The Triangular Libration Point $L_4$ in the the Restricted Three Body Problem when the Bigger Primary is a Triaxial Rigid Body and Source of Radiation, Perturbation Effects Act in Coriolis and Centrifugal Forces

B. K. Mandal

Department of Mathematics, Govt. Women’s Polytechnic, Bokaro, Jharkhand, India

Abstract: This paper deals with the stationary solutions of the triangular libration point $L_4$ of the planar restricted three body problem when the bigger primary is a triaxial rigid body and source of radiation, perturbation effects act in coriolis and centrifugal forces with one of the axes as the axis of symmetry and its equatorial plane coinciding with the plane of motion. It is seen that there are five libration points two triangular and three collinear. It is further observed that the collinear points are unstable, while the triangular points are stable for the mass parameter $0 \leq \mu < \mu_{cris}$ (the critical mass parameter). It is further seen that the triangular points have long or short periodic elliptical orbits in the same range of $\mu$.

Keywords: Restricted Three Body Problem; Libration Points; Rigid Body; Source of Radiation; Coriolis force; Centrifugal force

1. Introduction

It is well known that the classical planar restricted three body problem possesses five libration points, two triangular and three collinear. The collinear libration points $L_1$, $L_2$, $L_3$ are unstable, while the two equilateral libration points $L_4$, $L_5$ are stable for $\mu < \mu_{cris} = 0.0385208965 \ldots \ldots$ Szebehely. Winter showed that the stability of the two equilateral points is due to the existence of coriolis terms in the equations of motion written in a synodic co-ordinate system.

In recent times many perturbing forces i.e., oblateness and radiation forces of the primaries, coriolis and centrifugal forces, variation of the masses of the primaries and of the infinitesimal mass etc., have been included in the study of the restricted three body problem. In the case of restricted three body problem where both the primaries are oblate spheroids whose equatorial plane coincides with the plane of motion, the location of libration points and their stability in the Liapunov sense has been studied by Vidyakin. For the case, where the bigger primary is an oblate spheroid whose equatorial plane coincides with the plane of motion, Subba Rao and Sharma have studied the stability of libration points. A similar problems has been studied by El-Shaboury. Khanna and Bhatnagar have studied the problem when the smaller primary is a triaxial rigid body.

In this paper, we consider the bigger primary as a triaxial rigid body and a source of radiation, perturbation effects act in coriolis and centrifugal forces with one of the axes as the axis of symmetry and its equatorial plane coinciding with the plane of motion. Further we assume that both the primaries are moving without rotation about their centre of mass in circular orbits. An attempt is made to study the existence and stability of libration points.

Garain, Mandal and Ahikary (2007) found triangular equilibrium points and also examined the stability of restricted problem of three bodies when bigger primary is a triaxial rigid body and perturbation effects act in coriolis and centrifugal forces.


Mandal (2017) found triangular libration point $L_4$ in 2+2 body problem when perturbation effects act in coriolis and centrifugal forces, small primary is a radiating body and bigger primary is a triaxial rigid body.

2. Equations of Motion

We shall adopt the notation and terminology of Szebehely. As a consequence, the distance between the primaries does not change and is taken equal to one; the sum of the masses is also taken as one. The unit of time is so chosen as to make the gravitational constant unity. Using dimensionless variables, the equations of motion of the infinitesimal mass $m_3$ in a synodic co-ordinate system $(x, y)$ are

$$\ddot{x} - 2n \dot{y} = \frac{\partial \Omega}{\partial x},$$

$$\ddot{y} - 2n \dot{x} = \frac{\partial \Omega}{\partial y},$$

where $\Omega = \frac{1}{2} n^2 [(1 - \mu) r_1^2 + \mu r_2^2]$

$$+ (1 - p) \left[ \frac{1 - \mu}{r_1} \right] + \frac{1 - \mu}{2m_1 r_1^3} (l_1 + l_2 + l_3 - 3l) + \frac{\mu}{r_2},$$

McCusky
\[ r_i^2 = (x - \mu)^2 + y^2, \]
\[ r_i^2 = (x + 1 - \mu)^2 + y^2 \]  \hspace{0.5cm} (3)
and \( p = \frac{\text{Radiation pressure due to bigger primary}}{\text{Gravitation force due to bigger primary}} < 1. \)

Hence, we have applied effect of small perturbation in coriolis and centrifugal forces with the help of perturbations \( \alpha \) and \( \beta \) the unperturbed value of both being unity
\[ \alpha = 1 + \epsilon, |\epsilon| \ll 1 \]
\[ \beta = 1 + \epsilon', |\epsilon'| \ll 1 \]

\[
\Omega \text{ in the eq. (2) can also be written as }
\]
\[
\text{Here, we have also assumed that the principal axes of }
\text{respect to its principal axes }
\]
\[
\text{where } l_1, l_2, l_3 \text{ are the direction cosines of the line with respect to its principal axes.}
\]
\[
\text{Here, we have also assumed that the principal axes of } m_1 \text{ and } m_2 \text{ are parallel to the synodic axes } O(xyz).
\]
\[
\text{The mean motion, } n, \text{ is given by }
\]
\[
n^2 = 1 + \frac{3}{2}(2A_1 - A_2 - A_3). \]  \hspace{0.5cm} (4)
\[
\text{where } A_1 = \frac{a^2}{R^2}, A_2 = \frac{b^2}{R^2}, A_3 = \frac{c^2}{R^2} \text{ and } R \text{ is the distance between the primaries.}
\]
\[
\text{Here we are neglecting the perturbation in the potential between } m_1 \text{ and } m_2 \text{ due to radiation pressure because } m_2 \text{ is supposed to be sufficiently large.}
\]
\[
\Omega \text{ in the eq. (2) can also be written as }
\]
\[
\text{The mean motion gives in the eq. (4), becomes }
\]
\[
n^2 = 1 + \frac{3}{2}(2\sigma_1 - \sigma_2). \]
\[
\text{It may be noted that the mean motion, } n, \text{ is independent of the solar radiation pressure } p.
\]

3. Location of Libration Point L4

\[
\text{Equations (1) permits an integral analogous to Jacobi integral }
\]
\[ \dot{x}^2 + \dot{y}^2 - 2\Omega + C = 0 \]

The libration points are the singularities of the manifold
\[ F(x, y; x, y) = \dot{x}^2 + \dot{y}^2 - 2\Omega + C = 0 \]

Therefore, these points are the solutions of the equations
\[ \Omega = 0, \Omega' = 0 \]
\[ L_4 = x^4 \left( - \frac{74}{256} + \frac{\sigma_1}{512} \left( \frac{585}{512} \mu + \frac{2425}{512} \right) - \frac{25}{32 \mu} \right) + \frac{15}{32 \mu} \epsilon' \]
\[ + y^4 \left( \frac{3}{128} + \frac{\sigma_1}{512} \left( \frac{3705}{512} \mu + \frac{45}{32 \mu} \right) - \frac{1395}{32 \mu} \right) + \frac{128}{3 \mu} \epsilon' \]
\[ + x^3 y \left( \frac{105\sqrt{3}}{192} - \frac{300\sqrt{3}}{192} \mu + \frac{\sigma_1}{8} \left( - \frac{2555\sqrt{3}}{384} + \frac{1060\sqrt{3}}{384} \mu + \frac{320\sqrt{3}}{192 \mu} \right) + \frac{128}{3 \mu} \epsilon' \right) \]
\[ + x y^3 \left( \frac{90\sqrt{3}}{64} - \frac{45\sqrt{3}}{64} \mu + \frac{\sigma_1}{128} \left( - \frac{1505\sqrt{3}}{256} + \frac{2285\sqrt{3}}{256} \mu - \frac{5\sqrt{3}}{4 \mu} \right) - \frac{128}{3 \mu} \epsilon' \right) \]
\[ + x^2 y^2 \left( \frac{246}{128} + \frac{345}{128} \epsilon' + \frac{\sigma_1}{256} \left( - \frac{8835}{128} + \frac{1755}{128} \mu - \frac{840}{128 \mu} \right) + \frac{256}{128} \epsilon' \right) \]

4. Stability of Triangular Libration Points

Now, we write the variational equations by putting \( x = a + \xi \) and \( y = b + \eta \) in the equations of motion (1), where \( (a, b) \) are the co-ordinate of \( L_4 \) or \( L_3 \)

The characteristic equation is
\[ \lambda^4 + \left( 4n^2 - \Omega' x - \Omega' y \right) \lambda^2 + \Omega' x \Omega' y \right)^2 - \left( \Omega' x y \right)^2 = 0 \]  \hspace{0.5cm} (5)

where
\[ \Omega' x y = \frac{3\sqrt{3}}{2} \mu - \frac{11}{18} \epsilon' (2\mu - 1) + \frac{\sigma_1}{24 \mu} (847 \mu + 89 \mu^2) + \frac{\sigma_2}{24 \mu} (-8 + 9 \mu - 37 \mu^2) \]
\[\alpha' = \frac{3}{4} + \frac{5}{4} \epsilon' + \frac{3}{16\mu} \sigma_1 (15\mu^2 - 8 + 19\mu) + \frac{3}{16\mu} \sigma_2 (-31\mu^2 + 8 - \mu) > 0\]

\[\alpha'' = \frac{9}{4} - \frac{7}{4} \epsilon' + \frac{3}{16\mu} \sigma_1 (8 + 29\mu - 15\mu^2) + \frac{3}{16\mu} \sigma_2 (-8 - 7\mu + 15\mu^2) < 0\]

Replacing \(\lambda^2\) by \(\wedge\) in the equation (5), we get

\[\lambda^2 + \mu^2 + \kappa = 0\]  \hspace{1cm} (6)

Now we have the roots

\(\lambda_1 = \lambda_2 = \pm \sqrt{\gamma}, \lambda_3 = \pm \sqrt{\delta}\) and \(\lambda_4 = \pm \sqrt{\lambda_2}\)

depend in a simple manner, on the value of the mass parameter \(\mu, P, \sigma_1\) and \(\sigma_2\)

If \(P, q_1, \sigma_1\) and \(\sigma_2\) are equal to zero then \(\mu = \mu_0\) is a root of the given equation
where \(\mu_0 = 0.0385208965 \ldots \) \ldots \ldots Szebehely
when \(P, q_1, \sigma_1\) and \(\sigma_2\) are not equal to zero. We suppose

\[\mu_{crit} = \mu_0 + r_1 P + r_2 \sigma_1 + r_3 \sigma_2 + q_1 \epsilon\]

\[\mu_{crit} = 0.0385208965 \ldots \ldots -0.0089174706 P + 0.8126474 \sigma_1 - 1.09626653 \sigma_2 - 0.14267953 \epsilon\]

Now we shall treat the three regions of the values of \(\mu\) separately.

\(0 \leq \mu < \mu_{crit}\)
we have \(-\frac{1}{2} P \leq \lambda_1 \leq 0\) and \(-\frac{1}{2} P \geq \lambda_2 \geq -P\)

But \(P > 0\), therefore \(\lambda_1\) and \(\lambda_2\) are negative. In this case the four roots of the characteristic equation are written as

\(\lambda_{1,2} = \pm i (\lambda_1) = \pm i s_1\)
and \(\lambda_{3,4} = \pm i (\lambda_2) = \pm i s_2\)

This shows that the triangular libration points are stable in linear case.

5. Conclusion

In the restricted three body problems, when the bigger primary is a triaxial rigid body and a source of radiation, perturbation effects act in coriolis and centrifugal forces. There are five libration points, three collinear and two triangular. The collinear points are unstable for all values of the mass parameter \(\mu\). Such that \(0 < \mu \leq \frac{1}{2}\) and the triangular points are stable for \(\mu < \mu_{crit}\) and \(0 \leq \mu < \mu_{crit}\), where \(\mu_{crit} = 0.0385208965 \ldots \ldots -0.0089174706 P + 0.8126474 \sigma_1 - 1.09626653 \sigma_2 - 0.14267953 \epsilon\)

Hence \(L_4\) is stable.

For \(\mu_{crit} < \mu < \frac{1}{2}\) \hspace{1cm} \(L_4\) is unstable and \n\(\mu = \mu_{crit}\), \hspace{1cm} \(L_4\) is unstable when \(P = \sigma_1 = \sigma_2 = 0\) and also \(q_1 = 0\)

The results obtained are in conformity with the classical problem Szebehely.

References


[4] D.N.Garain, B.K.Mandal and S.N.Adhikary (2007) triangular equilibrium points and stability in the restricted three bodies when bigger primary is a triaxial rigid body and perturbation effects act in coriolis and centrifugal forces ARJPS 10 (1-2)2007, pp. 27-31