some $n \in Z_+$.If $m \in N + \psi(N)$, then $f(m) \in f(N) + f(\psi(N))$;that is $m' \in f(N) + f(\psi(N)) = f(N) + \psi'(f(N))$. If $r^n M \subseteq N + \psi(N)$, then $r^n f(M) \subseteq f(N) + f(\psi(N)) = f(N) + \psi'(f(N))$ implies that $r^n M' \subseteq f(N) + \psi'(f(N))$ for some $n \in Z_+$.

Corollary (2.25)
Let M be an R-module, let $K < N < M$ and N isaψ-primary of M. Then N/K is aψ'-primarysubmodule of $M / K$where ψ': $\delta(M / K) \longrightarrow \delta(M / K) \cup \{\phi\}$.

Proof: Let$\pi: M \longrightarrow M/K$be the natural mapping, then the result follows by proposition(2.25).

Proposition (2.26):
Let M be an R-module and let $K < N < M$.If$N$is aψ-primarysubmodule of M, thenN/K is aψ'-primarysubmodule of $M/K$and$\psi'(N/K) = \psi(N)/K$.

Proof: Let $r \in R$and $m \in N/K$ with $r m \in N/K$, where$m = m + K$, for some $m \in M$. So we have $rm \in N$, which gives that either $m \in N + \psi(N)$or $r M \subseteq N + \psi(N)$.Therefor either $m + K \in (N + \psi(N))/K = N/K + \psi(N)/K = N/K + \psi'(N/K)$ or $r^n M/K \subseteq (N + \psi(N))/K \subseteq N/K + \psi(N)/K = N/K + \psi'(N/K)$ for some $n \in Z_+$.Hence either $m \in N/K + \psi'(N/K)$ or $r^n M/K \subseteq N/K + \psi'(N/K)$ for some $\in Z_+$.Therefore N/K is a ψ'-primarysubmodule ofM/K.

Let S be a multiplicatively close subset of R and let R_s be the set of all fractional r/s where $r \in R$ and $s \in S$ and M_s be the set of all fractional x/s where $x \in M$and $s \in S$. For$x, x \in M$ and $s, s \in S$, $x/s = x/s$ if and only if there exists $t \in S$ such that $t(sx - sx) = 0$. So, we can make M_s in to R_s – module by setting $x/s + y/t = (tx + sy)/st$ and $r/t.x/s = rx/ts$ for every $x, y \in M$ and $s, t \in S, r \in R$. AndM_s is the module of fractions. If N be a submodule of an R – module M and S be a multiplicatively close subset of R so $N_s = \{n/s : n \in N, r \in S\}$be a submodule of the R_s – module M_s, [1, p.69].

Now, we state and prove the following proposition:

Proposition (2.27):
Let M be an R-module and N is a ψ-primary of M. Then N_s is a ψs—primarysubmodule of M_s, whereψ_s: $\delta(M_s) \longrightarrow \delta(M_s)) \cup \{\phi\}and[N + \psi(N)]_s = [N_s + \psi_s(N_s)]$and $[\psi(N)]_s = \psi_s(N_s)$.

Proof: Let $a/s \in R_s$ and $x/t \in M_s$ with $ax/st \in N_s$.Then$ax/st = n/u$ for some $n \in N, u \in S$ and so $axu = nst$ there exists $u \in S$ such that $uax \in N$. Since N is ψ-primary of M. Then either $x \in N + \psi(N)$ or$(ua)^n \in [N + \psi(N): M]$for some $n \in Z_+$. Therefore either $x/t \in [N + \psi(N)]_s = [N_s + \psi_s(N_s)]$ or $a^n/_s \in N+\psi N: Ms \subseteq N + \psi Ns: Ms = [Ns + \psi s Ns: Ms]$.Therefore$Ns$ is a ψ_s-- prime submodule of M_s.

References

[1] D.M.Burton, (1970), A first Course In Ring And Ideals, Addison – Wesley Publishing. Company.
[2] L, u, C.P., (1989), M-radicals of Submodule In Modules, math. Japon., 34, 211-219.
[3] L, u, C.P., (1981), Prime Submodule of Modules, Commutative Mathematics, University Spatula, 33, 16-69.
[4] A.Khaksari, and A.Jafari, (2011), ϕ -Prime Submodules, International Journal of Algebra, 5 (20), 1443-1440.
[5] M, E.Moore, S.J.Smith, (2002), Prime and Radical Submodules Of Modules Over Commutative Rings, Comm. Algebra, 30, 5073-5064.
[6] Adwia, J.A.(2005), Primary Modules, M.D. Thesis, University of Baghdad.

[7] Z.A. EI- Beast and P.F.Smith, (1988), Multiplication Modules, Comm. In Algebra, 16, 755-779.
[8] C. Faith, (1976), Ring Theory, Springer- Verlag, Berlin Heidelberg, New York.
[9] A.S.Mijbas, (1990), On Fully Stable Modules, Ph.D. Thesis, University of Baghdad.
[10] Ameen, Sh. A., (2002), Bounded Modules, M.sc. Thesis, University of Baghdad.